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Extremal Moment Methods and Stochastic Orders

Application in Actuarial Science

Chapters IV, V and VI

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With 60 Tables

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CHAPTER IV

STOCHASTICALLY ORDERED EXTREMAL RANDOM VARIABLES

1. Preliminaries.

Given a partial order between random variables and some class of random variables, it is possible to construct *extremal random variables* with respect to this partial order, which provide useful information about extreme situations in probabilistic modelling. For example the classical Chebyshev-Markov probability inequalities yield the extremal random variables with respect to the usual stochastic order for the class of random variables with a given range and moments known up to a fixed number.

Extremal random variables with respect to the increasing convex order, also called stop-loss order, are of similar general interest. Other probability inequalities induce other kinds of extremal random variables. By taking into account various geometric restrictions, it is possible to introduce further variation into the subject.

For several purposes, which the applications of Chapter VI will make clear, it is important to compare the obtained various extremal random variables with respect to the main stochastic orders. In Section 2, several elementary comparisons of this kind are stated. Mathematically more complex proofs of simple ordering comparisons are also presented in Sections 3 to 5. Finally, Section 6 shows the possibility to construct finite atomic stop-loss confidence bounds at the example of symmetric random variables. To start with, it is necessary to introduce a minimal number of notions, definitions, notations and results, which will be used throughout the present and next chapters.

Capital letters $X, Y, \ldots$ denote random variables with distribution functions $F_X(x), F_Y(x), \ldots$ and finite means $\mu_X, \mu_Y, \ldots$. The survival functions are denoted by $\overline{F}_X(x) = 1 - F_X(x), \ldots$. The stop-loss transform of a random variable X is defined by

(1.1) $$\pi_X(x) := E[(X-x)_+] = \int_x^\infty \overline{F}_X(t)dt, \quad x \text{ in the support of } X.$$

The random variable X is said to precede Y in *stochastic order* or *stochastic dominance of first order*, a relation written as $X \le_{st} Y$, if $\overline{F}_X(x) \le \overline{F}_Y(x)$ for all x in the common support of X and Y. The random variables X and Y satisfy the *stop-loss order*, or equivalently the *increasing convex order*, written as $X \le_{sl} Y$ (or $X \le_{icx} Y$), if $\pi_X(x) \le \pi_Y(x)$ for all x. A sufficient condition for a stop-loss order relation is the *dangerousness order* relation, written as $X \le_D Y$, defined by the once-crossing condition

(1.2) $$F_X(x) \le F_Y(x) \text{ for all } x<c, \qquad F_X(x) \ge F_Y(x) \text{ for all } x \ge c,$$

where c is some real number, and the requirement $\mu_X \le \mu_Y$. By equal means $\mu_X = \mu_Y$, the ordering relations $\le_{sl}$ and $\le_D$ are precised by writing $\le_{sl,=}$ and $\le_{D,=}$. The partial stop-loss order by equal means is also called *convex order* and denoted by $\le_{cx}$. The probabilistic attractiveness of the partial order relations $\le_{st}$ and $\le_{sl}$ is corroborated by several invariance properties (e.g. Kaas et al.(1994), chap. II.2 and III.2, or Shaked and Shanthikumar(1994)).

For example, both of $\le_{st}$ and $\le_{sl}$ are closed under convolution and compounding, and $\le_{sl}$ is additionally closed under mixing and conditional compound Poisson summing.

The class of all random variables with given range $[a,b]$, $-\infty \le a < b \le \infty$, and known momnets $\mu_1, \mu_2, \ldots, \mu_n$ is denoted by $D_n([a,b];\mu_1,\mu_2,\ldots,\mu_n)$ or simply D_n in case the context is clear. For each fixed n=2,3,4,..., we denote by $F_\ell^{(n)}(x), F_u^{(n)}(x)$ the Chebyshev-Markov extremal distributions, which are solutions of the extremal moment problems

$$F_\ell^{(n)}(x) = \min_{X \in D_n}\{F_X(x)\}, \quad F_u^{(n)}(x) = \max_{X \in D_n}\{F_X(x)\}, \tag{1.3}$$

and which have been studied in detail in Section III.4 for the most important special cases n=2,3,4. They satisfy the classical probability inequalities :

(1.4) $F_\ell^{(n)}(x) \le F_X(x) \le F_u^{(n)}(x)$, uniformly for all $x \in [a,b]$, for all $X \in D_n$.

Random variables with distributions $F_\ell^{(n)}(x), F_u^{(n)}(x)$ are denoted by $X_\ell^{(n)}, X_u^{(n)}$, and are extremal with respect to the usual stochastic order, that is one has $X_u^{(n)} \le_{st} X \le_{st} X_\ell^{(n)}$ for all $X \in D_n$. For each fixed n=2,3,4, the minimal and maximal stop-loss transforms over the space D_n, which are defined and denoted by $\pi_*^{(n)}(x) := \min_{X \in D_n}\{\pi_X(x)\}, \pi^{*(n)}(x) := \min_{Y \in D_n}\{\pi_Y(x)\}$, have been studied in detail in Section III.5. Sometimes, especially from Section 3 on, the upper index n, which distinguishes between the different spaces of random variables, will be omitted without possibility of great confusion. Since there is a one-to-one correspondence between a distribution and its stop-loss transform, this is (1.1) and the fact $\overline{F}_X(x) = -\pi_X'(x)$, one defines minimal and maximal stop-loss ordered random variables $X_*^{(n)}, X^{*(n)}$ by setting for their distributions

$$F_*^{(n)}(x) = 1 + \tfrac{d}{dx}\pi_*^{(n)}(x), \quad F^{*(n)}(x) = 1 + \tfrac{d}{dx}\pi^{*(n)}(x). \tag{1.5}$$

These are extremal in the sense that $X_*^{(n)} \le_{sl} X \le_{sl} X^{*(n)}$ for all $X \in D_n$.

The once-crossing condition or dangerousness order (1.2) is not a transitive relation. Though not a proper partial order, it is an important and main tool used to establish stop-loss order between two random variables. In fact, the *transitive (stop-loss-)closure* of the order $\le_D$, denoted by $\le_{D*}$, which is defined as the smallest partial order containing all pairs (X,Y) with $X \le_D Y$ as a subset, identifies with the stop-loss order. To be precise, X precedes Y in the transitive (stop-loss-)closure of dangerousness, written as $X \le_{D*} Y$, if there is a sequence of random variables $Z_1, Z_2, Z_3, \ldots$, such that $X = Z_1$, $Z_i \le_D Z_{i+1}$, and $Z_i \to Y$ in stop-loss convergence (equivalent to convergence in distribution plus convergence of the mean). The equivalence of $\le_{D*}$ and $\le_{sl}$ is described in detail by Müller(1996). In case there are finitely many sign changes between the distributions, the stated result simplifies as follows.

Theorem 1.1. (*Dangerousness characterization of stop-loss order*) If $X \le_{sl} Y$ and $F_X(x), F_Y(x)$ cross finitely many times, then there exists a finite sequence of random variables $Z_1, Z_2, \ldots, Z_n$ such that $X = Z_1, Y = Z_n$ and $Z_i \le_D Z_{i+1}$ for all i=1,...,n–1.

Proof. This is Kaas et al.(1994), Theorem III.1.3, and our later Remark 1.1.

The stop-loss order relation (by unequal means) can be separated into a stochastic order relation followed by a stop-loss order relation by equal means, a result sometimes useful.

Theorem 1.2. (*Separation theorem for stop-loss order*) If $X \le_{sl} Y$, then there exists a random variable Z such that $X \le_{st} Z \le_{sl,=} Y$.

Proof. This is shown in Kaas et al.(1994), Theorem IV.2.1, Shaked and Shanthikumar(1994), Theorem 3.A.3, Müller(1996), Theorem 3.7.

Besides the dangerousness characterization of stop-loss order, there exists a further characterization, which is sometimes applicable in practical work, and which consists of a generalized version of the once-crossing condition (1.2) originally introduced by Karlin and Novikoff(1963) (see Hürlimann(1997k) for some new applications).

Theorem 1.3. (*Karlin-Novikoff-Stoyan-Taylor crossing conditions for stop-loss order*) Let X, Y be random variables with means μ_X, μ_Y, distributions $F_X(x), F_Y(x)$ and stop-loss transforms $\pi_X(x), \pi_Y(x)$. Suppose the distributions cross $n \geq 1$ times in the crossing points $t_1 < t_2 < \ldots < t_n$. Then one has $X \le_{sl} Y$ if, and only if, one of the following is fulfilled :

Case 1 :

The first sign change of the difference $F_Y(x) - F_X(x)$ occurs from $-$ to $+$, there is an even number of crossing points n=2m, and one has the inequalities

$$\text{(1.6)} \qquad \pi_X(t_{2j-1}) \le \pi_Y(t_{2j-1}), \quad j = 1, \ldots, m$$

Case 2 :

The first sign change of the difference $F_Y(x) - F_X(x)$ occurs from $+$ to $-$, there is an odd number of crossing points n=2m+1, and one has the inequalities

$$\text{(1.7)} \qquad \mu_X \le \mu_Y, \quad \pi_X(t_{2j}) \le \pi_Y(t_{2j}), \quad j = 1, \ldots, m$$

Proof. Two cases must be distinguished.

Case 1 : the first sign change occurs from $-$ to $+$

If $X \le_{sl} Y$, then the last sign change occurs from $+$ to $-$ (otherwise $\pi_X(x) > \pi_Y(x)$ for some $x \ge t_n$), hence n=2m is even. Consider random variables $Z_0 = Y$, $Z_{m+1} = X$, and Z_j, $j = 1, \ldots, m$, with distribution functions

$$\text{(1.8)} \qquad F_j(x) = \begin{cases} F_X(x), & x \le t_{2j-1}, \\ F_Y(x), & x \ge t_{2j-1}. \end{cases}$$

For j=1,...,m, the Karlin-Novikoff once-crossing condition between Z_{j+1} and Z_j is fulfilled with crossing point t_{2j}. A partial integration shows the following mean formulas :

(1.9) $$\mu_j := E[Z_j] = \mu_X - \pi_X(t_{2j-1}) + \pi_Y(t_{2j-1}), \quad j = 1,...,m.$$

Now, by Karlin-Novikoff, one has $Z_{j+1} \le_D Z_j$, j=1,...,m, if, and only if, the inequalities $\mu_{j+1} \le \mu_j$ are fulfilled, that is

(1.10) $$\pi_X(t_{2j-1}) - \pi_Y(t_{2j-1}) \le \pi_X(t_{2j+1}) - \pi_Y(t_{2j+1}), \quad j = 1,...,m-1,$$

and

$$\pi_X(t_{2m-1}) - \pi_Y(t_{2m-1}) \le 0,$$

which is equivalent to (1.6). Since obviously $Z_1 \le_{st} Y$, one obtains the ordered sequence

(1.11) $$X = Z_{m+1} \le_D Z_m \le_D ... \le_D Z_1 \le_{st} Z_0 = Y,$$

which is valid under (1.6) and implies the result.

Case 2 : the first sign change occurs from + to –

If $X \le_{sl} Y$, then the last sign change occurs from + to –, hence n=2m+1 is odd. Similarly to Case 1, consider random variables $Z_0 = Y$, $Z_{m+1} = X$, and Z_j, $j = 1,...,m$, with distribution functions

(1.12) $$F_j(x) = \begin{cases} F_X(x), & x \le t_{2j}, \\ F_Y(x), & x \ge t_{2j}. \end{cases}$$

For j=0,1,...,m, the once-crossing condition between Z_{j+1} and Z_j is fulfilled with crossing point t_{2j+1}. Using the mean formulas

(1.13) $$\mu_j := E[Z_j] = \mu_X - \pi_X(t_{2j}) + \pi_Y(t_{2j}), \quad j = 1,...,m,$$

the conditions for $Z_{j+1} \le_D Z_j$, that is $\mu_{j+1} \le \mu_j$, j=0,1,...,m, are therefore

$$\mu_X - \mu_Y \le \pi_X(t_2) - \pi_Y(t_2),$$

(1.14) $$\pi_X(t_{2j}) - \pi_Y(t_{2j}) \le \pi_X(t_{2j+2}) - \pi_Y(t_{2j+2}), \quad j = 1,...,m-1,$$

and

$$\pi_X(t_{2m}) - \pi_Y(t_{2m}) \le 0,$$

which is equivalent to (1.7). One obtains the ordered sequence

(1.15) $$X = Z_{m+1} \le_D Z_m \le_D ... \le_D Z_1 \le_D Z_0 = Y,$$

which is valid under (1.7) and implies the result. ◊

Remark 1.1. The sequences (1.11) and (1.15) provide an alternative more detailed constructive proof of our preceding Theorem 1.1.

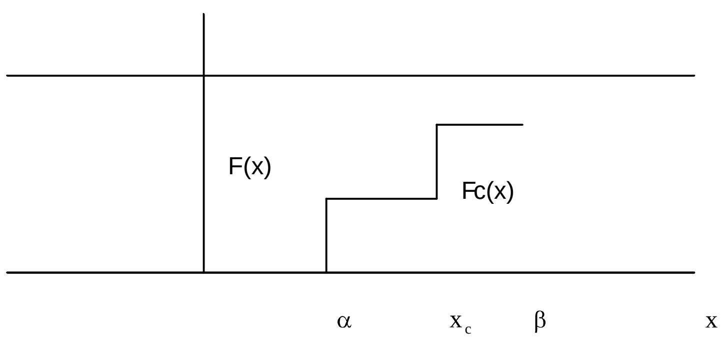
In general, the distributions of extremal random variables have a quite complex analytical structure. Therefore they require a computer algebra system for their numerical evaluations. If no implementation is available, this may be an obstacle for their use in practical work. However, relatively simple ordered discrete approximations can be constructed. By the well-known technique of *mass concentration* and *mass dispersion*, which allows to bound a given random variable by (finite atomic) less and more dangerous random variables such that in concrete applications the approximation error may be controlled.

Lemma 1.1. (*mass concentration over an interval*) Let X be a random variable with distribution $F(x)$ and stop-loss transform $\pi(x)$, and let $I=[\alpha,\beta]$ be a closed interval contained in the support of X. Then there exists a random variable $X_c \le_{D,=} X$ with distribution $F_c(x)$, obtained by concentrating the probability mass of X in I on an atom x_c of X_c, such that the mean of X over I is preserved. Its distribution function is determined as follows (see Figure 1.1) :

$$F_c(x)=\begin{cases} F(x), & x\notin I, \\ F(\alpha), & \alpha\le x< x_c, \\ F(\beta), & x_c\le x\le\beta, \end{cases} \tag{1.16}$$

$$x_c=\frac{\alpha\overline{F}(\alpha)-\beta\overline{F}(\beta)+\pi(\alpha)-\pi(\beta)}{\overline{F}(\alpha)-\overline{F}(\beta)}. \tag{1.17}$$

Figure 1.1 : mass concentration over an interval



Proof. The mean of X over I is preserved provided the atom x_c satisfies the condition $x_c\cdot\left(F(\beta)-F(\alpha)\right)=\int_\alpha^\beta x\,dF(x)$. A partial integration and a rearrangement yields (1.17). The ordering relation $X_c \le_{D,=} X$ follows from the once-crossing condition (1.2). ◊

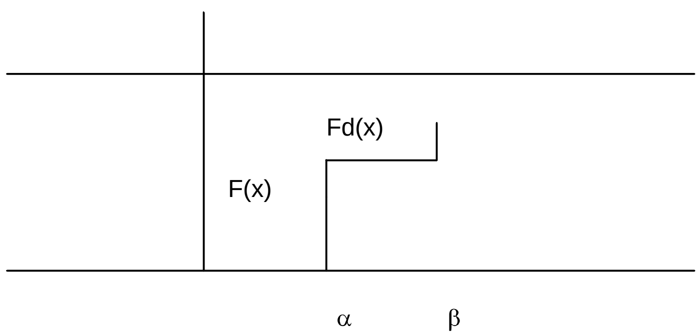
Lemma 1.2. (*mass dispersion over an interval*) Let X be a random variable with distribution $F(x)$ and stop-loss transform $\pi(x)$, and let $I=[\alpha,\beta]$ be a closed interval contained in the support of X. Then there exists a random variable $X_d \geq_{D,=} X$ with distribution $F_d(x)$, obtained by dispersing the probability mass of X in I on the pair of atoms $\{\alpha,\beta\}$ with probabilities $\{p_\alpha, p_\beta\}$, such that the probability mass and the mean of X over I are preserved. Its distribution function is determined as follows (see Figure 1.2) :

(1.18)
$$F_d(x)=\begin{cases} F(x), & x<\alpha, \\ F(\alpha)+p_\alpha = F(\beta)-p_\beta, & \alpha \le x<\beta, \\ F(x), & x \ge \beta, \end{cases}$$

(1.19)
$$p_\alpha = \frac{(\beta - x_c)\cdot(F(\beta)-F(\alpha))}{\beta-\alpha},$$
$$p_\beta = \frac{(x_c - \alpha)\cdot(F(\beta)-F(\alpha))}{\beta-\alpha},$$

(1.20)
$$x_c = \frac{\alpha\overline{F}(\alpha)-\beta\overline{F}(\beta)+\pi(\alpha)-\pi(\beta)}{\overline{F}(\alpha)-\overline{F}(\beta)}.$$

Figure 1.2 : mass dispersion over an interval



Proof. The probability mass and the mean of X over I are preserved provided the following system of equations is fulfilled :

(1.21)
$$p_\alpha + p_\beta = F(\beta)-F(\alpha),$$

(1.22)
$$p_\alpha \cdot \alpha + p_\beta \cdot \beta = x_c \cdot (F(\beta)-F(\alpha)),$$

where x_c is determined by (1.17). Its solution is straightforward. The ordering relation $X_d \geq_{D,=} X$ follows again from the once-crossing condition (1.2). ◊

2. Elementary comparisons of ordered extremal random variables.

An ultimate theoretical goal, which does not seem to be attainable in the near future, is a complete list of (elementary) stochastic ordering comparisons for the systems of Chebyshev-Markov and stop-loss ordered extremal random variables. The present Section is devoted to some elementary results in this area. Besides some of the material developed in the previous chapters, their proofs require only straightforward mathematics.

2.1. The Chebyshev-Markov extremal random variables.

As starting point, let us state the very intuitive and obvious fact that the Chebyshev-Markov stochastically ordered minimal and maximal random variables increase respectively decrease in stochastic order with an increasing number of known moments. Equivalently the range of variation of distributions with given range and known moments to a given order becomes smaller as one knows more about their moment structure. Applied to actuarial and financial theory, this means that the uncertainty about risks, when comparatively measured with respect to the stochastic order relation, is lessened in case more about their moments becomes known.

Theorem 2.1. (*Stochastic order between Chebyshev-Markov extremal random variables)* For all $X \in D_n$ and each $n \geq 3$ one has

(2.1) $$X_u^{(n-1)} \leq_{st} X_u^{(n)} \leq_{st} X \leq_{st} X_\ell^{(n)} \leq_{st} X_\ell^{(n-1)}.$$

Proof. This is trivial because $D_n \subset D_{n-1}$, and thus maxima are greater if the sets, over which maxima are taken, are enlarged. ◊

What happens by knowledge of only the mean μ? Let us complete the picture in case the range of the random variable consists of the whole real line. Markov's classical inequality gives an upper bound for the survival function, namely

(2.2) $$\overline{F}_X(x) \leq \min\left\{1, \frac{\mu}{x}\right\}, \text{ for all } x \in (-\infty,\infty), \text{ for all } X \in D_1 := D_1((-\infty,\infty);\mu).$$

The maximum is attained by a *Markov stochastically ordered maximal* random variable $X_\ell^{(1)}$, a random variable satisfying the property $X_\ell^{(1)} \geq_{st} X$ for all $X \in D_1$, and which is defined by the distribution function

(2.3) $$F_\ell^{(1)}(x) = \begin{cases} 0, & x \leq \mu, \\ 1-\dfrac{\mu}{x}, & x \geq \mu. \end{cases}$$

Since $E\left[X_\ell^{(1)}\right] = \infty$ this random variable does not belong to D_1.

Proposition 2.1. (*Chebyshev-Markov ordered maximum of order two versus Markov ordered maximum*) Let $X_\ell^{(2)}, X_\ell^{(1)}$ be the above random variables defined on $(-\infty,\infty)$. Then one has the dangerousness order relation $X_\ell^{(2)} \le_D X_\ell^{(1)}$.

Proof. Let μ, σ^2 be the mean and variance occuring in the definition of $X_\ell^{(2)}$. Recall that $X_\ell^{(2)}$ has distribution

(2.4) $$F_\ell^{(2)}(x) = \begin{cases} 0, & x \le \mu, \\ \dfrac{(x-\mu)^2}{\sigma^2 + (x-\mu)^2}, & x \ge \mu. \end{cases}$$

Given that $\mu > 0$ one shows without difficulty the once-crossing condition

(2.5) $$F_\ell^{(2)}(x) \le F_\ell^{(1)}(x), \text{ for all } x \le \mu + \frac{\sigma^2}{\mu},$$
$$F_\ell^{(2)}(x) \ge F_\ell^{(1)}(x), \text{ for all } x \ge \mu + \frac{\sigma^2}{\mu}.$$

Since $E\left[X_\ell^{(2)}\right] = \mu < E\left[X_\ell^{(1)}\right] = \infty$ one concludes that $X_l^{(2)} \le_D X_l^{(1)}$. ◊

2.2. The stop-loss ordered extremal random variables.

A next main elementary comparison states that the stop-loss ordered extremal random variables to any given order are in stop-loss order between the Chebyshev-Markov minimal and maximal random variables. Concerning applications, the use of the stop-loss ordered extremal distributions introduce a range of variation that is smaller than for the Chebyshev-Markov stochastically ordered extremal distributions. Since the stop-loss order reflects the common preferences of decision makers with a concave non-decreasing utility function, they are attractive in Actuarial Science, Finance and Economics.

Theorem 2.2. (*Stop-loss order between the Chebyshev-Markov and the stop-loss ordered extremal random variables*). For all $X \in D_n$ and any $n \ge 2$ one has

(2.6) $$X_u^{(n)} \le_{sl} X_*^{(n)} \le_{sl} X \le_{sl} X^{*(n)} \le_{sl} X_\ell^{(n)}.$$

Proof. By Theorem 2.1 one knows that $X_u^{(n)} \le_{st} X \le_{st} X_\ell^{(n)}$, for any $X \in D_n$. But the stochastic order implies the stop-loss order. Therefore one has the inequalities

(2.7) $$\pi_u^{(n)}(x) \le \pi_X(x) \le \pi_\ell^{(n)}(x), \text{ uniformly for all } x.$$

Since X is arbitrary one obtains a fortiori

(2.8) $$\pi_u^{(n)}(x) \le \min_{X \in D_n} \pi_X(x) = \pi_*(x) \le \pi_X(x) \le \max_{X \in D_n} \pi_X(x) = \pi^*(x) \le \pi_\ell^{(n)}(x),$$

uniformly for all x, which is equivalent to the affirmation. ◊

It seems that in general the sharper stochastic order comparisons

$$X_u^{(n)} \le_{st} X_*^{(n)}, \quad X^{*(n)} \le_{st} X_\ell^{(n)} \tag{2.9}$$

hold. As our application in Section VI.4 demonstrates, these sharper comparisons may indeed be required in real-life problems. For n=2,3,4 and any range $[a,b]$, $[a,\infty)$ and $(-\infty,\infty)$, a rather laborious proof of (2.9) is contained in the forthcoming Sections 3 to 5. In case n=2, a simple proof of a partial comparison result follows in Theorem 2.4.

2.3. The Hardy-Littlewood stochastic majorant.

Sometimes, as in Section VI.6, it is necessary to replace the stop-loss ordered extremal random variable $X^{*(n)}$ by a less tight majorant. An appropriate candidate is the least stochastic majorant $X^{*H(n)}$ of the family of all random variables, which precede $X^{*(n)}$ in stop-loss order. This so-called Hardy-Littlewood majorant is obtained from the following construction.

Theorem 2.3. (*Hardy-Littlewood stochastic majorant*) Given a random variable Z, let $S_Z = \{X : X \le_{sl} Z\}$ be the set of all random variables stop-loss smaller than Z. Then the least upper bound Z^H with respect to stochastic ordering for the family S_Z, that is such that $X \le_{st} Z^H$ for all $X \in S_Z$, is described by the random variable $Z^H = Z + m_Z(Z) = Z + \frac{\pi_Z(Z)}{\overline{F}_Z(Z)}$, where $m_Z(z) = E[Z - z | Z > z]$ is the mean residual life or mean excess function of Z. If $F^{-1}(u)$ is the quantile function of Z, then the quantile function of Z^H is given by

$$(F^H)^{-1}(u) = \begin{cases} \frac{1}{1-u}\int_u^1 F^{-1}(v)dv, u < 1, \\ F^{-1}(1), u = 1. \end{cases}$$

Proof. The first assertion is shown as in Meilijson and Nàdas(1979). For $x \in (E[Z], \sup\{Z\}]$, let $H(x) = x + m_Z(x)$ be the Hardy-Littlewood maximal function, and set $x_0 = H^{-1}(x) = \inf\{y : H(y) \ge x\}$. Since $\varphi(x) = (x - x_0)_+$ is a non-negative increasing convex function, one has using Markov's inequality

$$\overline{F}_X(x) = \overline{F}_{\varphi(X)}(\varphi(x)) \le \frac{E[\varphi(X)]}{\varphi(x)} \le \frac{E[\varphi(Z)]}{\varphi(x)} = \frac{\pi_Z(x_0)}{x - x_0} = \overline{F}_{Z^H}(x),$$

hence $X \le_{st} Z^H$. Sharpness of the ordering is shown as follows. With U a uniform random variable on $[0,1]$, set $X = E[F_Z^{-1}(U) | I\{U < u\}]$, where u is such that H(u)=x and I(A) is the indicator function of the event A. By Jensen's inequality, one has for each non-negative non-decreasing convex function φ that $E[\varphi(X)] \le E[E[\varphi \circ F_Z^{-1}(U) | I\{U < u\}]] = E[\varphi(Z)]$, hence $X \le_{sl} Z$. Further one has $\overline{F}_X(x) = \Pr(X = x) = \Pr(U \ge u) = \Pr(H(U) \ge x) = \overline{F}_{Z^H}(x)$, hence Z^H is the least stochatic majorant. The quantile function is already in Dubins and Gilat(1978), formula (1) (see also Kertz and Rösler(1990), p.181). ◊

In the special case n=2, a very simple proof of a partial comparison result of the type (2.9) follows. A proof of the missing comparison $X_u \le_{st} X^{*(2)}$ is postponed to Section 3.

Theorem 2.4. (*Stochastic comparisons of ordered extremal random variables*) Let X_l, X_u be the Chebyshev-Markov extremal random variables for D_2, let X_*, X^* be the stop-loss ordered extremal random variables for D_2, and let $(X^*)^H$ be the Hardy-Littlewood stochastic majorant of X^*. Then one has the stochastic ordering relations

$$X_u \le_{st} X_* \le_{st} X_\ell, \quad X^* \le_{st} (X^*)^H \le_{st} X_\ell. \tag{2.10}$$

Proof. It suffices to consider standard random variables taking values in an interval $[a,b]$ such that $1+ab \le 0$, which is the condition required for the existence of random variables with mean zero and variance one. The Chebyshev-Markov extremal standard survival functions are from Table III.4.1 described in tabular form as follows:

condition	$\overline{F}_l(x)$	$\overline{F}_u(x)$
$a < x \le \overline{b}$	1	$\frac{x^2}{1+x^2}$
$\overline{b} \le x \le \overline{a}$	$1-\frac{1+bx}{(b-a)(x-a)}$	$\frac{1+ax}{(b-a)(b-x)}$
$\overline{a} \le x < b$	$\frac{1}{1+x^2}$	0

The stop-loss ordered extremal standard survival functions are obtained from the extremal stop-loss transforms given in Tables II.5.1 and II.5.2. They are described in tabular form below. Based on Table 2.2 and Theorem 2.3, the Hardy-Littlewood majorant of X^* has survival function

$$(\overline{F}^{*H})(x) = \begin{cases} 1, & x < a, \\ \frac{(-a)}{x-a}, & a \le x < \overline{a}, \\ \frac{1}{1+x^2}, & \overline{a} \le x < b, \\ 0, & x \ge b. \end{cases} \tag{2.11}$$

It will be very useful to consider the simpler *modified Hardy-Littlewood majorant* X^{**} with survival function

$$\overline{F}^{**}(x) = \begin{cases} 1, & x < \overline{a}, \\ \frac{1}{1+x^2}, & \overline{a} \le x < b, \\ 0, & x \ge b. \end{cases} \tag{2.12}$$

Table 2.1 : Stop-loss ordered minimal standard survival function on $[a,b]$

condition	$\overline{F}_*(x)$	$\pi_*(x)$
$a \le x \le \overline{b}$	1	-x
$\overline{b} \le x \le \overline{a}$	$\frac{(-a)}{(b-a)}$	$\frac{1+ax}{b-a}$
$\overline{a} \le x \le b$	0	0

Table 2.2 : Stop-loss ordered maximal standard survival function on $[a,b]$

condition	$\overline{F}^*(x)$	$\pi^*(x)$
$a < x \le \frac{1}{2}(a+\overline{a})$	$\frac{a^2}{1+a^2}$	$(-a)\frac{1+ax}{1+a^2}$
$\frac{1}{2}(a+\overline{a}) \le x \le \frac{1}{2}(b+\overline{b})$	$\frac{1}{2}(1-\frac{x}{\sqrt{1+x^2}})$	$\frac{1}{2}(\sqrt{1+x^2}-x)$
$\frac{1}{2}(b+\overline{b}) \le x < b$	$\frac{1}{1+b^2}$	$\frac{b-x}{1+b^2}$

First, we prove the simpler fact $X_u \le_{st} X_* \le_{st} X_\ell$. A quick look at the above tables shows that the required inequalities $\overline{F}_u(x) \le_{st} \overline{F}_*(x) \le_{st} \overline{F}_\ell(x)$ are non-trivial only over the middle range $\overline{b} \le x \le \overline{a}$. An immediate calculation shows the inequalities are true provided $1+ab \le 0$, which is a required condition as stated above. The second fact $X^* \le_{st} (X^*)^H \le_{st} X_\ell$ is shown as follows. Since $X^* \le_{sl} X^*$ one has $X^* \le_{st} (X^*)^H$ by the defining property of the Hardy-Littlewood majorant. Further, the relation $X^{**} \le_{st} X_\ell$ is obvious in view of the obtained expressions for their survival functions. Since $(X^*)^H \le_{st} X^{**}$ the proof is complete. ◊

2.4. Another Chebyshev ordered maximal random variable.

Clearly it is possible to make comparisons of ordered extremal distributions for other kinds of stochastic ordering relations. We illustrate at the stochastic order induced by the classical (two-sided) Chebyshev inequality

$$\Pr(|X-\mu| \ge x) \le \min\left\{1, \frac{\sigma^2}{x^2}\right\}, \tag{2.13}$$

valid for all $x \in [0,\infty)$ and all $X \in D_2 = D_2((-\infty,\infty);\mu,\sigma^2)$. For two random variables $X, Y \in D_2$, we say that X precedes Y in *Chebyshev order*, written $X \le_T Y$, if the inequality

$$\Pr(|X-\mu| \ge x) \le \Pr(|Y-\mu| \ge x) \tag{2.14}$$

holds uniformly for all $x \in [0,\infty)$. It follows from Remark II.4.1 that the Chebyshev upper bound is attained by a triatomic random variable in D_2 with support $\{\mu - x, \mu, \mu + x\}$ and probabilities $\left\{\frac{\sigma^2}{2x^2}, 1-\frac{\sigma^2}{x^2}, \frac{\sigma^2}{2x^2}\right\}$ in case $x^2 \ge \sigma^2$, and by a diatomic random variable in D_2 with support $\{\mu - \sigma, \mu + \sigma\}$ and probabilities $\{\frac{1}{2}, \frac{1}{2}\}$ in case $x^2 \le \sigma^2$. It is less well-known that this maximum is attained by a *Chebyshev ordered maximal* random variable X^T, a random variable satisfying the property $X^T \ge_T X$ for all $X \in D_2$, whose distribution is

(2.15) $$F^T(x) = \begin{cases} \frac{1}{2}\left(\frac{\sigma}{x-\mu}\right)^2, & x \le \mu - \sigma, \\ \frac{1}{2}, & \mu - \sigma \le x \le \mu + \sigma, \\ 1 - \frac{1}{2}\left(\frac{\sigma}{x-\mu}\right)^2, & x \ge \mu + \sigma. \end{cases}$$

Indeed a calculation shows that

(2.16) $$\Pr(|X^T - \mu| \ge x) = 1 + F^T(\mu - x) - F^T(\mu + x) = \min\left\{1, \frac{\sigma^2}{x^2}\right\},$$

which shows that the Chebyshev upper bound (2.13) is attained at X^T. A probability density is

(2.17) $$f^T(x) = \begin{cases} -\frac{\sigma^2}{(x-\mu)^3}, & x \le \mu - \sigma, \\ 0, & \mu - \sigma \le x \le \mu + \sigma, \\ \frac{\sigma^2}{(x-\mu)^3}, & x \ge \mu + \sigma. \end{cases}$$

Through calculation one shows that $E[X^T] = \mu$, $Var[X^T] = \infty$, hence $X^T \notin D_2$.

Remark 2.1. It is interesting to note that (2.17) can be obtained from a first order differential equation. Consider the probability functional

(2.18) $$H_X(x) = \Pr(|X - \mu| \ge x), \; X \in D_2, \; x \in [0,\infty).$$

As stated above the maximum $H^*(x) = \max_{X \in D_2}\{H_X(x)\} = \min\left\{1, \frac{\sigma^2}{x^2}\right\}$ is attained at finite atomic *symmetric* random variables. Restrict the optimization over symmetric random variables X on $(-\infty,\infty)$ with distribution $F_X(x)$ and density $f_X(x) = F_X'(x)$. The relation $H_X(x) = 1 + F_X(\mu - x) - F_X(\mu + x)$ implies the property $f_X(\mu - x) + f_X(\mu + x) = -H_X'(x)$. In case X is symmetric around the mean, this implies the differential equation

(2.19) $$f_X(\mu + x) = -\tfrac{1}{2} H_X'(x).$$

This must also be satisfied at the extremum by a symmetric random variable X^* with probability density $f^*(x)$ such that

(2.20) $$f^*(\mu + x) = -\tfrac{1}{2} H^*{}'(x), \quad x \geq 0.$$

Through differentiation one verifies immediately that $X^* = X^T$. ◊

Concerning ordering comparisons, we obtain that X^T is in dangerousness order between $X^{*(2)}$ and $X_\ell^{(2)}$.

Proposition 2.2. The ordered extremal random variables $X_\ell^{(2)}, X_u^{(2)}, X^{*(2)}$ and X^T, all defined on $(-\infty, \infty)$, satisfy the following stochastic ordering relations :

(2.21) $$X_u^{(2)} \leq_{st} X^T \leq_{st} X_\ell^{(2)},$$

(2.22) $$X^{*(2)} \leq_{D,=} X^T.$$

Proof. Without loss of generality it suffices to consider the standardized situation $\mu = 0, \sigma = 1$. The extremal distributions are given as follows :

(2.23) $$F_\ell^{(2)}(x) = \begin{cases} 0, & x \leq 0 \\ \dfrac{x^2}{1+x^2}, & x \geq 0 \end{cases}, \qquad F_u^{(2)}(x) = \begin{cases} \dfrac{1}{1+x^2}, & x \leq 0 \\ 1, & x \geq 0 \end{cases}$$

(2.24) $$F^T(x) = \begin{cases} \dfrac{1}{2x^2}, & x \leq -1 \\ \dfrac{1}{2}, & -1 \leq x \leq 1 \\ 1 - \dfrac{1}{2x^2}, & x \geq 1 \end{cases}, \qquad F^{*(2)}(x) = \frac{1}{2}\left(1 + \frac{x}{\sqrt{1+x^2}}\right).$$

For (2.21) one shows that $F_\ell^{(2)}(x) \leq F^T(x) \leq F_u^{(2)}(x)$ uniformly for all x. To show (2.22) one verifies the once-crossing condition

(2.25) $$F^{*(2)}(x) \leq F^T(x),\ x \leq 0, \quad F^{*(2)}(x) \geq F^T(x),\ x \geq 0.$$ ◊

2.5. Ordered extremal random variables under geometric restrictions.

Finally, let us illustrate the influence of geometric restrictions on the comparison of ordered extremal random variables. Since a geometric condition, for example symmetry, unimodality, etc., imposes a restriction upon the shape of a distribution function, it is natural to expect that an ordered maximal (minimal) random variable will decrease (increase) in that order when the reference set satisfies the geometric constraint. We illustrate this point at the stop-loss ordered maximal standard random variables with infinite range $(-\infty, \infty)$.

From Table II.6.2 one derives via $\overline{F}_S^*(x) = -\pi_S^*{}'(x)$ the stop-loss ordered maximal symmetric distribution

$$(2.26) \qquad F_S^*(x) = \begin{cases} \dfrac{1}{8x^2}, & x \le -\dfrac{1}{2}, \\ \dfrac{1}{2}, & -\dfrac{1}{2} \le x \le \dfrac{1}{2}, \\ 1 - \dfrac{1}{8x^2}, & x \ge \dfrac{1}{2}, \end{cases}$$

Without the symmetric condition one has from (2.24)

$$(2.27) \qquad F^*(x) = \frac{1}{2}\left(1 + \frac{x}{\sqrt{1+x^2}}\right).$$

Let X_S^*, X^* be corresponding random variables with distribution functions $F_S^*(x), F^*(x)$. A calculation shows that the difference $F^*(x) - F_S^*(x)$ has n=3 proper sign changes, the first one from + to −, occuring at the crossing points $t_1 = -\frac{1}{4}\sqrt{7+\sqrt{17}}$, $t_2 = 0$, $t_3 = -t_1$. One observes that both means are zero (symmetric random variables) and that the stop-loss transforms are equal at $t_2 = 0$, namely $\pi^*(0) = \pi_S^*(0) = \frac{1}{2}$. Applying the extended Karlin-Novikoff crossing condition (1.7) in Theorem 1.3, one concludes that $X_S^* \le_{sl,=} X^*$.

3. The stop-loss ordered maximal random variables by known moments to order four.

Recall from Section III.5 the following structure for the maximal stop-loss transform of standard random variables by given range $[a,b]$, $-\infty \le a < b \le \infty$, and known moments to order four. There exists a finite partition $[a,b] = \bigcup_{i=1}^{m} [d_{i-1}, d_i]$ with $d_0 = a, d_m = b$, such that in each subinterval one finds a monotone increasing function $d_i(x) \in [d_{i-1}, d_i]$, the parameter x varying in some interval $[x_{i-1}, x_i]$, which one interprets as a deductible function. Then the maximal stop-loss transform on $[d_{i-1}, d_i]$ is attained at a finite atomic extremal random variable $X_i(x)$ with support $\{x_{i0}(x),..., x_{ir+1}(x)\}$ and probabilities $\{p_{i0}(x),..., p_{ir+1}(x)\}$, $x \in [x_{i-1}, x_i]$, and is given implicitely by the formula

$$(3.1) \qquad \pi^*(d_i(x)) = \sum_{j=0}^{r+1} p_{ij}(x) \cdot (x_{ij}(x) - d_i(x))_+, \; x \in [x_{i-1}, x_i], \; i = 1,...,m.$$

Applying the chain rule of differential calculus, one obtains

$$(3.2) \qquad F^*(d_i(x)) = 1 + \frac{\pi^*{}'(d_i(x))}{d_i'(x)}, \; x \in [x_{i-1}, x_i], \quad i = 1,\dots,m.$$

A thorough investigation of the analytical properties of the relation (3.1) shows then the validity of the following formula :

$$(3.3) \qquad F^*(d_i(x)) = 1 - \sum_{j=0}^{r+1} p_{ij}(x) \cdot 1_{\{x_{ij}(x) > d_i(x)\}}, \; x \in [x_{i-1}, x_i], \; i = 1,...,m.$$

The present Section contains a proof of the last relation in case the moments up to order four are given. It is based on a detailed analysis of the deductible functions $d_i(x)$ and the corresponding finite atomic extremal random variables at which $\pi^*(d_i(x))$ is attained.

Furthermore, a simple proof of the following stochastic dominance property is included:

$$X_u \le_{st} X^* \le_{st} X_\ell, \tag{3.4}$$

where X_ℓ, X_u are the Chebyshev-Markov extremal random variables by known moments up to the order four. Finally, by known mean and variance, one constructs less and more dangerous finite atomic approximations, which will be applied in Section VI.3.

3.1. The stop-loss ordered maximal random variables by known mean and variance.

The stop-loss ordered maximal distributions have already been described in tabular form in the proof of Theorem 2.4. For completeness the more structured and compact mathematical forms (3.1) and (3.3) are included here. It is also striking to observe that the deductible functions can be written as weighted averages of extremal atoms.

Theorem 3.1. The maximal stop-loss transform and the stop-loss ordered maximal distribution of an arbitrary standard random variable on $[a,b]$ are determined in Table 3.1.

Table 3.1: maximal stop-loss transform and stop-loss ordered maximal distribution on $[a,b]$

case	range of parameter	range of deductible	$\pi^*(d_i(x))$	$F^*(d_i(x))$	extremal support
(1)	$x \le a$	$a \le d_1(x) \le \frac{1}{2}(a+\bar{a})$	$p_{\bar{a}}^{(2)} \cdot (\bar{a} - d_1(x))$	$1 - p_{\bar{a}}^{(2)} = \dfrac{1}{1+a^2}$	$\{a, \bar{a}\}$
(2)	$a \le x \le \bar{b}$	$\frac{1}{2}(a+\bar{a}) \le d_2(x) \le \frac{1}{2}(b+\bar{b})$	$p_{\bar{x}}^{(2)} \cdot (\bar{x} - d_2(x))$	$1 - p_{\bar{x}}^{(2)} = \dfrac{1}{1+x^2}$	$\{x, \bar{x}\}$
(3)	$x \ge b$	$\frac{1}{2}(b+\bar{b}) \le d_3(x) \le b$	$p_b^{(2)} \cdot (b - d_3(x))$	$1 - p_b^{(2)} = \dfrac{b^2}{1+b^2}$	$\{\bar{b}, b\}$

The monotone increasing deductible functions are "weighted averages of extremal atoms" given by the formulas :

$$d_1(x) = \frac{(\bar{a}-x)a + (\bar{a}-a)\bar{a}}{(\bar{a}-x)+(\bar{a}-a)},\ d_2(x) = \frac{1}{2}(x+\bar{x}),\ d_3(x) = \frac{(b-\bar{b})\bar{b} + (x-\bar{b})b}{(b-\bar{b})+(x-\bar{b})}. \tag{3.5}$$

Proof. The formulas for the deductible functions $d_i(x)$ and the maximal stop-loss transform $\pi^*(d_i(x))$ have been described in Theorem III.5.1. To prove (3.3) one uses (3.2). In the cases (1) and (3) the relation is trivial because the atoms of the extremal support do not depend upon the parameter x. For case (2) set $d(x) = d_2(x)$, $p_{\bar{x}} = p_{\bar{x}}^{(2)}$. Then by (3.2) one sees that (3.3) holds if and only if the following identity is satisfied :

(3.6) $$\frac{d}{dx}p_{\bar{x}} \cdot (\bar{x} - d(x)) + p_{\bar{x}} \cdot \frac{d}{dx}\bar{x} = 0.$$

One concludes with elementary calculations, which show that

(3.7) $$\frac{d}{dx}\bar{x} = -\frac{\bar{x}}{x},$$

(3.8) $$\frac{d}{dx}p_{\bar{x}} = \frac{d}{dx}\left(\frac{-x}{\bar{x}-x}\right) = \frac{-2\bar{x}}{(\bar{x}-x)^2}. \ \Diamond$$

Remarks 3.1.

(i) Since the deductible functions are monotone increasing, they may be inversed, that is the parameter x may be expressed as function of the deductible $d=d(x)$. In case (2) one finds

(3.9) $$x = d - \sqrt{1+d^2},$$

which implies the explicit dependence

(3.10) $$F^*(d) = \frac{1}{2}\left(1 + \frac{d}{\sqrt{1+d^2}}\right), \quad \tfrac{1}{2}(a+\bar{a}) \le d \le \tfrac{1}{2}(b+\bar{b}),$$

as obtained previously.

(ii) For practical purposes it is useful to state the stop-loss ordered maximal distributions for the limiting cases of Table 3.1 letting $b \to \infty$ and $a \to -\infty$. For the interval $[a,\infty)$ one gets

(3.11) $$F^*(x) = \begin{cases} \dfrac{1}{1+a^2}, & a \le x \le \tfrac{1}{2}(a+\bar{a}), \\ \dfrac{1}{2}\left(1+\dfrac{x}{\sqrt{1+x^2}}\right), & x \ge \tfrac{1}{2}(a+\bar{a}), \end{cases}$$

and for the interval $(-\infty,\infty)$ one has

(3.12) $$F^*(x) = \frac{1}{2}\left(1+\frac{x}{\sqrt{1+x^2}}\right), \quad x \in (-\infty,\infty).$$

For later use, let us apply the technique of mass concentration and mass dispersion of Section 1 to derive ordered finite discrete approximations to the stop-loss ordered maximal random variables. First of all one observes that the stop-loss ordered minimal distribution over $[a,b]$ is already discrete, and thus it not necessary to find a discrete approximation to it. In fact X_* is a diatomic random variable with support $\{\bar{b},\bar{a}\}$ and probabilities $\left\{\frac{b}{b-a},\frac{(-a)}{b-a}\right\}$, as can be seen from Table 2.1. For the stop-loss ordered maximal distribution over $[a,b]$, we obtain the following result.

Proposition 3.1. Let X^* be the stop-loss ordered maximal standard random variable on $[a,b]$. Then there exists a triatomic random variable $X_c^* \le_{D,=} X^*$ with support $\left\{a, \frac{a+b}{1-ab}, b\right\}$ and probabilities $\left\{\frac{1}{1+a^2}, 1-\left(\frac{1}{1+a^2}+\frac{1}{1+b^2}\right), \frac{1}{1+b^2}\right\}$, and a 4-atomic random variable $X_d^* \ge_{D,=} X^*$ with support $\left\{a, \frac{1}{2}(a+\bar{a}), \frac{1}{2}(b+\bar{b}), b\right\}$ and probabilities $\left\{\frac{1}{1+a^2}, \frac{-(1+ab)(-a)}{(1+a^2)(b-a)}, \frac{-(1+ab)b}{(1+b^2)(b-a)}, \frac{1}{1+b^2}\right\}$.

Proof. We set $\alpha = \frac{1}{2}(a+\bar{a})$, $\beta = \frac{1}{2}(b+\bar{b})$, and apply the Lemmas 1.1 and 1.2. We use Table 2.2 and note that $F^*(x)$ is continuous over the whole open range (a,b).

Step 1 : construction of the less dangerous discrete approximation

One concentrates the probability mass of the interval $[\alpha,\beta]$ on an atom x_c of X_c^* with probability $F^*(\beta) - F^*(\alpha) = 1 - \left(\frac{1}{1+a^2}+\frac{1}{1+b^2}\right)$, where x_c is chosen such that the mean over $[\alpha,\beta]$ is preserved (use Lemma 1.1):

(3.13) $$x_c = \frac{\alpha\overline{F}^*(\alpha) - \beta\overline{F}^*(\beta) + \pi^*(\alpha) - \pi^*(\beta)}{F^*(\beta) - F^*(\alpha)}.$$

Using that $\pi^*(\alpha) = \frac{1}{2}(-a)$, $\pi^*(\beta) = \frac{1}{2}(-\bar{b})$, an elementary calculation shows that $x_c = \frac{a+b}{1-ab}$. Since $F^*(x)$ has jumps in a and b, it follows that X_c^* is the displayed triatomic random variable.

Step 2 : construction of the more dangerous discrete approximation

Mass dispersion over the interval $[\alpha,\beta]$ on the pair of atoms $\{\alpha,\beta\}$ with probabilities $\{p_\alpha, p_\beta\}$ yields by Lemma 2.2 :

(3.14) $$p_\alpha = \frac{(\beta - x_c)(F^*(\beta) - F^*(\alpha))}{\beta - \alpha} = \frac{-(1+ab)(-a)}{(1+a^2)(b-a)},$$

(3.15) $$p_\alpha = F^*(\beta) - F^*(\alpha) - p_\alpha = \frac{-(1+ab)b}{(1+b^2)(b-a)}.$$

Since $F^*(x)$ has jumps in a and b, one concludes that X_d^* is the displayed 4-atomic random variable. ◊

3.2. The stop-loss ordered maximal random variables by known skewness.

The structured form (3.1) of the maximal stop-loss transform of random variables by given range $[a,b]$ and known mean, variance and skewness γ has been described in Theorem III.5.2.

Theorem 3.2. The stop-loss ordered maximal distribution of an arbitrary standard random variale on $[a,b]$ by known skewness γ is determined in Table 3.2.

Table 3.2 : stop-loss ordered maximal distribution on $[a,b]$ by known skewness γ

case	range of parameter	$F^*(d_i(x))$	extremal support
(1)	$x \le a$	$p_a^{(3)} = \dfrac{1+\gamma b - b^2}{(b-a)(\gamma - 2a - (1+a^2)b)}$	$\{a, \varphi(a,b), b\}$
(2)	$a \le x \le c$	$p_x^{(3)} = \dfrac{1+\gamma b - b^2}{(b-x)(\gamma - 2x - (1+x^2)b)}$	$\{x, \varphi(x,b), b\}$
(3)	$x \ge b$	$1 - p_{\bar{c}}^{(2)} = \dfrac{1}{1+c^2}$	$\{c, \bar{c}\}$
(4)	$x \le a$	$1 - p_{\bar{c}}^{(2)} = \dfrac{1}{1+c^2}$	$\{c, \bar{c}\}$
(5)	$\bar{c} \le x \le b$	$1 - p_x^{(3)} = 1 - \dfrac{1+\gamma a - a^2}{(x-a)(2x - \gamma + (1+x^2)a)}$	$\{a, \varphi(a,x), x\}$
(6)	$x \ge b$	$1 - p_b^{(3)} = 1 - \dfrac{1+\gamma a - a^2}{(b-a)(2b - \gamma + (1+b^2)a)}$	$\{a, \varphi(a,b), b\}$

The monotone increasing deductible functions are "weighted averages" and given by the formulas following Table III.5.2.

Proof. Since the expressions for the maximal stop-loss transform are known, it remains to show (3.3) using (3.2) and (3.1). The cases (1), (3), (4), (6) are trivial because the extremal support does not depend upon the parameter x.

<u>Case (2)</u> :

Setting $d(x) = d_2(x)$ one has the relation $\pi^*(d(x)) = -x - p_x^{(3)}(x - d(x))$. Then (3.3) holds if and only if the identity

(3.16) $$\frac{d}{dx} p_x^{(3)} \cdot (x - d(x)) + p_x^{(3)} = 0$$

holds, or equivalently

(3.17) $$\frac{d}{dx} \ln\{p_x^{(3)}\} = \frac{1}{d(x) - x}.$$

Using the relation

(3.18) $$\frac{d}{dx} \varphi(x,b) = -\left(\frac{1 + b\varphi(x,b)}{1+bx}\right),$$

one shows with elementary calculations that

(3.19) $$\frac{d}{dx} \ln\{p_x^{(3)}\} = \frac{d}{dx}\{\ln\{1 + b\varphi(x,b)\} - \ln\{b - x\} - \ln\{\varphi(x,b) - x\}\} = \frac{1}{b-x} + \frac{2}{\varphi(x,b) - x}.$$

On the other side one has

(3.20) $$d(x)-x=\frac{(b-x)(\varphi(x,b)-b)}{2(b-x)+(\varphi(x,b)-b)},$$

from which (3.17) follows through comparison.

Case (5) :

With $d(x)=d_5(x)$ and $\pi^*(d(x))=p_x^{(3)}(x-d(x))$ one sees that (3.3) is equivalent with the identity

(3.21) $$\frac{d}{dx}\ln\{p_x^{(3)}\}=\frac{1}{d(x)-x}.$$

As above one shows that

(3.22) $$\begin{aligned}\frac{d}{dx}\ln\{p_x^{(3)}\}&=\frac{d}{dx}\{\ln\{1+a\varphi(a,x)\}-\ln\{x-a\}-\ln\{x-\varphi(a,x)\}\}\\&=-\left\{\frac{1}{x-a}+\frac{2}{x-\varphi(a,x)}\right\},\end{aligned}$$

(3.23) $$d(x)-x=-\frac{(x-a)(x-\varphi(a,x))}{2(x-a)+(x-\varphi(a,x))},$$

from which (3.21) follows through comparison. ◊

Again it is useful to state the above result for the limiting cases $b\to\infty$ or/and $a\to-\infty$. One observes that for the limiting range $(-\infty,\infty)$ one recovers (3.12). In this situation there is no improvement by additional knowledge of the skewness. For the interval $[a,\infty)$ the obtained distribution is of a reasonable mathematical tractability.

Table 3.2'' : stop-loss ordered maximal distribution on $[a,\infty)$ by known skewness γ

case	range of parameter	$F^*(d_i(x))$	extremal support
(1)	$x\le a$	$1-p_{\bar a}^{(2)}=\frac{1}{1+a^2}$	$\{a,\bar a\}$
(2)	$a\le x\le c$	$1-p_{\bar x}^{(2)}=\frac{1}{1+x^2}$	$\{x,\bar x\}$
(3)	$x\le a$	$1-p_{\bar c}^{(2)}=\frac{1}{1+c^2}$	$\{c,\bar c\}$
(4)	$x\ge\bar c$	$1-p_x^{(3)}=1-\frac{1+\gamma a-a^2}{(x-a)(2x-\gamma+(1+x^2)a)}$	$\{a,\varphi(a,x),x\}$

The deductible functions take the weighted average forms following Table III.5.2".

3.3. The stop-loss ordered maximal random variables by known skewness and kurtosis.

The structured form (3.1) of the maximal stop-loss transform of distributions by given range $[a,b]$ and known mean, variance, skewness and kurtosis, has been described in Theorem III.5.3.

Theorem 3.3. The stop-loss ordered maximal distribution of an arbitrary standard random variable on $[a,b]$ by known skewness γ and kurtosis $\gamma_2 = \delta - 3$ is given as follows :

Table 3.3 : stop-loss ordered maximal distribution on $[a,b]$ by known skewness and kurtosis, $\Delta = \delta - (\gamma^2 + 1)$

case	range of parameter	$F^*(d_i(x))$	extremal support
(1)	$x \le a$	$p_a^{(3)} = \dfrac{\Delta}{q(a)^2 + \Delta(1+a^2)}$	$\{a, \varphi(a,a^*), a^*\}$
(2)	$a \le x \le b^*$	$p_x^{(3)} = \dfrac{\Delta}{q(x)^2 + \Delta(1+x^2)}$	$\{x, \varphi(x,x^*), x^*\}$
(3)	$x \ge b$	$p_{b^*}^{(3)} = \dfrac{\Delta}{q(b^*)^2 + \Delta(1+b^{*2})}$	$\{b^*, \varphi(b^*,b), b\}$
(4)	$x \le a$	$p_{b^*}^{(3)} = \dfrac{\Delta}{q(b^*)^2 + \Delta(1+b^{*2})}$	$\{b^*, \varphi(b^*,b), b\}$
(5)	$b^* \le x \le \varphi(a,a^*)$	$p_a^{(4)} + p_x^{(4)} = \dfrac{1+b\psi}{(b-a)(\psi-a)} - \dfrac{(1+ab)(\psi-\varphi)[(b-a)+(\psi-x)]}{(b-a)(\psi-a)(b-x)(\psi-x)}$	$\{a, x, \psi, b\}$ $\psi = \psi(x;a,b)$, $\varphi = \varphi(a,b)$
(6)	$x \ge b$	$1 - p_{a^*}^{(3)} = 1 - \dfrac{\Delta}{q(a^*)^2 + \Delta(1+a^{*2})}$	$\{a, \varphi(a,a^*), a^*\}$
(7)	$x \le a$	$1 - p_{a^*}^{(3)} = 1 - \dfrac{\Delta}{q(a^*)^2 + \Delta(1+a^{*2})}$	$\{a, \varphi(a,a^*), a^*\}$
(8)	$a \le x \le b^*$	$1 - p_{x^*}^{(3)} = 1 - \dfrac{\Delta}{q(x^*)^2 + \Delta(1+x^{*2})}$	$\{x, \varphi(x,x^*), x^*\}$
(9)	$x \ge b$	$1 - p_b^{(3)} = 1 - \dfrac{\Delta}{q(b)^2 + \Delta(1+b^2)}$	$\{b^*, \varphi(b^*,b), b\}$

The monotone increasing deductible functions are defined by the "weighted averages" following Table III.5.3.

Proof. It remains to show (3.3). Clearly the cases (1), (3), (4), (6), (7), (9) are trivial. We show first the simpler cases (2) and (8), then (5).

<u>Case (2)</u> :

With $d(x) = d_2(x)$ one has the relation $\pi^*(d(x)) = -d(x) + p_x^{(3)} \cdot (d(x) - x)$, from which one deduces that (3.3) holds if and only if the following identity holds :

(3.24) $$\frac{d}{dx}\ln\{p_x^{(3)}\}=\frac{1}{d(x)-x}.$$

According to Theorem I.5.3, the value $z = x^*$ can be viewed as a real algebraic function z=z(x) obtained as unique solution in the interval $[a^*, b]$ of the quadratic equation

(3.25) $$q(x)q(z)+\Delta(1+xz)=0, \text{ with}$$
$$q(t)=1+\gamma t-t^2, \quad \Delta=\delta-\gamma^2-1.$$

Taking derivatives in (3.25) and rearranging, the derivative of the algebraic function z=z(x) can be written as

(3.26) $$\begin{aligned} z' = z'(x) &= -\frac{q'(x)q(z)+\Delta z}{q(x)q'(z)+\Delta x} = -\frac{q(z)}{q(x)}\left\{\frac{zq(x)-q'(x)(1+xz)}{xq(z)-q'(z)(1+xz)}\right\} \\ &= \frac{q(z)}{q(x)}\left\{\frac{\gamma-(x+z)-x(1+xz)}{z(1+xz)-\gamma+(x+z)}\right\} = \frac{(1+z\varphi(x,z))(\varphi(x,z)-x)}{(1+x\varphi(x,z))(z-\varphi(x,z))}, \end{aligned}$$

where, for the last equality, use has been made of the relations

(3.27) $$\begin{aligned} &(1+z\varphi(x,z))(1+xz)=q(z), \quad (1+x\varphi(x,z))(1+xz)=q(x), \\ &(\gamma-(x+z))(1+xz)=\varphi(x,z). \end{aligned}$$

It follows that

(3.28) $$\frac{d}{dx}\varphi(x,z)=-\left(\frac{1+z\varphi(x,z)}{1+xz}\right)\left(1+\frac{\varphi(x,z)-x}{z-\varphi(x,z)}\right).$$

Some laborious but elementary calculations show that

(3.29) $$\frac{d}{dx}\ln\{p_x^{(3)}\}=\frac{d}{dx}\left[\ln\{1+z\varphi(x,z)\}-\ln\{\varphi(x,z)-x\}-\ln\{z-x\}\right]=2\left[\frac{1}{z-x}+\frac{1}{\varphi(x,z)-x}\right].$$

On the other side it is immediate that

(3.30) $$d(x)-x=\frac{1}{2}\left\{\frac{(\varphi(x,z)-x)(z-x)}{(\varphi(x,z)-x)+(z-x)}\right\},$$

from which (3.24) follows through comparison.

Case (8) :

With $d(x)=d_8(x)$, $z=z(x)=x^*$, one deduces from $\pi^*(d(x))=p_z^{(3)}\cdot(z-d(x))$ that (3.3) holds exactly when

(3.31) $$\frac{1}{z'(x)}\cdot\frac{d}{dx}\ln\{p_z^{(3)}\}=\frac{1}{d(x)-z}$$

is fulfilled. Similarly to the above case (2), one shows that

(3.32)
$$\frac{1}{z'}\cdot\frac{d}{dx}\ln\{p_z^{(3)}\}=\frac{1}{z'}\cdot\frac{d}{dx}\left[\ln\{1+x\varphi(x,z)\}-\ln\{z-\varphi(x,z)\}-\ln\{z-x\}\right]$$
$$=-2\left[\frac{1}{z-x}+\frac{1}{z-\varphi(x,z)}\right],$$

(3.33)
$$d(x)-z=-\frac{1}{2}\left\{\frac{(z-\varphi(x,z))(z-x)}{(z-\varphi(x,z))+(z-x)}\right\},$$

which implies (3.31) through comparison.

Case (5) :

With $d(x)=d_5(x)$, $\psi=\psi(x;a,b)$, one has for the maximal stop-loss transform

(3.34)
$$\pi^*(d(x))=-d(x)+p_a^{(4)}\cdot(d(x)-a)+p_x^{(4)}\cdot(d(x)-x).$$

Using (3.2) one sees that (3.3) holds exactly when the following identity is fulfilled :

(3.35)
$$p_x^{(4)}\cdot\left\{1+\frac{d}{dx}\ln\{p_x^{(4)}\}\cdot(x-a)\right\}=(d(x)-a)\cdot\frac{d}{dx}(p_a^{(4)}+p_x^{(4)}).$$

To calculate the left hand side of this expression, observe that

(3.36)
$$p_x^{(4)}=\frac{\gamma-(a+b+\psi)-ab\psi}{(x-a)(b-x)(\psi-x)}=\frac{(1+ab)(\varphi(a,b)-\psi)}{(x-a)(b-x)(\psi-x)}.$$

An elementary calculation shows that

(3.37)
$$\frac{d}{dx}\ln\{p_x^{(4)}\}=\frac{d}{dx}\left[\ln\{\varphi(a,b)-\psi\}-\ln\{x-a\}-\ln\{b-x\}-\ln\{\psi-x\}\right]$$
$$=\frac{2}{\psi-x}+\frac{1}{b-x}-\frac{1}{x-a},$$

where one uses the fact that

(3.38)
$$\frac{d}{dx}\psi=-\left(\frac{\varphi(a,b)-\psi}{\varphi(a,b)-x}\right).$$

It follows that

(3.39)
$$p_x^{(4)}\cdot\left\{1+\frac{d}{dx}\ln\{p_x^{(4)}\}\cdot(x-a)\right\}=\frac{(1+ab)(\varphi(a,b)-\psi)(2(b-x)+(\psi-x))}{(b-x)^2(\psi-x)^2}.$$

Using the weighted average representation of $d(x)$ one shows without difficulty that

(3.40)
$$d(x)-a=\frac{(b-a)\left[2(b-x)+(\psi-x)\right](\psi-a)^2}{\left[2(b-x)+(\psi-x)\right](\psi-a)^2+\left[2(\psi-a)+(\psi-x)\right](b-x)^2}.$$

Combining (3.39) and (3.40) one obtains

(3.41)
$$\frac{p_x^{(4)}+\frac{d}{dx}p_x^{(4)}\cdot(x-a)}{d(x)-a}=(1+ab)(\varphi(a,b)-\psi)\cdot$$
$$\left\{\frac{2}{(b-x)(\psi-x)^2}+\frac{1}{(b-x)^2(\psi-x)}+\frac{2}{(\psi-a)(\psi-x)^2}+\frac{1}{(\psi-a)^2(\psi-x)}\right\},$$

which by (3.35) must be equal to $\frac{d}{dx}(p_a^{(4)}+p_x^{(4)})$. To show this rewrite $p_a^{(4)}$ as follows :

(3.42)
$$p_a^{(4)}=\frac{\gamma-(x+b+\psi)-xb\psi}{(b-a)(x-a)(a-\psi)}=\frac{\gamma-(a+b+\psi)-ab\psi+(1+b\psi)(a-x)}{(b-a)(x-a)(a-\psi)}$$
$$=\frac{(1+ab)(\varphi(a,b)-\psi)}{(b-a)(x-a)(a-\psi)}+\frac{1+ab}{(b-a)(\psi-a)}+\frac{b}{b-a}.$$

From (3.36) one obtains

(3.43)
$$p_x^{(4)}=\frac{(1+ab)(\varphi(a,b)-\psi)}{(b-a)}\left[\frac{1}{(x-a)(\psi-x)}+\frac{1}{(b-x)(\psi-x)}\right].$$

(3.44)
$$p_a^{(4)}+p_x^{(4)}=\frac{(1+ab)}{(b-a)}\left[\frac{\varphi(a,b)-\psi}{(b-x)(\psi-x)}+\frac{\varphi(a,b)-\psi}{(\psi-a)(\psi-x)}\right]+\frac{b}{b-a},$$

from which one gets in particular the expression displayed in Table 3.3. Through application of (3.38) and some elementary calculations one shows that

(3.45)
$$\frac{d}{dx}\frac{\varphi(a,b)-\psi}{(b-x)(\psi-x)}=(\varphi(a,b)-\psi)\cdot\left[\frac{2}{(b-x)(\psi-x)^2}+\frac{1}{(b-x)^2(\psi-x)}\right],$$

(3.46)
$$\frac{d}{dx}\frac{\varphi(a,b)-\psi}{(\psi-a)(\psi-x)}=(\varphi(a,b)-\psi)\cdot\left[\frac{2}{(\psi-a)(\psi-x)^2}+\frac{1}{(\psi-a)^2(\psi-x)}\right]-\frac{d}{dx}\frac{1}{(\psi-a)}.$$

With (3.44) one obtains that $\frac{d}{dx}(p_a^{(4)}+p_x^{(4)})$ coincides with the right-hand side of (3.41). ◊

The limiting cases $b\to\infty$ or/and $a\to-\infty$ simplify considerably. Mathematical details of the limiting process are parallel to those required to derive Tables III.5.3' and III.5.3''.

Table 3.3' : stop-loss ordered maximal distribution on $(-\infty,\infty)$ by known skewness and kurtosis

case	range of parameter	range of deductible	$F^*(d(x))$	extremal support
(1)	$x\le c$	$d(x)\le\frac{1}{2}\gamma$	$p_x^{(3)}=\frac{\Delta}{q(x)^2+\Delta(1+x^2)}$	$\{x,\varphi(x,z),z\}$
(2)	$x\ge\overline{c}$	$d(x)\ge\frac{1}{2}\gamma$	$1-p_x^{(3)}=1-\frac{\Delta}{q(x)^2+\Delta(1+x^2)}$	$\{z,\varphi(z,x),x\}$

The monotone increasing deductible function is defined by the weighted average

$$d(x)=\frac{1}{2}\left\{\frac{[\varphi(x,z)-x](x+z)+2(z-x)x}{[\varphi(x,z)-x]+(z-x)}\right\},$$

where z=z(x) is the unique solution of the quadratic equation $q(x)q(z)+\Delta(1+xz)=0$, with $q(t)=1+\gamma t-t^2$, $\Delta=\delta-\gamma^2-1$, such that $z\in[\bar{c},\infty)$ if $x\in(-\infty,c]$, respectively $z\in(-\infty,c]$ if $x\in[\bar{c},\infty)$.

Table 3.3'' : stop-loss ordered maximal distribution on $[a,\infty)$ by known skewness and kurtosis

case	range of parameter	$F^*(d_i(x))$	extremal support
(1)	$x\le a$	$p_a^{(3)}=\dfrac{\Delta}{q(a)^2+\Delta(1+a^2)}$	$\{a,\varphi(a,a^*),a^*\}$
(2)	$a\le x\le c$	$p_x^{(3)}=\dfrac{\Delta}{q(x)^2+\Delta(1+x^2)}$	$\{x,\varphi(x,x^*),x^*\}$
(3)	$x\le a$	$1-p_{\bar{c}}^{(2)}=\dfrac{1}{1+c^2}$	$\{c,\bar{c}\}$
(4)	$c\le x\le\varphi(a,a^*)$	$1-p_{\varphi}^{(4)}=$ $1-\dfrac{(1+ax)^3}{\{\gamma-2a-(1+a^2)x\}\{\gamma-2x-(1+x^2)a\}}$	$\{a,x,\varphi(a,x)\}$
(5)	$x\le a$	$1-p_{a^*}^{(3)}=1-\dfrac{\Delta}{q(a^*)^2+\Delta(1+a^{*2})}$	$\{a,\varphi(a,a^*),a^*\}$
(6)	$a\le x\le c$	$1-p_{x^*}^{(3)}=1-\dfrac{\Delta}{q(x^*)^2+\Delta(1+x^{*2})}$	$\{x,\varphi(x,x^*),x^*\}$

The monotone increasing deductible functions are defined by the "weighted averages" formulas following Table III.5.3".

Example 3.1 : skewness and kurtosis of a standard normal distribution

In the special case $\gamma=0$, $\gamma_2=0$, one gets the very simple distribution function

$$F^*(d(x))=\begin{cases}\dfrac{2}{3+x^4}, & x\le -1,\\[2mm] 1-\dfrac{2}{3+x^4}, & x\ge 1,\end{cases}\tag{3.47}$$

where the deductible function is defined by

$$d(x)=\frac{3}{4}\left\{\frac{x^4-1}{x^3}\right\}, x\in(-\infty,-1]\cup[1,\infty).\tag{3.48}$$

3.4. Comparisons with the Chebyshev-Markov extremal random variables.

Based on the simple analytical structure (3.1) and (3.3) for the maximal stop-loss transform and its associated stop-loss ordered maximal distribution, we present a simple proof of the stochastic order relation $X_u \le_{st} X^* \le_{st} X_\ell$, which appears to hold by known moments up to the fourth order and any given range $[a,b]$, $[a,\infty)$, $(-\infty,\infty)$.

Theorem 3.4. Let X_ℓ, X_u, X^* be the Chebyshev-Markov extremal and the stop-loss ordered maximal random variables by given range and known moments up to the order four, the first two assumed to be standardized. Then the stochastic order relation $X_u \le_{st} X^* \le_{st} X_\ell$ holds under each possible combination of the moment constraints.

Proof. The formulas (3.1) and (3.3) tell us that the maxima $\pi^*(d_i(x))$ and $F^*(d_i(x))$, i=1,...,m, are attained at the same finite atomic extremal random variable, say $X_i(x)$. But the last random variable is defined on the given range and satisfies the required moment constraints. By the classical Chebyshev-Markov probability inequalities, it follows that

(3.49) $$F_\ell(d_i(x)) \le F_{X_i(x)}(d_i(x)) = F^*(d_i(x)) \le F_u(d_i(x)), x \in [x_{i-1}, x_i], i = 1,...,m.$$

Since $d_i(x) \in [d_{i-1}, d_i]$ is arbitrary and $\bigcup_{i=1}^{m}[d_{i-1}, d_i]$ is a finite partition of the given range, one concludes that $F_\ell(x) \le F^*(x) \le F_u(x)$ uniformly for all x in the given range. ◊

4. The stop-loss ordered minimal random variables by known moments to order three.

The stop-loss ordered minimal standard distribution for arbitrary standard random variables with range $[a,b]$ has been described in Table 2.1, and a comparison with the Chebyshev-Markov extremal random variables has been stated and proved in Theorem 2.4.

4.1. Analytical structure of the stop-loss ordered minimal distribution.

In Theorem III.4.2 the Chebyshev-Markov extremal distributions for a standard distribution by known skewness and range $[a,b]$ have been stated and derived. In the present Section we will need their explicit analytical expressions, which are obtained in an elementary way using the explicit characterization of triatomic random variables given in Theorem III.5.2. First, the analytical structure of the minimal stop-loss transform is derived. Then, by differentiation, the stop-loss ordered minimal distribution is obtained.

Theorem 4.1. The minimal stop-loss transform $\pi_*(x) = \min\{E[(X-x)_+]\}$ over the set of all standard random variables X defined on $[a,b]$ with known skewness γ is given and attained as in Table 4.1.

Proof. From Kaas and Goovaerts(1986), Theorem 1, one knows that the minimal stop-loss transform is attained for the finite atomic random variables, at which the Chebyshev-Markov maximum $F_u(x)$ is attained. The extremal triatomic random variables, at which $\pi_*(x)$ is attained, have been given in Table III.4.2. The values of $\pi_*(x)$ in the cases (1) and (4) are immediate because all atoms are either above or below the deductible. In case (2) the formula follows from

$$\pi_*(x) = p^{(3)}_{\varphi(a,x)} \cdot (\varphi(a,x) - x) = \frac{1+ax}{\varphi(a,x)-a}, \text{ and in case (3) from}$$

$$\pi_*(x) = p^{(3)}_{b} \cdot (b - x) = \frac{1+x\varphi(x,b)}{b-\varphi(x,b)}. \ \Diamond$$

Table 4.1 : Minimal stop-loss transform for standard distributions by known skewness γ and range $[a,b]$

case	condition	$\pi_*(x)$	extremal support
(1)	$a < x \le c$	-x	$\{x, \varphi(x,b), b\}$
(2)	$c \le x \le \varphi(a,b)$	$\frac{(1+ax)^2}{\gamma - 2a - (1+a^2)x}$	$\{a, x, \varphi(a,x)\}$
(3)	$\varphi(a,b) \le x \le \bar{c}$	$\frac{1+\gamma x - x^2}{2b - \gamma + (1+b^2)x}$	$\{\varphi(x,b), x, b\}$
(4)	$\bar{c} \le x < b$	0	$\{a, \varphi(a,x), x\}$

Theorem 4.2. The stop-loss ordered minimal distribution associated to any standard random variable on $[a,b]$ by known skewness γ is determined in Table 4.2.

Table 4.2 : Stop-loss ordered minimal distribution by skewness γ and range $[a,b]$

case	condition	$F_*(x)$
(1)	$a < x \le c$	0
(2)	$c \le x \le \varphi(a,b)$	$\frac{1}{1+a^2}\left\{1+\left(\frac{1+\gamma a - a^2}{\gamma - 2a - (1+a^2)x}\right)^2\right\}$
(3)	$\varphi(a,b) \le x \le \bar{c}$	$1-\frac{1}{1+b^2}\left\{1+\left(\frac{b^2 - \gamma b - 1}{2b - \gamma + (1+b^2)x}\right)^2\right\}$
(4)	$\bar{c} \le x < b$	1

Proof. Only the cases (2) and (3) are non-trivial.

Case (2) : $x \in [c, \varphi(a,b)]$

Using Table 4.1 one obtains after elementary calculations

$$F_*(x) = 1 + \frac{d}{dx}\pi_*(x) = \frac{(a-\gamma+x)^2 + (1+ax)^2}{(\gamma - 2a - (1+a^2)x)^2}.$$

One the other side one has the partial fraction expansions

$$\frac{x+a-\gamma}{(1+a^2)x+2a-\gamma} = \frac{1}{1+a^2}\left\{1 + a\cdot\frac{a^2-\gamma a-1}{(1+a^2)x+2a-\gamma}\right\},$$

$$\frac{ax+1}{(1+a^2)x+2a-\gamma} = \frac{1}{1+a^2}\left\{a - \frac{a^2-\gamma a-1}{(1+a^2)x+2a-\gamma}\right\}.$$

Inserted in the expression for $F_*(x)$ one gets the desired formula.

Case (3) : $x \in [\varphi(a,b), \bar{c}]$

An elementary calculation shows that

$$\overline{F}_*(x) = -\frac{d}{dx}\pi_*(x) = \frac{(b-\gamma+x)^2 + (1+bx)^2}{(2b - \gamma + (1+b^2)x)^2}.$$

The same calculations as in case (2) with a replaced by b shows the desired formula. ◊

4.2. Comparisons with the Chebyshev-Markov extremal random variables.

As stated in the stochastic ordering relation (2.9), it is natural to ask if the stochastic order relation $X_u \le_{st} X_* \le_{st} X_\ell$ holds, or equivalently

(4.1) $$F_\ell(x) \le F_*(x) \le F_u(x), \text{ for all } x \in [a,b].$$

A detailed analysis shows the following sharper result.

Theorem 4.3. By known skewness and range $[a,b]$, the standard Chebyshev-Markov extremal distributions and the stop-loss ordered minimal distribution satisfy the following inequalities :

Case (1) : $0 = F_\ell(x) = F_*(x) \le F_u(x)$, for all $x \in (a,c]$

Case (2) : $F_\ell(x) \le \frac{1}{1+a^2} \le F_*(x) \le F_u(x)$, for all $x \in [c, \varphi(a,b)]$

Case (3) : $F_\ell(x) \le F_*(x) \le \frac{b^2}{1+b^2} \le F_u(x)$, for all $x \in [\varphi(a,b), \bar{c}]$

Case (4) : $F_\ell(x) \le F_*(x) = F_u(x) = 1$, for all $x \in [\bar{c}, b)$

Proof. Clearly only cases (2) and (3) require a proof. The idea is to exploit the explicit dependence upon the skewness $\gamma \in [\gamma_{min}, \gamma_{max}] = [a + \bar{a}, b + \bar{b}]$ (see Theorem I.4.1).

Case (2) : $x \in [c, \varphi(a,b)]$

Step 1 : $F_\ell(x) \le \frac{1}{1+a^2} \le F_*(x)$

The second inequality follows immediately from the expression in Table 4.2. To show the first inequality, observe that

$$p_a^{(3)}(\gamma_{min}) = 1 - p_{\varphi(a,x)}^{(3)}(\gamma_{min}) = \frac{1}{1+a^2}.$$

Since

$$\frac{\partial}{\partial\gamma} p_a^{(3)}(\gamma) = -\frac{(1+ax)^2}{(x-a)(\gamma - 2a - (1+a^2)x)} \le 0,$$

the function $p_a^{(3)}(\gamma)$ is decreasing in γ. It follows that

$$F_\ell(x) = p_a^{(3)}(\gamma) \le p_a^{(3)}(\gamma_{min}) = \frac{1}{1+a^2}.$$

Step 2 : $F_*(x) \le 1 - p_{\varphi(a,x)}^{(3)} = F_u(x)$

Using Table III.4.2, Theorem I.5.2 and Table 4.2, one must show the inequality

$$h(\gamma) := \xi(\gamma) + \frac{1}{1+a^2}\eta(\gamma)^2 \le \frac{a^2}{1+a^2},$$

where one defines

$$\xi(\gamma) := \frac{(1+ax)^3}{(\gamma - 2a - (1+a^2)x)(\gamma - 2x - (1+x^2)a)},$$

$$\eta(\gamma) := \frac{1+\gamma a - a^2}{\gamma - 2a - (1+a^2)x}.$$

But one has $\xi'(\gamma) \le 0$, $\eta'(\gamma) \le 0$, $\eta(\gamma) \ge 0$, hence $h'(\gamma) \le 0$. Since $h(\gamma)$ is monotone decreasing in γ, the affirmation follows from

$$h(\gamma) \le h(\gamma_{min}) = \xi(\gamma_{min}) + \frac{1}{1+a^2}\eta(\gamma_{min})^2 = \frac{a^2}{1+a^2} + 0 = \frac{a^2}{1+a^2}.$$

Case (3) : $x \in [\varphi(a,b), \bar{c}]$

Step 1 : $F_*(x) \le \frac{b^2}{1+b^2} \le F_u(x)$

The first inequality follows immediately from the expression in Table 4.2. To show the second inequality, observe that

$$1 - p_b^{(3)}(\gamma_{max}) = p_{\varphi(b,x)}^{(3)}(\gamma_{max}) = \frac{b^2}{1+b^2}.$$

Since

$$\frac{\partial}{\partial\gamma} p_b^{(3)}(\gamma) = \frac{(1+bx)^2}{(b-x)(2b-\gamma+(1+b^2)x)} \geq 0,$$

the function $p_b^{(3)}(\gamma)$ is increasing in γ. It follows that

$$\frac{b^2}{1+b^2} = 1 - p_b^{(3)}(\gamma_{\max}) \leq 1 - p_b^{(3)}(\gamma) = F_u(x).$$

<u>Step 2</u> : $F_\ell(x) = p^{(3)}_{\varphi(x,b)} \leq F_*(x)$

Using Table III.4.2, Theorem I.5.2 and Table 4.2, one must show the inequality

$$h(\gamma) := \xi(\gamma) + \frac{1}{1+b^2}\eta(\gamma)^2 \leq \frac{b^2}{1+b^2},$$

where one defines

$$\xi(\gamma) := \frac{(1+bx)^3}{(2b-\gamma+(1+b^2)x)(2x-\gamma+(1+x^2)b)},$$

$$\eta(\gamma) := \frac{1+\gamma b - b^2}{2b-\gamma+(1+b^2)x}.$$

But one has $\xi'(\gamma) \geq 0$, $\eta'(\gamma) \geq 0$, $\eta(\gamma) \geq 0$, hence $h'(\gamma) \geq 0$. Since $h(\gamma)$ is monotone increasing in γ, the affirmation follows from

$$h(\gamma) \leq h(\gamma_{\max}) = \xi(\gamma_{\max}) + \frac{1}{1+b^2}\eta(\gamma_{\max})^2 = \frac{b^2}{1+b^2} + 0 = \frac{b^2}{1+b^2}. \ \Diamond$$

4.3. Small atomic ordered approximations to the stop-loss ordered minimum.

First, a less dangerous finite atomic random variable $X_*^c \leq_{D,=} X_*$ is constructed from Table 4.2 by concentrating the probability masses in the subintervals $[c, \varphi(a,b)]$ and $[\varphi(a,b), \bar{c}]$ on two atoms x_0, y_0. The resulting random variable X_*^c is in general based on a 5-atomic support.

Proposition 4.1. Let X_* be the stop-loss ordered minimal random variable defined on $[a,b]$ by known skewness γ. Then there exists a 5-atomic random variable $X_*^c \leq_{D,=} X_*$ with support $\{c, x_0, \varphi(a,b), y_0, \bar{c}\}$ and probabilities
$\{F_*(c), F_*(\varphi(a,b)^-) - F_*(c), F_*(\varphi(a,b)^+) - F_*(\varphi(a,b)^-), F_*(\bar{c}) - F_*(\varphi(a,b)^-), \overline{F}_*(\bar{c})\}$,
where the displayed quantities are given by the following formulas :

(4.2) $$F_*(c) = \frac{\gamma - 2c}{\gamma - 2a - (1+a^2)c}, \ \overline{F}_*(\bar{c}) = \frac{2\bar{c} - \gamma}{2b - \gamma + (1+b^2)\bar{c}}, \ c = \tfrac{1}{2}(\gamma - \sqrt{4+\gamma^2}),$$

(4.3) $$F_*(\varphi(a,b)^-) = \frac{1+b^2}{(b-a)^2}, \ \overline{F}_*(\varphi(a,b)^+) = \frac{1+a^2}{(b-a)^2},$$

(4.4) $$x_0 = \frac{c\overline{F}_*(\overline{c}) - \varphi(a,b)\overline{F}_*(\varphi(a,b)^-) + \pi_*(c) - \pi_*(\varphi(a,b))}{F_*(\varphi(a,b)^-) - F_*(c)}$$

(4.5) $$y_0 = \frac{\varphi(a,b)\overline{F}_*(\varphi(a,b)^+) - \overline{c}\overline{F}_*(\overline{c}) + \pi_*(\varphi(a,b)) - \pi_*(\overline{c})}{F_*(\overline{c}) - F_*(\varphi(a,b)^+)}$$

Proof. Formulas (4.4) and (4.5) follow by applying Lemma 2.1 to the intervals $[c, \varphi(a,b)]$ and $[\varphi(a,b), \overline{c}]$. The formulas (4.2) are checked by observing that $c, \overline{c}$ are zeros of the quadratic equation $z^2 - \gamma z - 1 = 0$. The formulas (4.3) are obtained most simply from the equivalent analytical representations

(4.6) $$F_*(x) = \frac{1}{1+a^2}\left\{1 + \left(\frac{1+a\varphi(a,x)}{\varphi(a,x)-a}\right)^2\right\},\ x \in [c, \varphi(a,b)],$$

(4.7) $$\overline{F}_*(x) = \frac{1}{1+b^2}\left\{1 + \left(\frac{1+b\varphi(x,b)}{b-\varphi(x,b)}\right)^2\right\},\ x \in [\varphi(a,b), \overline{c}].$$

Indeed putting $x = \varphi(a,b)$ into (4.6) and using that $\varphi(a, \varphi(a,b)) = b$ yields immediately the value of $F_*(\varphi(a,b)^-)$. Simlarly putting $x = \varphi(a,b)$ into (4.7) and using that $\varphi(\varphi(a,b), b) = a$ shows the second formula in (4.3). Finally the support of X_*^c is in general 5-atomic because $F_*(x)$ may have discontinuities at $x = c, \varphi(a,b), \overline{c}$. ◊

Remark 4.1. The distribution $F_*(x)$ is continuous at $x = \varphi(a,b)$ only if $b = \overline{a}$, $\gamma = \gamma_{min} = \gamma_{max} = a + \overline{a} = b + \overline{b}$. This degenerate situation must be analyzed separately.

Next, a more dangerous finite atomic random variable $X_*^d \geq_{D,=} X_*$ is constructed from Table 4.2 by dispersing the probability masses in the subintervals $[c, \varphi(a,b)]$ and $[\varphi(a,b), \overline{c}]$ on the two pairs of atoms $\{c, \varphi(a,b)\}$ and $\{\varphi(a,b), \overline{c}\}$. The resulting random variable X_*^d is triatomic with support $\{c, \varphi(a,b), \overline{c}\}$.

Proposition 4.2. Let X_* be the stop-loss ordered minimal random variable defined on $[a,b]$ by known skewness γ. Then there exists a triatomic random variable $X_*^d \geq_{D,=} X_*$ with support $\{c, \varphi(a,b), \overline{c}\}$ and probabilities

$$\left\{1 - \frac{\pi_*(c) - \pi_*(\varphi(a,b))}{\varphi(a,b) - c}, \frac{\pi_*(c) - \pi_*(\varphi(a,b))}{\varphi(a,b) - c} - \frac{\pi_*(\varphi(a,b)) - \pi_*(\overline{c})}{\overline{c} - \varphi(a,b)}, \frac{\pi_*(\varphi(a,b)) - \pi_*(\overline{c})}{\overline{c} - \varphi(a,b)}\right\}$$

Proof. Mass dispersion over the subinterval $[c, \varphi(a,b)]$ on the pair of atoms $\{c, \varphi(a,b)\}$ with probabilities $\{p_c, p_{\varphi^-}\}$ yields through application of Lemma 2.2 the formulas

$$p_c = \overline{F}_*(c) - \frac{\pi_*(c) - \pi_*(\varphi(a,b))}{\varphi(a,b) - c},\quad p_{\varphi^-} = \frac{\pi_*(c) - \pi_*(\varphi(a,b))}{\varphi(a,b) - c} - \overline{F}_*(\varphi(a,b)^-).$$

A similar mass dispersion over $[\varphi(a,b), \overline{c}]$ yields probabilities

$$p_{\varphi^+} = \overline{F}_*(\varphi(a,b)^+) - \frac{\pi_*(\varphi(a,b)) - \pi_*(\overline{c})}{\overline{c} - \varphi(a,b)}, \quad p_{\overline{c}} = \frac{\pi_*(\varphi(a,b)) - \pi_*(\overline{c})}{\overline{c} - \varphi(a,b)} - \overline{F}_*(\overline{c}).$$

Through addition one finds that the random variable X_*^d with support $\{c, \varphi(a,b), \overline{c}\}$ has probabilities equal to the desired ones. ◊

4.4. The special case of a one-sided infinite range.

All of the preceding results, valid for an arbitrary finite interval $[a,b]$, can be formulated for the important limiting case $b \to \infty$. Of main help for this is the limiting relation $\lim_{b\to\infty} \varphi(a,b) = \overline{a}$. One finds that the corresponding stop-loss ordered minimal random variable X_* has distribution given in Table 4.2''.

Table 4.2'' : Stop-loss ordered minimal distribution by skewness γ and range $[a,\infty)$

case	condition	$F_*(x)$
(1)	$a < x \le c$	0
(2)	$c \le x \le \overline{a}$	$\frac{1}{1+a^2}\left\{1+\left(\frac{1+\gamma a - a^2}{\gamma - 2a - (1+a^2)x}\right)^2\right\}$
(3)	$x \ge \overline{a}$	1

In this situation the construction of the finite atomic stop-loss ordered confidence bounds $X_*^c \le_{D,=} X_* \le D_{D,=} X_*^d$ simplifies considerably. In particular there exist diatomic less and more dangerous bounds.

Proposition 4.3. Let X_* be the stop-loss ordered minimal random variable on $[a,\infty)$ with known skewness $\gamma \in [a+\overline{a},\infty)$. Then there exists a diatomic random variable $X_*^c \le_{D,=} X_*$ with support $\left\{c, (-c)\frac{F_*(c)}{\overline{F}_*(c)}\right\} = \left\{c, \frac{1+c^2}{c-2a-a^2c}\right\}$ and probabilities $\{F_*(c), \overline{F}_*(c)\}$, with $F_*(c) = \frac{\gamma - 2c}{\gamma - 2a - (1+a^2)c}$, and a diatomic random variable $X_*^d \ge_{D,=} X_*$ with support $\{c, \overline{a}\}$ and probabilities $\left\{\frac{\overline{a}}{\overline{a}-c}, \frac{-c}{\overline{a}-c}\right\}$.

Proof. Concentrating the probability mass over $[c,\overline{a}]$ on the atom x_0, one finds, taking into account that $\overline{F}_*(\overline{a}) = 0$, that

$$x_0 = c + \frac{\int_c^{\overline{a}} \overline{F}_*(x)dx}{\overline{F}_*(c)}.$$

Since $E\left[X_*^\ell\right] = E\left[X_*\right] = c + \int_c^{\bar{a}} \overline{F}_*(x)dx = 0$, one gets the desired value of x_0. The less dangerous bound is diatomic because $\overline{F}_*(\bar{a}) = 0$. Calculation of the more dangerous diatomic bound through mass dispersion presents no difficulty and is left to the reader. Observe that it is also possible to take the limit as $b \to \infty$ in the first part of the proof of Proposition 4.2. ◊

Remark 4.2. Clearly the stochastic order properties of Theorem 4.3 carry over to the limiting situation $[a,\infty)$. In particular one has the inequalities

$$F_\ell(x) \le \frac{1}{1+a^2} \le F_*(x) \le F_u(x), \quad x \in [c,\bar{a}].$$

This suggest to consider the diatomic random variable $\tilde{X}$ with support $\{c,\bar{a}\}$ and probabilities $\left\{\frac{1}{1+a^2},\frac{a^2}{1+a^2}\right\}$ as discrete approximation of X_*. Comparing means, one finds that $E[X_*] = 0 \le E[\tilde{X}]$. A comparison of the stop-loss premiums of X_*^d and $\tilde{X}$ shows that

$$\pi_*^d(x) = (-c)\left(\frac{\bar{a}-x}{\bar{a}-c}\right) \le \frac{a^2}{1+a^2}(\bar{a}-x)$$

uniformly for all $x \in [c,\bar{a}]$. Therefore one has $X_*^d \le_{sl} \tilde{X}$, which means that the diatomic approximation X_*^d is stop-loss tighter than $\tilde{X}$.

Example 4.1. Of special importance is the special case $a = -\frac{\mu}{\sigma} = -\frac{1}{k},\ \gamma \in \left[\frac{k^2-1}{k},\infty\right)$, where μ, σ, k are the mean, variance and coefficient of variation of a random variable with known skewness γ, which in the non-standard scale is defined on the positive real line $[0,\infty)$. The corresponding stop-loss ordered minimal distribution is given by

$$F_*(x) = \begin{cases} 0, \quad x \in \left[-\frac{1}{k}, c\right], \\ \frac{k^2}{1+k^2}\left\{1+\left(\frac{k^2-\gamma k-1}{\gamma k^2+2k-(1+k^2)x}\right)^2\right\}, \quad x \in [c,k], \\ 1, \quad x \in [k,\infty) \end{cases}$$

Its less and more dangerous diatomic bounds are found from Proposition 4.3.

5. The stop-loss ordered minimal random variables by known skewness and kurtosis.

Similarly to Section 4, the explicit analytical structure of the stop-loss ordered minimal distribution is required in order to compare it with the Chebyshev-Markov extremal distributions. Then small atomic ordered discrete approximations are displayed.

5.1. Analytical structure of the stop-loss ordered minimal distribution.

Recall that the minimal stop-loss transform values are attained at the finite atomic random variables, which solve the Chebyshev-Markov problem. Using Table III.4.3 , one obtains Table 5.1.

According to Theorem I.5.3, the value $z = x^*$ can be viewed as a real algebraic function z=z(x) obtained as the unique solution of the quadratic equation

(5.1) $$q(x)q(z)+\Delta(1+xz)=0, \text{ with } q(t)=1+\gamma t-t^2, \quad \Delta=\delta-(\gamma^2+1),$$

which satisfies the condition $z\in[a^*,b]$ if $x\in[a,b^*]\cup[\varphi(a,a^*),\varphi(b,b^*)]$, respectively $z\in[a,b^*]$ if $x\in[a^*,b]$. The analytical structure of the stop-loss ordered minimal distribution obtained from the defining property $F_*(x)=1+\pi_*'(x)$, is quite complex.

Table 5.1 : Minimal stop-loss transform for standardized distributions on $[a,b]$ by known skewness and kurtosis

case	condition	minimum $\pi_*(x)$	extremal support
(1)	$a\le x\le b^*$	-x	$\{x,\varphi(x,z),z\}$
(2)	$b^*\le x\le \varphi(a,a^*)$	$-x+p_a^{(4)}\cdot(x-a)$	$\{a,x,\psi(x;a,b),b\}$
(3)	$\varphi(a,a^*)\le x\le\varphi(b,b^*)$	$p_z^{(3)}\cdot(z-x)$	$\{\varphi(x,z),x,z\}$
(4)	$\varphi(b,b^*)\le x\le a^*$	$p_b^{(4)}\cdot(b-x)$	$\{a,\psi(x;a,b),x,b\}$
(5)	$a^*\le x\le b$	0	$\{z,\varphi(z,x),x\}$

Theorem 5.1. The stop-loss ordered minimal distribution for standard distributions on $[a,b]$ by known skewness and kurtosis is determined in Table 5.2.

Table 5.2 : stop-loss ordered minimal distribution on $\left[a,b\right]$ by known skewness and kurtosis

case	condition	$F_*(x)$
(1)	$a \le x \le b^*$	0
(2)	$b^* \le x \le \varphi(a,a^*)$	$\dfrac{1+b\psi}{(b-a)(\psi-a)}+\dfrac{(1+ab)(\psi-\varphi)}{(b-a)(\psi-a)^2}$
(3)	$\varphi(a,a^*) \le x \le \varphi(b,b^*)$	$1-\dfrac{1+\varphi(x,z)^2}{(z-\varphi(x,z))^2}-\dfrac{2(x-\varphi(x,z))(1+z\varphi(x,z))}{(z-\varphi(x,z))^3}$
(4)	$\varphi(b,b^*) \le x \le a^*$	$1-\left\{\dfrac{1+a\psi}{(b-a)(b-\psi)}+\dfrac{(1+ab)(\varphi-\psi)}{(b-a)(b-\psi)^2}\right\}$
(5)	$a^* \le x \le b$	1

For concrete calculations recall that $\varphi(x,z)=\dfrac{\gamma-(x+z)}{1+xz}$, with $z=z(x)$ defined by (5.1) and

$$\psi(x;a,b)=\frac{\varphi(a,b)x-\omega(a,b)}{x-\varphi(a,b)},\ \text{ with }\ \omega(a,b)=\frac{\delta-(a+b)\gamma+ab}{1+ab}. \tag{5.2}$$

In Table 5.2 and in the following one uses the abbreviations $\varphi=\varphi(a,b)$, $\psi=\psi(x;a,b)$.

Proof. We will need the derivatives

$$\begin{aligned}
\psi_x &= \frac{d}{dx}\psi(x;a,b)=-\left(\frac{\varphi-\psi}{\varphi-x}\right),\\
\varphi(a,x)_x &= \frac{d}{dx}\varphi(a,x)=-\left(\frac{1+a\varphi(a,x)}{1+ax}\right),\\
\varphi(x,b)_x &= \frac{d}{dx}\varphi(x,b)=-\left(\frac{1+b\varphi(x,b)}{1+bx}\right),
\end{aligned} \tag{5.3}$$

as well as the identities

(5.4) $(1+ax)(b-\varphi(a,x))=(1+ab)(x-\varphi)$, $(1+bx)(\varphi(x,b)-a)=(1+ab)(\varphi-x)$.

Clearly the cases (1) and (5) are trivial. We first show the simpler cases (2) and (4), then (3).

<u>Case (2)</u> :

From Table 5.1 and the explicit expression for $p_a^{(4)}$ one has

$$\pi_*(x)=-x+\frac{(1+bx)(\psi-\varphi(x,b))}{(b-a)(\psi-a)}. \tag{5.5}$$

Elementary calculations with the above formulas (5.3) and (5.4) show the desired expression for $F_*(x)=1+\pi_*'(x)$.

Case (4) :

This follows similarly to case (2) using the explicit expression

$$\pi_*(x) = \frac{(1+ax)(\varphi(a,x)-\psi)}{(b-a)(b-\psi)}. \tag{5.6}$$

Case (3) :

An analytical expression for the minimal stop-loss transform is

$$\pi_*(x) = \frac{1+x\varphi(x,z)}{z-\varphi(x,z)} = \frac{1+\gamma x - x^2}{2z-\gamma+(1+z^2)x}. \tag{5.7}$$

Through elementary calculations and rearrangements one obtains for the survival function

$$\begin{aligned}\overline{F}_*(x) = -\pi_*'(x) &= \frac{(1+xz)^2+(x+z-\gamma)^2+2z'(1+xz)(1+\gamma-x^2)}{\left[2z-\gamma+(1+z^2)x\right]^2} \\ &= \frac{1+\varphi(x,z)^2+2z'(1+x\varphi(x,z))}{(z-\varphi(x,z))^2}\end{aligned} \tag{5.8}$$

Taking derivatives with respect to x in (5.1) and making use of the latter identity, one obtains successively

$$\begin{aligned} z' &= -\frac{q'(x)q(z)+\Delta z}{q'(z)q(x)+\Delta x} = -\frac{q(z)}{q(x)}\cdot\frac{\left[q'(x)q(x)q(z)+\Delta z q(x)\right]}{\left[q'(z)q(x)q(z)+\Delta x q(z)\right]} \\ &= \frac{q(z)}{q(x)}\cdot\left\{\frac{\gamma-(x+z)-x(1+xz)}{z(1+xz)-(\gamma-(x+z))}\right\} = \frac{(1+z\varphi(x,z))(\varphi(x,z)-x)}{(1+x\varphi(x,z))(z-\varphi(x,z))}, \end{aligned} \tag{5.9}$$

where, for the last equality, use has been made of the relations

$$\begin{aligned} (1+z\varphi(x,z))(1+xz) &= q(z), \\ (1+x\varphi(x,z))(1+xz) &= q(x), \\ (\gamma-(x+z))(1+xz) &= \varphi(x,z). \end{aligned} \tag{5.10}$$

Inserting (5.9) into (5.8) one gets the desired expression for $F_*(x)$. ◊

Since in practical applications the limiting ranges $[a,\infty)$ and $(-\infty,\infty)$ are of great importance, let us write down the resulting distributions in these situations.

Table 5.2' : stop-loss ordered minimal distribution on $(-\infty,\infty)$ by known skewness and kurtosis

case	condition	$F_*(x)$
(1)	$x \le c$	0
(2)	$c \le x \le \overline{c}$	$1-\frac{1+\varphi(x,z)^2}{(z-\varphi(x,z))^2}-\frac{2(x-\varphi(x,z))(1+z\varphi(x,z))}{(z-\varphi(x,z))^3}$
(3)	$x \ge \overline{c}$	1

Table 5.2'' : stop-loss ordered minimal distribution on $[a,\infty)$ by known skewness and kurtosis

case	condition	$F_*(x)$
(1)	$a \le x \le c$	0
(2)	$c \le x \le \varphi(a,a^*)$	$\frac{1+\varphi(a,x)^2}{(\varphi(a,x)-a)^2}$
(3)	$\varphi(a,a^*) \le x \le \bar{c}$	$1-\frac{1+\varphi(x,z)^2}{(z-\varphi(x,z))^2}-\frac{2(x-\varphi(x,z))(1+z\varphi(x,z))}{(z-\varphi(x,z))^3}$
(4)	$\bar{c} \le x \le a^*$	1
(5)	$x \ge a^*$	1

Proof of Table 5.2''. Only the limiting cases (2) and (4) must be checked. One notes that the supports of the finite atomic extremal random variables in Table 5.1 are in these limiting cases equal to $\{a, x, \varphi(a,x), \infty\}$ respectively $\{a, \varphi(a,x), x, \infty\}$ (in the sense of Table 4.3"). Then case (4) is trivial because $\pi_*(x)=0$. In case (2) one has

$$\pi_*(x) = -x + p_a^{(3)}(x-a) = -x + \frac{1+x\varphi(a,x)}{\varphi(a,x)-a}, \tag{5.11}$$

from which $F_*(x)$ follows by differentiation and some elementary calculations.

5.2. Comparisons with the Chebyshev-Markov extremal random variables.

A proof of the stochastic order relation $X_u \le_{st} X_* \le_{st} X_\ell$, or equivalently $F_\ell(x) \le F_*(x) \le F_u(x)$, uniformly over the considered range, where the bounds are the standard Chebyshev-Markov extremal distributions, is provided in the order of increasing complexity for the ranges $(-\infty,\infty)$, $[a,\infty)$ and $[a,b]$. The expressions for the standard Chebyshev-Markov extremal distributions are those of Tables III.4.3, III.4.3', and III.4.3".

Theorem 5.2. By known skewness, kurtosis and given range $(-\infty,\infty)$, the stochastic order relation $F_\ell(x) \le F_*(x) \le F_u(x)$ holds uniformly for all $x \in (-\infty,\infty)$.

Proof. Clearly only the case (2) of Table 5.4 is non-trivial. Setting $\varphi = \varphi(x,z)$ one must show the inequalities, valid for $x \in [c,\bar{c}]$,

$$\begin{aligned} p_\varphi^{(3)} &= \frac{1+xz}{(x-\varphi)(z-\varphi)} \le \\ F_*(x) &= 1-\frac{1+\varphi^2}{(z-\varphi)^2}+\frac{2(x-\varphi)(1+z\varphi)}{(z-\varphi)^3} \le \\ 1-p_z^{(3)} &= 1-\frac{1+x\varphi}{(z-x)(z-\varphi)}. \end{aligned} \tag{5.12}$$

The first inequality is shown to be equivalent with the inequality

(5.13) $$\frac{(1+z\varphi)(2\varphi-x-z)}{(x-\varphi)(z-\varphi)^2}+\frac{2(x-\varphi)(1+z\varphi)}{(z-\varphi)^3}\ge 0,$$

or rearranged

(5.14) $$\frac{-(1+z\varphi)}{(x-\varphi)(z-\varphi)^2}\left\{[(x-\varphi)+(z-\varphi)](z-\varphi)-2(x-\varphi)^2\right\}\ge 0.$$

The expression in curly bracket can be rewritten as

(5.15) $$(z-x)[2(x-\varphi)+(z-\varphi)],$$

and is non-negative because the atoms of the support of the extremal distribution, which minimizes the stop-loss transform are ordered as $\varphi<x<z$. Since this support defines a feasible triatomic random variable, the condition $p_x^{(3)}\ge 0$ implies in particular that $1+z\varphi\le 0$. These facts show that (5.14) is fulfilled. The second inequality can be rewritten as

(5.16) $$\frac{(1+z\varphi)(\varphi-x)}{(z-x)(z-\varphi)^2}-\frac{2(x-\varphi)(1+z\varphi)}{(z-\varphi)^3}\ge 0,$$

or rearranged

(5.17) $$-(1+z\varphi)\left\{\frac{x-\varphi}{(z-x)(z-\varphi)^2}+\frac{2(x-\varphi)}{(z-\varphi)^3}\right\}\ge 0.$$

One concludes by observing that $\varphi<x<z$ and $1+z\varphi\le 0$. ◊

Theorem 5.3. By known skewness, kurtosis and given range $[a,\infty)$, the stochastic order relation $F_\ell(x)\le F_*(x)\le F_u(x)$ holds uniformly for all $x\in[a,\infty)$.

Proof. Only the cases (2) and (3) of Table 5.3 are non-trivial, where case (3) holds by the same proof as in Theorem 5.2. Setting $\varphi=\varphi(a,x)$ one must show the inequalities, valid for $x\in[c,\varphi(a,a^*)]$,

(5.18) $$p_a^{(3)}=\frac{1+x\varphi}{(x-a)(\varphi-a)}\le F_*(x)=\frac{1+\varphi^2}{(\varphi-a)^2}\le 1-p_\varphi^{(3)}=1-\frac{1+ax}{(\varphi-a)(\varphi-x)}.$$

The first inequality is equivalent with

(5.19) $$-(1+a\varphi)(\varphi-x)\ge 0.$$

But $\{a,x,\varphi\}$ is the ordered support of the triatomic extremal distribution, which minimizes the stop-loss transform. In particular one has $(1+a\varphi)\le 0$ and $\varphi>x$, hence (5.19) holds. The second inequality can be rearranged to

(5.20) $$-(1+a\varphi)\{(\varphi-x)+(\varphi-a)\}\ge 0,$$

and is thus also fulfilled. ◊

Theorem 5.4. By known skewness, kurtosis and given range $[a,b]$, the stochastic order relation $F_\ell(x) \le F_*(x) \le F_u(x)$ holds uniformly for all $x \in [a,b]$.

Proof. Only the cases (2), (3) and (4) of Table 5.2 are non-trivial, where case (3) holds by the same proof as in Proposition 7.1.

Case (2) :

Setting $\varphi = \varphi(a,b)$, $\psi = \psi(x;a,b)$ one must show for $x \in [b^*, \varphi(a,a^*)]$ the inequalities

$$
\begin{aligned}
&p_a^{(4)} = \frac{(1+bx)(\psi-\varphi(x,b))}{(b-a)(x-a)(\psi-a)} \le \\
(5.21) \qquad &F_*(x) = \frac{1+b\psi}{(b-a)(\psi-a)} + \frac{(1+ab)(\psi-\varphi)}{(b-a)(\psi-a)^2} \le \\
&p_a^{(4)} + p_x^{(4)} = \frac{1+b\psi}{(b-a)(\psi-a)} - \frac{(1+ab)(\psi-\varphi)[(b-x)+(\psi-a)]}{(b-a)(\psi-a)(b-x)(\psi-x)}.
\end{aligned}
$$

Using the second identity in (5.4), the first inequality is equivalent with

$$1+b\psi - \frac{(1+bx)(\psi-\varphi(x,b))}{x-a} + \frac{(1+bx)(\psi-\varphi)(\varphi(x,b)-a)}{(\psi-a)(\varphi-x)} \ge 0, \tag{5.22}$$

or rearranged

$$-\frac{(1+ab)(\psi-x)}{x-a} + \frac{(1+bx)(\varphi(x,b)-a)(\varphi-a)(\psi-x)}{(x-a)(\psi-a)(\varphi-x)} \ge 0. \tag{5.23}$$

One concludes by using that $1+ab \le 0$, $a < x < \psi < b$, $x \le \varphi(a,a^*) < \varphi(a,b) = \varphi$, $a < \varphi(x,b)$, $1+bx \ge 0$. The second inequality is seen to be equivalent with

$$-(1+ab)(\psi-\varphi)\left\{\frac{(b-x)+(\psi-a)}{(b-x)(\psi-x)} + \frac{1}{\psi-a}\right\} \ge 0. \tag{5.24}$$

Under the assumption of case (2) one has

$$p_x^{(4)} = \frac{-(1+ab)(\psi-\varphi)}{(x-a)(b-x)(\psi-x)}. \tag{5.25}$$

This probability is positive, hence in particular $\psi > \varphi$. It follows that (5.24) holds.

Case (4) :

The inequalities for the survival functions, which are symmetric to case (2) and must be valid for $x \in [\varphi(b,b^*), a^*]$, read

$$
\begin{aligned}
&p_b^{(4)} = \frac{(1+ax)(\varphi(a,x)-\psi)}{(b-a)(b-x)(b-\psi)} \le \\
(5.26)\qquad &\overline{F}_*(x) = \frac{1+a\psi}{(b-a)(b-\psi)} + \frac{(1+ab)(\varphi-\psi)}{(b-a)(b-\psi)^2} \le \\
&p_x^{(4)} + p_b^{(4)} = \frac{1+a\psi}{(b-a)(b-\psi)} - \frac{(1+ab)(\varphi-\psi)\left[(x-a)+(b-\psi)\right]}{(b-a)(b-\psi)(x-a)(x-\psi)}.
\end{aligned}
$$

Using the first identity in (5.4) one shows that the first inequality is equivalent with

$$
(5.27)\qquad -\frac{(1+ab)(x-\psi)}{b-x} + \frac{(1+ax)(b-\varphi(a,x))(b-\varphi)(x-\psi)}{(b-x)(b-\psi)(x-\varphi)} \ge 0,
$$

and is satisfied because $1+ab \le 0$, $a < \psi < x < b$, $x \ge \varphi(b,b^*) > \varphi(a,b) = \varphi$, $b > \varphi(a,x)$, $1+ax \ge 0$. The second inequality can be rearranged to

$$
(5.28)\qquad -(1+ab)(\varphi-\psi)\left\{\frac{(x-a)+(b-\psi)}{(x-a)(x-\psi)} + \frac{1}{b-\psi}\right\} \ge 0.
$$

Since $p_x^{(4)}$ is positive one must have $\psi < \varphi$ in case (4). One concludes that (5.28) holds. ◊

5.3. Small atomic ordered approximations over the range $(-\infty,\infty)$.

To simplify models and calculations, one is interested in less and more dangerous finite atomic approximations of X_* with equal mean, which satisfy the dangerousness relation (stop-loss confidence bounds) :

$$
(5.29)\qquad X_*^c \le_{D,=} \le X_* \le_{D,=} X_*^d.
$$

It is not difficult to show that $\overline{F}_*(x)$ is a continuous function. In particular one has the relations $\overline{F}_*(c)=1, \overline{F}_*(\overline{c})=0$. A concentration of the probability mass of X_* over $\left[c,\overline{c}\right]$ on a single atom yields a trivial lower bound, and a dispersion of the mass of $\left[c,\overline{c}\right]$ on the two atoms $c, \overline{c}$ yields similarly a trivial upper bound. The simplest non-trivial way to concentrate and disperse probability masses is over the two subintervals $\left[c,0\right]$ and $\left[0,\overline{c}\right]$.

Proposition 5.1. Let X_* be the stop-loss ordered minimal random variable on $(-\infty,\infty)$ by known skewness γ and kurtosis $\gamma_2 = \delta - 3$. Then there exists a diatomic random variable $X_*^c \le_{D,=} \le X_*$ with support $\{x_0, y_0\} = \left\{-\frac{\pi_*(0)}{F_*(0)}, \frac{\pi_*(0)}{\overline{F}_*(0)}\right\}$ and probabilities $\{F_*(0), \overline{F}_*(0)\}$, and a triatomic random variable $X_*^d \ge_{D,=} \le X_*$ with support $\{c, 0, \overline{c}\}$ and probabilities $\{\overline{c}\pi_*(0), 1-(\overline{c}-c)\pi_*(0), (-c)\pi_*(0)\}$, where one has

$$
(5.30)\qquad \pi_*(0) = \frac{1}{(4\delta-3\gamma^2)^{1/2}}, \quad \overline{F}_*(0) = \frac{1}{2}\left\{1 - \gamma \cdot \frac{2\delta-\gamma^2+2}{(4\delta-3\gamma^2)^{3/2}}\right\}.
$$

Proof. The result is shown in three steps.

Step 1 : construction of the less dangerous lower bound

Concentrating the probability mass of $[c,0]$ on a single point, one gets the atom $x_0 = -\frac{\pi_*(0)}{F_*(0)}$ of X_*^c with probability $F_*(0)$. Similarly mass concentration of $[0,\bar{c}]$ yields an atom $y_0 = \frac{\pi_*(0)}{\overline{F}_*(0)}$ with probability $\overline{F}_*(0)$.

Step 2 : construction of the more dangerous upper bound

Dispersing the probability mass of $[c,0]$ on the pair of atoms $\{c,0\}$ with probabilities $\{p_c, p_0\}$, one obtains $p_c = \bar{c}\pi_*(0)$, $p_0 = F_*(0) - \bar{c}\pi_*(0)$. Similarly, through mass dispersion of $[0,\bar{c}]$ on the atoms $\{0,\bar{c}\}$ with probabilities $\{p_0, p_{\bar{c}}\}$, one finds $p_{\bar{c}} = (-c)\pi_*(0)$, $p_0 = \overline{F}_*(0) - (-c)\pi_*(0)$. Combining all these atoms one gets X_*^d.

Step 3 : determination of $\pi_*(0)$ and $\overline{F}_*(0)$

For x=0 one obtains from equation (5.1) that $z = z(0) = 0^*$ is solution of the quadratic equation $z^2 - \gamma z - (\Delta + 1) = 0$. Since $\Delta = \delta - \gamma^2 - 1$ one finds $z = 0^* = \frac{1}{2}(\gamma + \sqrt{4\delta - 3\gamma^2})$. From formula (5.7) one gets immediately the desired value of $\pi_*(0)$. Setting x=0 in the relation (5.1) one has $q(z = 0^*) = -\Delta$. Inserted in the first expression of (5.9) one obtains

$$z'(x=0) = \frac{\Delta(z-\gamma)}{2z-\gamma}, \quad z = 0^*. \tag{5.31}$$

From the first expression in (5.8) one gets

$$\overline{F}_*(0) = \frac{1 + (z-\gamma)^2 + \dfrac{2\Delta(z-\gamma)}{(2z-\gamma)}}{(2z-\gamma)^2}, \quad z = 0^*. \tag{5.32}$$

Rearranging by using that $2z - \gamma = \sqrt{4\delta - 3\gamma^2}$ and $q(z = 0^*) = -\Delta$, one obtains the desired expression for $\overline{F}_*(0)$. ◊

In applications one is interested in the quality of these simple di- and triatomic bounds. For example the minimal stop-loss transform can be bounded as follows, where the elementary check using Proposition 5.1 is left to the reader.

Table 5.3: dangerous confidence bounds for the minimal stop-loss transform of standard random variables by known skewness, kurtosis and range $(-\infty,\infty)$

case	condition	$\pi_*^l(x)$	$\pi_*^u(x)$
(1)	$x \le c$	-x	-x
(2)	$c \le x \le x_0$	-x	$-x+(1+\bar{c}x)\pi_*(0)$
(3)	$x_0 \le x \le 0$	$\pi_*(0)-x\overline{F}_*(0)$	$-x+(1+\bar{c}x)\pi_*(0)$
(4)	$0 \le x \le y_0$	$\pi_*(0)-x\overline{F}_*(0)$	$(1+cx)\pi_*(0)$
(5)	$y_0 \le x \le \bar{c}$	0	$(1+cx)\pi_*(0)$
(6)	$x \ge \bar{c}$	0	0

Since the true value of $\pi_*(x)$ lies between the two bounds, a straightforward piecewise linear estimator is the average :

(5.33) $$\hat{\pi}_*(x)=\frac{1}{2}(\pi_*^u(x)+\pi_*^\ell(x))\approx\pi_*(x)$$

In concrete situations the approximation error can be estimated.

Example 5.1 : skewness and kurtosis of a standard normal distribution

With $\gamma=0, \delta=3$, one obtains immediately the values

(5.34) $$\pi_*(0)=\frac{\sqrt{3}}{6},\quad F_*(0)=\frac{1}{2},\quad c=-1,\quad \bar{c}=1,\quad x_0=-\frac{\sqrt{3}}{3},\quad y_0=\frac{\sqrt{3}}{3}.$$

From (5.33) and Table 5.5 one gets for example

(5.35) $$\hat{\pi}_*(y_0)=\frac{1}{2}(1-\frac{\sqrt{3}}{3})\cdot\frac{\sqrt{3}}{6}=\frac{\sqrt{3}-1}{12}=0.061\,.$$

On the other side the exact value of the minimal stop-loss transform is for $x\in[-1,1]$:

(5.36) $$\pi_*(x)=\frac{1-x^2}{2z+(1+z^2)x},\quad z=z(x)=\frac{x+\sqrt{x^4-3x^2+3}}{1-x^2}.$$

For $x=y_0$ one has $z=z(y_0)=\frac{1}{2}(\sqrt{3}+\sqrt{19})$, hence $\pi_*(y_0)=\dfrac{4}{19\sqrt{3}+9\sqrt{19}}=0.055$. In this case the approximation error is less than 0.006, which is quite satisfactory.

6. Small atomic stop-loss confidence bounds for symmetric random variables.

For an arbitrary real symmetric random variable we construct a diatomic stop-loss lower bound, and a "generalized" or modified triatomic stop-loss upper bound. These bounds are used to obtain an optimal piecewise linear approximation to the stop-loss transform of an arbitrary symmetric random variable. A numerical illustration for the stop-loss ordered maximal distribution by known mean and variance is also given.

Given is a real random variable X taking values in $(-\infty,\infty)$ and symmetric around a symmetry center, which can be assumed to be zero by a location transformation. The problem, we are interested in, consists to find finite atomic random variables X_ℓ, X_u with the smallest possible number of atoms such that the stop-loss transforms of X_ℓ, X, X_u are ordered as

(6.1) $\pi_\ell(x) \le \pi(x) \le \pi_u(x) + \varepsilon$, uniformly for all $x \in (-\infty,\infty)$, for all small $\varepsilon > 0$,

where $\pi(x) = \int_x^\infty \overline{F}(x)dx$ denotes the stop-loss transform of a random variable with survival function $\overline{F}(x)$.

The applications in mind concern primarily the stop-loss ordered maximal random variables considered in Section 3, but of course the method has a wider scope of application. Note that the construction of stop-loss ordered confidence bounds for the stop-loss ordered minimal random variables is much more straightforward and has been considered in detail in Sections 4 and 5.

In Subsection 6.1, respectively Subsection 6.2, we construct the lower stop-loss bound, respectively the upper stop-loss bound. In Subsection 6.3 these are used to determine an optimal piecewise linear approximation to the stop-loss transform of an arbitrary symmetric distribution. A numerical illustration for the stop-loss ordered maximal distribution associated to an arbitrary standard distribution on $(-\infty,\infty)$ is given in Subsection 6.4.

6.1. A diatomic stop-loss ordered lower bound for symmetric random variables.

Let X be a real random variable defined on $(-\infty,\infty)$ symmetric around zero with mean zero, survival function $\overline{F}(x)$, and stop-loss transform $\pi(x) = \int_x^\infty \overline{F}(x)dx$. Concentrating the probability mass over the interval $(-\infty,0]$ on an atom x_0 with probability $F(0) = \frac{1}{2}$, one finds through partial integration

$$x_0 = \frac{\int_{-\infty}^0 x dF(x)}{F(0)} = \frac{-\int_0^\infty \overline{F}(x)dx}{F(0)} = -2\pi(0). \tag{6.2}$$

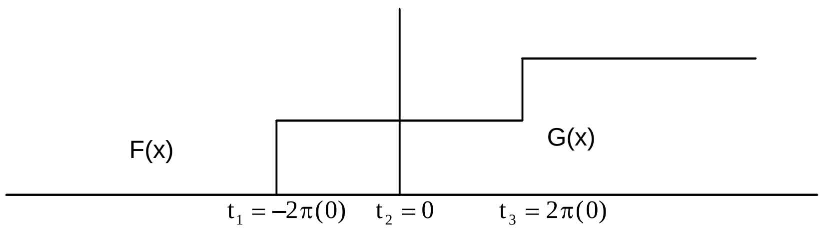
Similarly concentration of the mass over $[0,\infty)$ yields an atom $y_0 = 2\pi(0)$ with probability $\overline{F}(0) = \frac{1}{2}$. One obtains a diatomic random variable X_ℓ with support $\{-2\pi(0), 2\pi(0)\}$ and probabilities $\{\frac{1}{2},\frac{1}{2}\}$, with mean zero and piecewise linear stop-loss transform

$$\pi_\ell(x) = \begin{cases} -x, & x \le -2\pi(0) \\ \pi(0) - \frac{1}{2}x, & -2\pi(0) \le x \le 2\pi(0) \\ 0, & x \ge 2\pi(0) \end{cases} \tag{6.3}$$

The fact that $\pi_\ell(x) \le \pi(x)$ for all x, or equivalently $X_\ell \le_{sl,=} X$, follows from the Karlin-Novikoff-Stoyan-Taylor crossing conditions for stop-loss order stated in Theorem 1.3.

Proposition 6.1. Let X be a random variable on $(-\infty,\infty)$, which is symmetric around zero. Then the diatomic random variable X_ℓ with support $\{-2\pi(0), 2\pi(0)\}$ and probabilities $\{\frac{1}{2},\frac{1}{2}\}$ is stop-loss smaller than X with equal mean, that is $X_\ell \le_{sl,=} X$.

Proof. It suffices to apply case 2 of Theorem 1.3 with Y= X_ℓ. One observes that F(x) and G(x) cross n=3 times as in the figure :



Since $\mu_X = \mu_Y = 0$ and $\pi_Y(0) = \pi(0)$ the assertion follows immediately. ◊

6.2. A modified triatomic stop-loss upper bound.

The random variables X is assumed to have the same properties as above. In a *first step* one replaces X by a *double-cut tail* random variable X_b defined on $[-b, b]$ for some b>0 with *double-cut tail distribution* $F_b(x)$ given by

(6.4)
$$F_b(x) = \begin{cases} 0, & x < -b, \\ F(x), & -b \le x < b, \\ 1, & x \ge b. \end{cases}$$

To compare stop-loss transforms, one requires the following elementary result.

Lemma 6.1. The stop-loss transform of X_b equals

(6.5)
$$\pi_b(x) = \begin{cases} -x, & x < -b, \\ \pi(x) - \pi(b), & -b \le x < b, \\ 0, & x \ge b, \end{cases}$$

and satisfies the following inequality

(6.6) $\pi(x) \le \pi_b(x) + \pi(b)$, uniformly for all x.

Proof. Since X is symmetric around zero, its stop-loss transform satisfies the identity

(6.7) $\pi(-x) = x + \pi(x)$, for all x.

One obtains successively for

$x < -b$: $\pi_b(x) = \int_x^{-b} dx + \int_{-b}^{b} \overline{F}(x)dx = -b - x + \pi(-b) - \pi(b) = -x$,

$-b \le x < b$: $\pi_b(x) = \int_x^{b} \overline{F}(x)dx = \pi(x) - \pi(b)$,

$x \ge b$: $\pi_b(x) = 0$.

It remains to show (6.6) for $x \ge b$ and $x < -b$. Since $\pi(x)$ is a decreasing function of x, this is immediate for $x \ge b$. Since $\frac{d}{dx}(x + \pi(x)) = \overline{F}(x) \ge 0$ the function $x + \pi(x)$ is monotone increasing. For $x < -b$ one obtains using (6.7) :

$$x + \pi(x) \le -b + \pi(-b) = \pi(b).$$

But this inequality says that for $x < -b$ one has $\pi(x) \le -x + \pi(b) = \pi_b(x) + \pi(b)$. ◊

In a *second step* we construct a triatomic random variable X_b^u on $[-b, b]$, which is more dangerous than X_b with equal mean, and is in particular such that $X_b \le_{sl,=} X_b^u$. Dispersing the probability mass of X_b in the subinterval $[-b, 0]$ on the pair of atoms $\{-b, 0\}$ with probabilities $\{p_{-b}, p_0^-\}$ one obtains from Lemma 2.2 :

$$p_{-b} = \frac{\pi(0) - \pi(b)}{b} - F(-b), \quad p_0^- = \tfrac{1}{2} - \left(\frac{\pi(0) - \pi(b)}{b}\right). \tag{6.8}$$

A similar dispersion in $[0, b]$ on the pair of atoms $\{0, b\}$ with probabilities $\{p_0^+, p_b\}$ yields

$$p_0^+ = \tfrac{1}{2} - \left(\frac{\pi(0) - \pi(b)}{b}\right), \quad p_b = \frac{\pi(0) - \pi(b)}{b} - \overline{F}(b). \tag{6.9}$$

Combining both mass dispersions one obtains a triatomic random variable $X_b^u \ge_{sl,=} X_b$ with support $\{-b, 0, b\}$ and probabilities $\left\{\frac{\pi(0) - \pi(b)}{b}, 1 - 2(\frac{\pi(0) - \pi(b)}{b}), \frac{\pi(0) - \pi(b)}{b}\right\}$. From (6.6) one obtains furthermore the stop-loss inequality, valid uniformly for all $x \in (-\infty, \infty)$:

$$\pi(x) \le \pi_b^u(x) + \pi(b). \tag{6.10}$$

Since $\pi(b) \to 0$ as $b \to \infty$, choose b such that $\varepsilon = \pi(b)$. Then $X_u := X_b^u$ is a stop-loss upper bound for X, which satisfies (6.1).

Proposition 6.2. Let X be a random variable on $(-\infty, \infty)$, which is symmetric around zero. Then there exists a triatomic random variable X_b^u with support $\{-b, 0, b\}$ and probabilities $\left\{\frac{\pi(0) - \pi(b)}{b}, 1 - 2(\frac{\pi(0) - \pi(b)}{b}), \frac{\pi(0) - \pi(b)}{b}\right\}$, which satisfies the stop-loss inequality

$$\pi(x) \le \pi_b^u(x) + \pi(b), \text{ for all } x \in (-\infty, \infty), \tag{6.11}$$

where the upper bound is determined by the piecewise linear stop-loss transform

$$\pi_b^u(x) = \begin{cases} -x, & x \le -b, \\ \pi(0)-\pi(b)+(\frac{\pi(0)-\pi(b)}{b})x - x, & -b \le x \le 0 \\ \pi(0)-\pi(b)+(\frac{\pi(0)-\pi(b)}{b})x, & 0 \le x \le b \\ 0, & x \ge b. \end{cases} \tag{6.12}$$

Proof. It remains to show (6.12), which presents no difficulty and is left to the reader.

6.3. Optimal piecewise linear approximations to stop-loss transforms.

Again X denotes a real random variable on $(-\infty,\infty)$, which is symmetric around zero. It has been shown in Subsections 6.1 and 6.2 that the true value of the stop-loss transform $\pi(x)$ lies between the two piecewise linear bounds $\pi_\ell(x)$ and $\pi_b^u(x)+\pi(b)$. As a straightforward approximation, one can consider the average

$$\hat{\pi}(x;b) := \frac{1}{2}(\pi_\ell(x)+\pi_b^u(x)+\pi(b)) \tag{6.13}$$

and try to find an optimal value b^* for b, which minimizes the *stop-loss distance*

$$d(b) := \max_{x\in(-\infty,\infty)} \left|\hat{\pi}(x;b)-\pi(x)\right|. \tag{6.14}$$

To determine d(b) one has to consider $d(x;b) := \left|\hat{\pi}(x;b)-\pi(x)\right|$. We formulate conditions under which the stated optimization problem can be solved.

Choose $b \ge 2\pi(0)$ and use (6.3) and (6.12) to get the following table for the signed distance between $\hat{\pi}(x;b)$ and $\pi(x)$:

Table 6.1 : signed distance between the stop-loss transform and its approximation

case	condition	$\hat{\pi}(x;b)-\pi(x)$
(1)	$x \le -b$	$\frac{1}{2}\pi(b)-\pi(-x)$
(2)	$-b \le x \le -2\pi(0)$	$\frac{1}{2}\left\{\pi(0)+\left(\frac{\pi(0)-\pi(b)}{b}\right)x-2x-2\pi(x)\right\}$
(3)	$-2\pi(0) \le x \le 0$	$\frac{1}{2}\left\{2\pi(0)+\left(\frac{\pi(0)-\pi(b)}{b}\right)x-\frac{3}{2}x-2\pi(x)\right\}$
(4)	$0 \le x \le 2\pi(0)$	$\frac{1}{2}\left\{2\pi(0)-\left(\frac{\pi(0)-\pi(b)}{b}\right)x-\frac{1}{2}x-2\pi(x)\right\}$
(5)	$2\pi(0) \le x \le b$	$\frac{1}{2}\left\{\pi(0)-\left(\frac{\pi(0)-\pi(b)}{b}\right)x-2\pi(x)\right\}$
(6)	$x \ge b$	$\frac{1}{2}\pi(b)-\pi(x)$

In case (1) one has $-x \ge b$, $0 \le \pi(-x) \le \pi(b)$, and thus one gets $d(x;b) \le \frac{1}{2}\pi(b)$, where equality is attained for x=–b. Similarly in case (6) one obtains also $d(x;b) \le \frac{1}{2}\pi(b)$ with equality when x=b. In each other case (i) we determine $x_i, b_i \ge 2\pi(0), i = 2,3,4,5$, such that $d(x;b_i) \le \frac{1}{2}\pi(b_i)$ and equality is attained at $x = x_i$. Setting $b^* = \min_{i=2,3,4,5}\{b_i\} \ge 2\pi(0)$, this implies the uniform best upper bound (solution of our optimization problem)

(6.15) $$d(x;b^*) \le \tfrac{1}{2}\pi(b^*),\text{ for all } x \in (-\infty,\infty).$$

In the following one sets $h(x;b) := 2\cdot(\hat{\pi}(x;b) - \pi(x))$. Assume a probability density f(x)=F'(x) exists, and note that $\pi'(x) = -\overline{F}(x)$. Table 4.1 shows that $h''(x;b) = -2f(x) < 0$ in all cases (2) to (5). It follows that h(x;b) is maximal at the value x=x(b), which is solution of the first order condition h'(x(b);b)=0 provided x(b) belongs to the range of the considered case. In each seperate case the maximizing value x(b) is implicitely given as follows :

Case (2) : $$2\overline{F}(x(b)) = 2 - \left(\frac{\pi(0)-\pi(b)}{b}\right)$$

Case (3) : $$2\overline{F}(x(b)) = \frac{3}{2} - \left(\frac{\pi(0)-\pi(b)}{b}\right)$$

Case (4) : $$2\overline{F}(x(b)) = \frac{1}{2} - \left(\frac{\pi(0)-\pi(b)}{b}\right)$$

Case (5) : $$2\overline{F}(x(b)) = \left(\frac{\pi(0)-\pi(b)}{b}\right)$$

The corresponding values of h(x;b) are then as follows :

Case (2) : $$h(x(b);b) = \pi(0) - 2\{x(b)\cdot\overline{F}(x(b)) + \pi(x(b))\}$$

Case (3) : $$h(x(b);b) = 2\pi(0) - 2\{x(b)\cdot\overline{F}(x(b)) + \pi(x(b))\}$$

Case (4) : $$h(x(b);b) = 2\pi(0) - 2\{x(b)\cdot\overline{F}(x(b)) + \pi(x(b))\}$$

Case (5) : $$h(x(b);b) = \pi(0) - 2\{x(b)\cdot\overline{F}(x(b)) + \pi(x(b))\}$$

Independently of the considered case one obtains

(6.16) $$\frac{\partial}{\partial b}h(x(b);b) = 2x(b)x'(b)f(x(b)).$$

One the other side one has

$$\frac{\partial}{\partial b}\left(\frac{\pi(b)-\pi(0)}{b}\right) = \frac{\pi(0)-\pi(b)-b\overline{F}(b)}{b^2} =: \frac{g(b)}{b^2}.$$

Since g'(b)=bf(b)>0 and b>0, it follows that g(b)>g(0)=0. Taking derivatives with respect to b in the defining equations for x(b), one gets the relations

Cases (2) and (3) : $$-f(x(b))x'(b) = \frac{g(b)}{b^2}$$

Cases (4) and (5) : $$-f(x(b))x'(b) = -\frac{g(b)}{b^2}$$

Taking into account the conditions in Table 6.1, which the values $x(b)$ must satisfy, one sees that in any case $\operatorname{sgn}\{x'(b)\} = \operatorname{sgn}\{x(b)\}$. From (6.16) it follows that $h(x(b);b)$ is monotone increasing in b. Since $\pi(b)$ is monotone decreasing in b, the optimal value b_i of b is in each case necessarily solution of the implicit equation $h(x(b_i);b_i) = \pi(b_i)$, i=2,3,4,5. Gathering all details together, a solution to the above optimization problem, under the assumption it exists, is determined by the following algorithmic result.

Proposition 6.3. Let X be a random variable on $(-\infty,\infty)$, which is symmetric around zero. Then an optimal piecewise linear approximation to the stop-loss transform with a minimal stop-loss distance

$$d(b^*) = \min_{b \ge 2\pi(0)} d(b) = \min_{b \ge 2\pi(0)} \left\{ \max_{x \in (-\infty,\infty)} \left| \hat{\pi}(x;b) - \pi(x) \right| \right\} \tag{6.17}$$

exists provided the following system of equations and conditions in x_i, b_i can be satisfied :

Table 6.2 : conditions for optimal piecewise linear approximations to stop-loss transforms of symmetric distributions

case	range of x_i, b_i	$2\overline{F}(x_i)$	$2\{x_i\overline{F}(x_i) + \pi(x_i)\}$
(2)	$-b_2 \le x_2 \le -2\pi(0)$	$2 - \left(\dfrac{\pi(0) - \pi(b_2)}{b_2}\right)$	$\pi(0) - \pi(b_2)$
(3)	$-b_3 \le -2\pi(0) \le x_3 \le 0$	$\dfrac{3}{2} - \left(\dfrac{\pi(0) - \pi(b_3)}{b_3}\right)$	$2\pi(0) - \pi(b_3)$
(4)	$0 \le x_4 \le 2\pi(0) \le b_4$	$\dfrac{1}{2} - \left(\dfrac{\pi(0) - \pi(b_4)}{b_4}\right)$	$2\pi(0) - \pi(b_4)$
(5)	$2\pi(0) \le x_5 \le b_5$	$\left(\dfrac{\pi(0) - \pi(b_5)}{b_5}\right)$	$\pi(0) - \pi(b_5)$

If Table 6.2 has a solution and $b^* = \min_{i=2,3,4,5}\{b_i\}$, then the minimal stop-loss distance equals $d(b^*) = \frac{1}{2}\pi(b^*)$.

6.4. A numerical example.

We illustrate at a simple concrete situation how optimal piecewise linear approximations to stop-loss transforms of symmetric random variables can be obtained.

To an arbitrary standard random variable Z on $(-\infty,\infty)$, one can associate a random variable $X \ge_{sl,=} Z$ such that the stop-loss transform $\pi(x)$ of X coincides with the maximum of $\pi_Z(x)$ over all Z uniformly for all $x \in (-\infty,\infty)$. From Sections 2 and 3, one knows that the stop-loss ordered maximal distribution is defined by

$$\pi(x) = \frac{1}{2}(\sqrt{1+x^2} - x), \quad F(x) = \frac{1}{2}\left(1 + \frac{x}{\sqrt{1+x^2}}\right). \tag{6.18}$$

To solve the system of conditions in Table 6.2, one sets $\pi(b)=\varepsilon$, hence $b=\frac{1-4\varepsilon^2}{4\varepsilon}$. A calculation shows that

(6.19) $$2\{x\overline{F}(x)+\pi(x)\}=\frac{1}{\sqrt{1+x^2}}.$$

Proceed now case by case.

Case (2) : $-b \le x \le -1$

The second equation to satisfy reads $\frac{1}{\sqrt{1+x^2}}=\frac{1}{2}-\varepsilon$, and is satisfied by $x=-\frac{\sqrt{3+4\varepsilon(1-\varepsilon)}}{1-2\varepsilon}$. Inserting into the first equation and using that $b=\frac{1-4\varepsilon^2}{4\varepsilon}$, one finds the condition $(1+2\varepsilon)\sqrt{3+4\varepsilon(1-\varepsilon)}=2$, which is equivalent to the biquadratic equation $16\varepsilon^4-24\varepsilon^2-16\varepsilon+1=0$. Neglecting the fourth power term, one gets as a sufficiently accurate quadratic approximate solution the value

(6.20) $$\varepsilon_2=\frac{\sqrt{22}-4}{12}=0.05753.$$

A numerical checks shows that $-b_2=-4.288\le x_2=-2.0268\le -1$.

Case (3) : $-1 \le x \le 0$

Solving the equation $\frac{1}{\sqrt{1+x^2}}=1-\varepsilon$, one gets $x=-\frac{\sqrt{\varepsilon(2-\varepsilon)}}{1-\varepsilon}$. The first equation leads then to the condition $2(1+2\varepsilon)\sqrt{\varepsilon(2-\varepsilon)}=1-2\varepsilon$, which is equivalent to $16\varepsilon^4-16\varepsilon^3-24\varepsilon^2-12\varepsilon+1=0$. The quadratic approximate solution is

(6.21) $$\varepsilon_3=\frac{\sqrt{15}-3}{12}=0.07275.$$

As a numerical check one has $-b_3=-3.3637\le -1\le x_3=-0.4038\le 0$.

Case (4) : $0 \le x \le 1$

Through calculation one verifies the symmetries $\varepsilon_4=\varepsilon_3$, $x_4=-x_3$, $b_4=b_3$.

Case (5) : $1 \le x \le b$

Through calculation one verifies the symmetries $\varepsilon_5=\varepsilon_2$, $x_5=-x_2$, $b_5=b_2$.

Our approximation method shows that the maximal stop-loss transform $\pi(x)$ can be approximated by the piecewise linear function $\hat{\pi}(x;b_3)$ up to the optimal uniform stop-loss error bound $d(b_3)=\frac{1}{2}\pi(b_3)=\frac{1}{2}\varepsilon_3=0.036375$, which may be enough accurate for some practical purposes.

7. Notes.

The theory of stochastic orders is a growing branch of Applid Probability and Statistics with an important impact on applications including many fields as Reliability, Operations Research, Biology, Actuarial Science, Finance and Economics. Extensive literature has been classified by Mosler and Scarsini(1993), and useful books include Mosler and Scarsini(1991), Shaked and Shanthikumar(1994) and Kaas et al.(1994).

General facts about extremal random variables with respect to a partial order are found in Stoyan(1977), chapter 1. By given range, mean and variance, the extremal random variables for the increasing convex order have been constructed first by Stoyan(1973), and have been rediscovered by the author(1995/96a) under the terminology "stop-loss ordered extremal distributions". In actuarial science, the identification of the transitive closure of dangerousness with the stop-loss order, as well as the separation theorem, goes back to van Heerwaarden and Kaas(1990) and Kaas and van Heerwaarden(1992) (see also van Heerwaarden(1991)). The practical usefulness of the Karlin-Novikoff-Stoyan-Taylor crossing conditions for stop-loss order has been demonstrated by Taylor(1983), which attributes the result to Stoyan(1977). However, a proof seemed to be missing. A systematic approach to higher degree stop-loss transforms and stochastic orders, together with some new applications, is proposed in Hürlimann(1997e).

In actuarial science, the construction of ordered discrete approximations through mass concentration and mass dispersion has been widely applied. Exposés of this technique are in particular found in Gerber(1979), Examples 3.1 and 3.2, p. 98-99, Heilmann(1987), p.108-109, and Kaas et al.(1994), Example III.1.2, p. 24. These transformed distributions are particular cases of fusions of probability measures studied by Elton and Hill(1992) (see also Szekli(1995)).

The Hardy-Littlewood stochastic majorant is closely related to the Hardy-Littlewood(1930) maximal function and has been considered by several authors (e.g. Blackwell and Dubins(1963), Dubins and Gilat(1978), Meilijson and Nàdas(1979), Kertz and Rösler(1990/92), Rüschendorf(1991)).

CHAPTER V

BOUNDS FOR BIVARIATE EXPECTED VALUES

1. Introduction.

General methods to derive (best) bounds for univariate expected values (bivariate expected values) of univariate transforms f(X) (bivariate transforms f(X,Y)) when the random variable X (bivariate pair of random variables (X,Y)) belong(s) to some specific set are numerous in the literature on Applied Probability and Statistics. However, a detailed and exhaustive catalogue of analytically solvable problems together with their solutions does not seem to be available, even for the simpler case when only means and variances are known.

Extremal values of univariate expected values $E[f(X)]$, where f(x) is some real function and the random variable X belongs to some specific set, have been studied extensively. In general the univariate case is better understood than the corresponding extremal problem for multivariate expected values $E[f(X_1,...,X_n)]$, where f is some multivariate real function and the random vector $\mathbf{X}=(X_1,...,X_n)$ varies over some set.

In general, by known mean-covariance structure, one often bounds the expected value of a multivariate transfom $f(\mathbf{X}):=f(X_1,...,X_n)$ by constructing a multivariate quadratic polynomial $q(\mathbf{x}):=q(x_1,...,x_n)=a_0+\sum_{i=1}^{n}a_ix_i+\sum_{i,j=1}^{n}a_{ij}x_ix_j$ such that $q(\mathbf{x})\ge f(\mathbf{x})$ to obtain a maximum, respectively $q(\mathbf{x})\le f(\mathbf{x})$ to obtain a minimum. If a multivariate finite atomic random vector $\mathbf{X}$ (usually a multivariate di- or triatomic random vector) can be found such that $\Pr(q(\mathbf{X})=f(\mathbf{X}))=1$, that is all mass points of the multivariate quadratic transform $q(\mathbf{X})$ are simultaneously mass points of $f(\mathbf{X})$, then $E[q(\mathbf{X})]=E[f(\mathbf{X})]$, which depends only on the mean-covariance structure, is necessarily the maximum, respectively the minimum. In the univariate case, a systematic study of this approach has been offered in Chapter II. In Sections 2 and 3, we consider the bivariate quadratic polynomial majorant/minorant method for the two most illustrative examples, namely the bivariate Chebyshev-Markov inequality and stop-loss bounds for bivariate random sums.

In the bivariate case, an alternative method to derive bounds for expected values is by means of the Hoeffding-Fréchet extremal distributions for the set of all bivariate distributions with fixed marginals. It is considered in Section 4. This general method allows to determine, under some regularity assumptions, bounds for expected values of the type $E[f(X,Y)]$, where f(x,y) is either a quasi-monotone (sometimes called superadditive) or a quasi-antitone right-continuous function. Its origin lies in an inequality for rearrangements by Lorentz(1953) (see Theorem 2.8 in Whitt(1976)) and has been further studied by Tchen(1980), Cambanis, Simons and Stout(1976), and Cambanis and Simons(1982).

We study in detail the illustrative example of the quasi-antitone function $f(x,y)=(x-y)_+$, which in passing solves by linear transformation the bivariate stop-loss transform case $f(x,y)=(x+y-D)_+$, $D\in R$. A combined Hoeffding-Fréchet upper bound for $E[(X-Y)_+]$ is determined in Theorem 4.1. In Section 5 it is shown that this upper bound can be obtained alternatively by minimizing a simple linear function of the univariate stop-loss transforms of X and Y. Through an unexpected link with the theory of stop-loss ordered extremal random variables considered in Chapter IV, the detailed calculation of the upper bound by given arbitrary ranges, means and variances of the marginals is made possible.

2. A bivariate Chebyshev-Markov inequality.

The simplest bivariate extension $f(X,Y) = I_{\{X \le x, Y \le y\}}(X,Y)$ of the original Chebyshev problem for $f(X) = I_{\{X \le x\}}(X)$ seems not to have been exhaustively analyzed.

In the following denote by $H(x,y) = \Pr(X \le x, Y \le y)$ the bivariate distribution of a couple (X,Y) of random variables with marginal distributions $F(x) = \Pr(X \le x)$, $G(y) = \Pr(Y \le y)$, marginal means μ_X, μ_Y, marginal variances σ_X^2, σ_Y^2, and correlation coefficient $\rho = Cov[X,Y]/\sigma_X \sigma_Y$. Consider the following sets of bivariate distributions :

(2.1)

$BD_1 = BD(F,G) = \{$ H(x,y) with fixed marginals F(x), G(y) $\}$

$BD_2 = BD(\mu_X, \mu_Y, \sigma_X^2, \sigma_Y^2) = \{$ H(x,y) with fixed marginal means and variances $\}$

$BD_3 = BD(\mu_X, \mu_Y, \sigma_X^2, \sigma_Y^2, \rho) = \{$ H(x,y) with fixed marginal means, variances and correlation coefficient $\}$

They generate six different extremal problems of Chebyshev type

$$\min_{(X,Y)\in BD_i} \{E[f(X,Y)]\}, \quad \max_{(X,Y)\in BD_i} \{E[f(X,Y)]\}, \quad i = 1,2,3, \tag{2.2}$$

whose solutions in the particular case $f(X,Y) = I_{\{X \le x, Y \le y\}}(X,Y)$ will be compared in the present Section. The extremal problems over BD_1 have been solved by Hoeffding(1940) and Fréchet(1951), which have shown that the best bivariate distribution bounds are given by

$$H_*(x,y) = \max\{F(x) + G(y) - 1, 0\} \le H(x,y) \le H^*(x,y) = \min\{F(x), G(y)\}. \tag{2.3}$$

In practical work, however, often only incomplete information about X, Y is available. This results in a wider range of variation of the extremal bounds for H(x,y), at least over BD_2 since $BD_1 \subseteq BD_2$. A solution to the optimization problem over BD_3 seems in general quite complex. Since $BD_3 \subseteq BD_2$ it generates a solution to the problem over BD_2.

A method of first choice (not necessarily the most adequate one) for solving optimization problems of Chebyshev type over BD_3 is the bivariate quadratic polynomial majorant/minorant method, which consists to bound the expected value of a bivariate random function f(X,Y) by constructing a bivariate quadratic polynomial

$$q(x,y) = ax^2 + by^2 + cxy + dx + ey + f \tag{2.4}$$

such that $q(x,y) \ge f(x,y)$ to obtain a maximum, respectively $q(x,y) \le f(x,y)$ to obtain a minimum, which is the special case n=2 of the method explained above.

2.1. Structure of diatomic couples.

Random variables are assumed to take values on the whole real line. Recall the structure in the univariate case.

Lemma 2.1. The set $D_2^{(2)} = D_2^{(2)}(\mu,\sigma)$ of all non-degenerate diatomic random variables with mean μ and standard deviation σ is described by a one-parametric family of supports $\{x_1, x_2\}$, $x_1 < x_2$, and probabilities $\{p_1, p_2\}$ such that

(2.5) $$x_2 = \mu + \frac{\sigma^2}{\mu - x_1}, \quad p_1 = \left(\frac{x_2-\mu}{x_2-x_1}\right), \quad p_2 = \left(\frac{\mu - x_1}{x_2-x_1}\right), \quad x_1 < \mu,$$

or equivalently

(2.6) $$x_1 = \mu - \sigma\sqrt{\frac{p_2}{p_1}}, \quad x_2 = \mu + \sigma\sqrt{\frac{p_1}{p_2}}, \quad 0 < p_1 < 1.$$

Proof. Apply Theorem I.5.1 and Remark (I.5.3). ◊

To clarify the structure of bivariate diatomic random variables, also called diatomic couples, consider the set denoted

(2.7) $$BD_3^{(2)} = \{(X,Y) : X \in D_2^{(2)}(\mu_X, \sigma_X), Y \in D_2^{(2)}(\mu_Y, \sigma_Y), Cov[X,Y] = \rho\sigma_X\sigma_Y\}.$$

The marginal X has support $\{x_1, x_2\}, x_1 < x_2$, and probabilities $\{p_1, p_2\}$, and Y has support $\{y_1, y_2\}$, $y_1 < y_2$, and probabilities $\{q_1, q_2\}$. By Lemma 1.1 one has the relations

(2.8) $$\begin{aligned} x_1 &= \mu_X - \sigma_X\sqrt{\frac{p_2}{p_1}}, \quad x_2 = \mu_X + \sigma_X\sqrt{\frac{p_1}{p_2}}, \\ y_1 &= \mu_Y - \sigma_Y\sqrt{\frac{q_2}{q_1}}, \quad y_2 = \mu_Y + \sigma_Y\sqrt{\frac{q_1}{q_2}}. \end{aligned}$$

The bivariate distribution of a couple (X,Y) is uniquely determined by the distribution of X and the conditional distribution of $(Y|X)$, and is thus given by a triple (α, β, p_1) such that

(2.9) $$\begin{aligned} &\alpha = P(Y = y_1 | X = x_1), \quad \beta = P(Y = y_1 | X = x_2), \quad p_1 = P(X = x_1), \\ &0 < \alpha + \beta < 2, \quad 0 < p_1 < 1 \end{aligned}$$

Then the joint probabilities $p_{ij} = P(X = x_i, Y = y_j)$, i,j=1,2, are given by

(2.10) $$\begin{aligned} p_{11} &= \alpha p_1, \quad p_{12} = (1-\alpha)p_1, \\ p_{21} &= \beta p_2, \quad p_{22} = (1-\beta)p_2. \end{aligned}$$

An equivalent representation in terms of the marginal probabilities and the correlation coefficient, that is in terms of the triple (p_1, q_1, ρ) is obtained as follows.

The marginal probability of Y satisfies the relation

$$\alpha p_1 + \beta p_2 = q_1, \tag{2.11}$$

and the correlation coefficient the relation

$$(\alpha - \beta) p_1 p_2 = \rho \sqrt{p_1 p_2 q_1 q_2}\,. \tag{2.12}$$

Solving the linear system (2.11), (2.12) and inserting into (2.10), one gets the following *canonical representation.*

Lemma 2.2. A diatomic couple $(X,Y) \in BD_3^{(2)}$ is uniquely characterized by its support $\{x_1, x_2\} x \{y_1, y_2\}$. The marginal probabilities are $p_1 = \left(\frac{x_2 - \mu_X}{x_2 - x_1}\right)$, $q_1 = \left(\frac{y_2 - \mu_Y}{y_2 - y_1}\right)$, the variances $\sigma_X^2 = (\mu_X - x_1)\cdot(x_2 - \mu_X)$, $\sigma_Y^2 = (\mu_Y - y_1)\cdot(y_2 - \mu_Y)$, and the joint probabilities are given by

$$\begin{aligned}
p_{11} &= p_1 q_1 + \rho\sqrt{p_1 p_2 q_1 q_2}\,, \\
p_{12} &= p_1 q_2 - \rho\sqrt{p_1 p_2 q_1 q_2}\,, \\
p_{21} &= p_2 q_1 - \rho\sqrt{p_1 p_2 q_1 q_2}\,, \\
p_{22} &= p_2 q_2 + \rho\sqrt{p_1 p_2 q_1 q_2}\,.
\end{aligned} \tag{2.13}$$

For calculations with diatomic couples (X,Y), it suffices to consider a unique *canonical arrangement* of its atoms.

Lemma 2.3. Without loss of generality the atoms of a couple $(X,Y) \in BD_3^{(2)}$ can be rearranged such that $x_1 < x_2$, $y_1 < y_2$, $y_2 - y_1 \le x_2 - x_1$.

Proof. By Lemma 2.1 one can assume $x_1 < x_2$, $y_1 < y_2$. If $y_2 - y_1 > x_2 - x_1$ then exchange the role of X and Y. ◊

2.2. A bivariate version of the Chebyshev-Markov inequality.

It suffices to consider standardized couples $(X,Y) \in BD_3 = BD(0,0,1,1,\rho)$. Indeed, the property

$$H(x,y) = \Pr(X \le x, Y \le y) = \Pr\left(\frac{X - \mu_X}{\sigma_X} \le \frac{x - \mu_X}{\sigma_X}, \frac{Y - \mu_Y}{\sigma_Y} \le \frac{y - \mu_Y}{\sigma_Y}\right)$$

shows the invariance of the probability distribution function under a standard transformation of variables. The (bivariate) Chebyshev-Markov maximal distribution over BD_3, if it exists, is denoted by

$$H_u(x.y) = \max_{(X,Y)\in BD_3} \{H(x,y)\}. \tag{2.14}$$

Theorem 2.1 (*Bivariate Chebyshev-Markov inequality*) Let $(X,Y) \in BD_3 = BD(0,0,1,1,\rho)$ be a standard couple with correlation coefficient ρ. Then the Chebyshev-Markov maximal distribution (2.14) satisfies the properties listed in Table 2.1.

In Table 2.1 and the subsequent discussion, one uses the notation $\bar{x} = -1/x$, which defines an involution mapping, whose square is by definition the identity mapping. Before the details of the derivation are presented, the obtained result is somewhat discussed in the Remarks 2.1.

Table 2.1 : bivariate Chebyshev-Markov inequality over $(-\infty,\infty)$

case	conditions	$H_u(x,y)$	bivariate extremal support	bivariate quadratic polynomial majorant
(1)	$x \le 0,\ y \ge 0$	$\frac{1}{1+x^2}$	$\{x,\bar{x}\}x\{\bar{y},y\}$	$\left(\frac{X-\bar{x}}{\bar{x}-x}\right)^2$
(2)	$x \ge 0,\ y \le 0$	$\frac{1}{1+y^2}$	$\{\bar{x},x\}x\{y,\bar{y}\}$	$\left(\frac{Y-\bar{y}}{\bar{y}-y}\right)^2$
(3)	$x < 0,\ y < 0$			
(3a)	$\lvert y\rvert \le \rho\lvert x\rvert$	$\frac{1}{1+x^2}$	$\{x,\bar{x}\}x\{\rho x,\overline{\rho x}\}$	$\left(\frac{X-\bar{x}}{\bar{x}-x}\right)^2$
(3b)	$\lvert x\rvert \le \rho\lvert y\rvert$	$\frac{1}{1+y^2}$	$\{\rho y,\overline{\rho y}\}x\{y,\bar{y}\}$	$\left(\frac{Y-\bar{y}}{\bar{y}-y}\right)^2$
(4)	$x > 0,\ y > 0$	1	$\{\bar{x},x\}x\{\bar{y},y\}$	1

Remarks 2.1.

(i) In the cases (1), (2), (4) there is no restriction on the correlation coefficient.

(ii) In case (3), when $\rho \ne 0$ and $\rho < \min\left\{\left|\frac{y}{x}\right|,\left|\frac{x}{y}\right|\right\}$, the maximum cannot be attained at a diatomic couple because there does not exist a quadratic majorant $q(X,Y) \ge I_{\{X\le x, Y\le y\}}(X,Y)$ such that $\Pr(q(X,Y) = f(X,Y)) = 1$. It is actually not clear what happens in this situation. Does there exist a maximum over BD_3 ?

(iii) If X and Y are independent, hence $\rho=0$, there exists a more precise statement in case (3). From the univariate Chebyshev-Markov inequality, one knows that

$$H(x,y) = F(x)G(y) \le \frac{1}{(1+x^2)(1+y^2)},$$

and the upper bound is attained at the diatomic couple with support $\{x,\bar{x}\}x\{y,\bar{y}\}$.

(iv) The bivariate Chebyshev-Markov extremal upper bound $H_u(x,y)$ is uniformly attained for all (x,y) if and only if ρ=1, which is complete dependence. Indeed $H_u(x,x)=\frac{1}{1+x^2}$ for x<0 is only attained provided ρ=1. In other words a maximizing extremal distribution to the problem $\max_{(X,Y)\in BD_2}\{H(x,y)\}$ *exists uniformly* for all (x,y) by setting ρ=1 in Table 2.1.

(v) In general $H_u(x,y)$ is not attained by a diatomic Hoeffding-Fréchet extremal upper bound $H^*(x,y)=\min\{F(x),G(y)\}$, which exists only if ρ>0 and is described by the following joint probabilities :

$$\begin{aligned} &p_{11}=p_1, \quad p_{12}=0, \quad p_{21}=q_1-p_1, \quad p_{22}=q_2, \quad \text{if} \quad p_1\le q_1, \\ &p_{11}=q_1, \quad p_{12}=p_1-q_1, \quad p_{21}=0, \quad p_{22}=p_2, \quad \text{if} \quad p_1\ge q_1. \end{aligned}$$

This affirmation also holds under the restriction ρ>0. The condition $p_{12}=0$, respectively $p_{21}=0$, implies the relation $y_1=\rho x_1$, respectively $x_1=\rho y_1$. An elementary check of these relations is done using the canonical representation (2.13) of Lemma 2.2. It follows that only one of x, y can be atom of such a $H^*(x,y)$. The four possible distributions have support :

Case 1 : $p_1\le q_1$: $\{x,\bar{x}\}x\{\rho x,\overline{\rho x}\}$, $\{\overline{\rho y},\rho y\}x\{\bar{y},y\}$

Case 2 : $p_1\ge q_1$: $\{x,\bar{x}\}x\left\{\frac{1}{\rho}x,\rho\bar{x}\right\}$, $\left\{\rho\bar{y},\frac{1}{\rho}y\right\}x\{\bar{y},y\}$

One finds now pairs of x, y such that $H^*(x,y)<H_u(x,y)$. For example if $x<\min\left\{\rho\bar{y},\frac{1}{\rho}\bar{y}\right\}<0, y\ge 0$, one has always $H^*(x,y)=0<H_u(x,y)$.

Proof of Theorem 2.1. In the following we set $f(X,Y)=I_{\{X\le x,Y\le y\}}(X,Y)$ and consider the half-planes

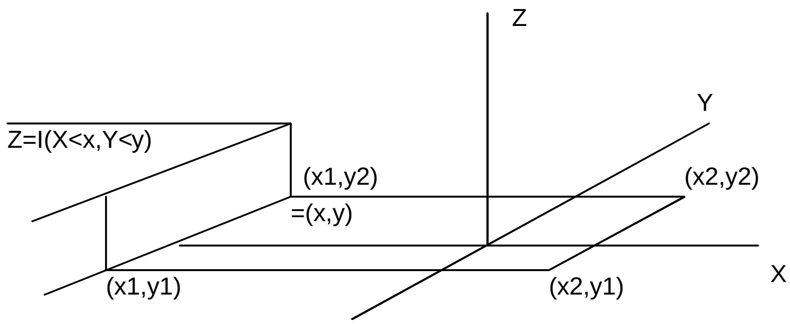
$$H_1=\{(X,Y)\in R^2: f(X,Y)=1\}, \quad H_2=\{(X,Y)\in R^2: f(X,Y)=0\}.$$

As seen in Lemma 1.3, a diatomic couple (X,Y) can be uniquely described by its support $\{x_1,x_2\}x\{y_1,y_2\}$ such that $x_1<x_2$, $y_1<y_2$, $y_2-y_1\le x_2-x_1$. The corresponding joint probabilities are given by the relations (2.13) of Lemma 2.2. To derive Table 2.1 we proceed case by case and construct in each case a quadratic polynomial majorant $q(X,Y)\ge f(X,Y)$ with all diatomic couples at zeros of $q(X,Y)-f(X,Y)$.

Case (1) : $x\le 0, y\ge 0$

One constructs a diatomic couple (X,Y) and q(X,Y) such that $q(x_i,y_j)=f(x_i,y_j)$, $i,j=1,2$, q(X,Y)≥1 on H_1, and q(X,Y)≥0 on H_2 as in Figure 2.1 :

Figure 2.1 : quadratic majorant in case (1)



Since $(x_2, y_2) \in H_2$, an appropriate choice for q(X,Y), together with its first partial derivatives, is given by

$$q(X,Y) = a(X-x_2)^2 + b(Y-y_2)^2 + c(X-x_2)(Y-y_2) + d(X-x_2) + e(Y-y_2) + f$$
$$q_X(X,Y) = 2a(X-x_2) + c(Y-y_2) + d$$
$$q_Y(X,Y) = 2b(Y-y_2) + c(X-x_2) + e$$

Since $(x_2, y_1), (x_2, y_2)$ are inner points of H_2 and q(X,Y)≥0 on H_2, these must be tangent at the quadratic surface Z=q(X,Y). The necessary conditions

$$q(x_2, y_2) = q_X(x_2, y_2) = q_Y(x_2, y_2) = q_X(x_2, y_1) = q_Y(x_2, y_1) = 0$$

imply that b=c=d=e=f=0, hence the form $q(X,Y) = a(X-x_2)^2$. The remaining points $(x_1, y_1), (x_1, y_2) \in H_1$ are zeros of q(X,Y)–1, hence

$$q(X,Y) = \left(\frac{X-x_2}{x_2-x_1}\right)^2.$$

Moreover one must have $x_1 \le x < x_2$, $y_1 < y_2 \le y$. The choice $\{x_1, x_2\}x\{y_1, y_2\} = \{x, \overline{x}\}x\{\overline{y}, y\}$ implies that

$$q(X,Y) = \left(\frac{X-\overline{x}}{\overline{x}-x}\right)^2$$

is a required quadratic majorant of f(X,Y). The inequality q(X,Y)≥1 follows because $\overline{x} - X \ge \overline{x} - x$ on H_1. Finally one obtains the extremal value as

$$H_u(x,y) = E\left[q(X,Y)\right] = \frac{1+\overline{x}^2}{(\overline{x}-x)^2} = \frac{1}{1+x^2}.$$

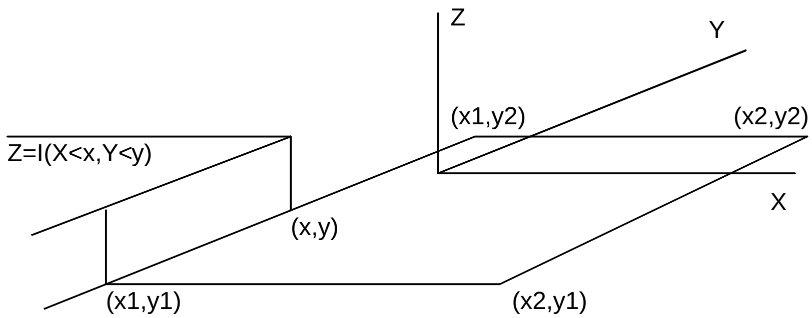
Case (2) : $x \ge 0, y \le 0$

This follows directly from case (1) by exchanging the variables x and y.

Case (3) : $x < 0, y < 0$

One constructs a diatomic couple (X,Y) and a q(X,Y) such that $q(x_i, y_j) = f(x_i, y_j)$, except in (x_1, y_2), where one sets $p_{12} = 0$, and with the properties q(X,Y)≥1 on H_1, and q(X,Y)≥0 on H_2 as in Figure 2.2 :

Figure 2.2 : quadratic majorant in case (3)



The couples $(x_2, y_1), (x_2, y_2)$ are inner points of H_2 and q(X,Y)≥0 on H_2, hence they must be tangent at the quadratic surface Z=q(X,Y). As in case (1) it follows that $q(X,Y) = a(X - x_2)^2$. Since $(x_1, y_1) \in H_1$ must be zero of q(X,Y)–1, one gets

$$q(X,Y) = \left(\frac{X - x_2}{x_2 - x_1}\right)^2 .$$

The choice $x_1 = x, x_2 = \overline{x}$ implies that $\overline{x} - X \ge \overline{x} - x$ on H_1, hence q(X,Y)≥1 on H_1. The condition $p_{12} = 0$ implies the relations (use (2.13)) : $y_1 = \rho x_1 = \rho x$, $y_2 = \overline{y}_1 = \overline{\rho}\overline{x}$. Since $x < 0, y_1 < 0$ one must have ρ>0. Furthermore the condition $(x_1, y_1) \in H_1$ implies $y_1 = \rho x \le y$, hence $-y = |y| \le \rho(-x) = \rho|x|$. The subcase (3a) has been shown. Exchanging the role of x and y, one gets subcase (3b).

Case (4) : $x > 0, y > 0$

It is trivial that for the diatomic couple (X,Y) with support $\{\overline{x}, x\} x \{\overline{y}, y\}$ one has H(x,y)=1, which is clearly maximal. ◊

3. Best stop-loss bounds for bivariate random sums.

As a next step, and similarly to the adopted approach in the univariate case, we consider the bivariate stop-loss function $f(x,y)=(x+y-D)_+$, where D is the deductible.

It will be shown in Subsection 3.1 that a bivariate quadratic polynomial majorant is a separable function q(x,y)=q(x)+q(y), where q(x), q(y) are quadratic polynomials, and thus does not contain the mixed term in xy. In particular, the maximum does not depend on the given (positive) correlation and is only attained by complete dependence. In contrast to this the minimal stop-loss bound over all bivariate sums by known means, variances and fixed negative correlation exists, at least over a wide range of deductibles, as shown in Subsection 3.2. Our result shows that the trivial best lower stop-loss bound is attained by diatomic couples with any possible negative correlation.

3.1. A best upper bound for bivariate stop-loss sums.

The identity $(X+Y-D)_+=((X-\mu_X)+(Y-\mu_Y)-(D-\mu))_+ \quad \mu=\mu_X+\mu_Y$, shows that without loss of generality one can assume that $\mu_X=\mu_Y=\mu=0$. A bivariate quadratic polynomial majorant of $f(x,y)=(x+y-D)_+$, as defined by

(3.1) $$q(x,y)=ax^2+by^2+cxy+dx+ey+f,$$

depends on 6 unknown coefficients. Consider the identities

$$(D-x-y)_+=\int_0^D I\{x\le u, y\le D-u\}du, \quad (X+Y-D)_+=X+Y-D+(D-X-Y)_+.$$

Inserting the first one into the second one and taking expectation, one obtains

$$E[(X+Y-D)_+]=E[X]+E[Y]-D+\int_{-\infty}^{D} H(x,D-x)dx.$$

Therefore, the maximum of $E[f(X,Y)]$ over arbitrary couples (X,Y), by given marginals, is attained at the Hoeffding-Fréchet maximal distribution $H^*(x,y)=\min\{F(x),G(y)\}$. A count of the number of unknowns and corresponding conditions (given below), which must be fulfilled in order to get a bivariate quadratic majorant, shows that the immediate candidates to consider first are diatomic couples.

In the special situation $(X,Y)\in BD_3^{(2)}=BD(0,0,\sigma_X,\sigma_Y,\rho), \rho>0$, the Hoeffding-Fréchet upper bound is described by the following joint probabilities :

(3.2) $$p_{11}=p_1, \quad p_{12}=0, \quad p_{21}=q_1-p_1, \quad p_{22}=q_2, \quad \text{if} \quad p_1\le q_1,$$

(3.3) $$p_{11}=q_1, \quad p_{12}=p_1-q_1, \quad p_{21}=0, \quad p_{22}=p_2, \quad \text{if} \quad p_1\ge q_1.$$

Restrict first the attention to the form (3.2) with

(3.4) $$p_1=\frac{x_2}{x_2-x_1}, \quad q_1=\frac{y_2}{y_2-y_1}.$$

Using (2.13), one sees that the marginal probabilities satisfy the following constraint :

(3.5) $$\sqrt{\frac{q_2}{q_1}}=\rho\sqrt{\frac{p_2}{p_1}}, \quad \text{if} \quad p_1\le q_1,$$

which expressed in terms of the atoms yields the relation

(3.6) $$y_2 = \frac{1}{\rho}\cdot\frac{\sigma_Y}{\sigma_X}\cdot x_2, \quad 0<\rho\le 1.$$

On the other side the equations of marginal variances imply the further constraints

(3.7) $$x_1x_2 = -\sigma_X^2, \quad y_1y_2 = -\sigma_Y^2.$$

Thus a possible extremal diatomic couple is completely specified by a single unknown atom, say x_1. Using the above facts and summarizing, one can restrict attention to the subset of all $(X,Y)\in BD_3^{(2)}$ of the form

(3.8) $$\begin{aligned} &x_1 = -\sigma_X\cdot x, \quad x_2 = \sigma_X\cdot\frac{1}{x}, \quad p_1 = \frac{1}{1+x^2}, \quad x>0, \\ &y_1 = -\rho\sigma_Y\cdot x, \quad y_2 = \sigma_Y\cdot\frac{1}{\rho x}, \quad q_1 = \frac{1}{1+(\rho x)^2}, \\ &p_{11} = p_1, \quad p_{12} = 0, \quad p_{21} = q_1 - p_1, \quad p_{22} = q_2. \end{aligned}$$

Since $p_{12}=0$, the relevant bivariate sum mass points are $x_1+y_1, x_2+y_1, x_2+y_2$. With (3.8) and Lemma 2.3 assume that (otherwise exchange X and Y)

(3.9) $$(x_2-x_1)-(y_2-y_1)\ge 0.$$

Therefore one can suppose that $x_1+y_1\le d< x_2+y_2$ (otherwise the calculation is trivial).

Consider z=q(x,y) as a quadratic surface in the (x,y,z)-space, and $z=f(x,y)=(x+y-D)_+$ as a bivariate piecewise linear function with the two pieces $z=l_1(x,y)=0$ defined on the half-plane $H_1=\{(x,y): x+y\le D\}$ and $z=l_2(x,y)=x+y-D$ on the half-plane $H_2=\{(x,y): x+y\ge D\}$, then one must have $Q_1(x,y):=q(x,y)-\ell_1(x,y)\ge 0$ on H_1, and $Q_2(x,y):=q(x,y)-\ell_2(x,y)\ge 0$ on H_2. To achieve Pr(q(X,Y)=f(X,Y)=1)=1 one must satisfy the 3 conditions

(3.10) $$\begin{aligned} &Q_1(x,y)=0 \\ &Q_i(x_2,y_1)=0, \quad (x_2,y_1) \text{ in one of } H_i, \ i=1,2 \\ &Q_2(x_2,y_2)=0 \end{aligned}$$

The inequalities constraints $Q_i(x,y)\ge 0$ imply that (x_i,y_i) must be tangent at the hyperplane $z=\ell_i(x,y),\ i=1,2$, hence the 4 further conditions

(3.11) $$\begin{aligned} &\frac{\partial}{\partial x}Q_i(x,y)\Big|_{(x_i,y_i)}=0, \\ &\frac{\partial}{\partial y}Q_i(x,y)\Big|_{(x_i,y_i)}=0, \quad i=1,2. \end{aligned}$$

Together (3.10) and (3.11) imply 7 conditions for 7 unknowns (6 coefficients plus one mass point), a necessary system of equations to determine a bivariate quadratic majorant, which can eventually be solved. To simplify calculations, let us replace q(x,y) by the equivalent form

(3.12) $$q(x,y)=a(x-x_1)^2+b(y-y_1)^2+c(x-x_1)(y-y_1)+d(x-x_1)+e(y-y_1)+f.$$

The required partial derivatives are

(3.13) $$\begin{aligned} q_x(x,y) &= 2a(x-x_1)+c(y-y_1)+d \\ q_y(x,y) &= 2b(y-y_1)+c(x-x_1)+e \end{aligned}$$

Then the 7 conditions above translate to the system of equations in $x_i, y_i, i=1,2$:

(C1) $$q(x_1,y_1)=f=0$$

(C2) $$q(x_2,y_1)=a(x_2-x_1)^2+d(x_2-x_1)=(x_2+y_1-D)_+$$

(C3) $$\begin{aligned} &q(x_2,y_2)=a(x_2-x_1)^2+b(y_2-y_1)^2+c(x_2-x_1)(y_2-y_1) \\ &+d(x_2-x_1)+e(y_2-y_1)=x_2+y_2-D \end{aligned}$$

(C4) $$q_x(x_1,y_1)=d=0$$

(C5) $$q_y(x_1,y_1)=e=0$$

(C6) $$q_x(x_2,y_2)=2a(x_2-x_1)+c(y_2-y_1)=1$$

(C7) $$q_y(x_2,y_2)=2b(y_2-y_1)+c(x_2-x_1)=1$$

In particular one has d=e=f=0. The conditions (C6), (C7) can be rewritten as

(C6) $$a(x_2-x_1)=\tfrac{1}{2}(1-c(y_2-y_1))$$

(C7) $$b(y_2-y_1)=\tfrac{1}{2}(1-c(x_2-x_1))$$

Insert these values into (C3) to see that the following relation must hold :

(3.14) $$(x_1+y_1)+(x_2+y_2)=2D.$$

It says that the sum of the two extreme maximizing couple sums equals two times the deductible. Observe in passing that the similar constraint holds quite generally in the univariate case (proof of Theorem II.2.1 for type (D1)).

Now try to satisfy (C2). If $(x_2,y_1)\in H_1$ one must have a=0, hence $c(y_2-y_1)=1$ by (C6), and $b=\frac{1}{2}\left\{\frac{(y_2-y_1)-(x_2-x_1)}{(y_2-y_1)^2}\right\}$ by (C7). Similarly, if $(x_2,y_1)\in H_2$ one obtains $a(x_2-x_1)^2=x_2+y_1-D$, hence $c(x_2-x_1)=1$ by (C6) using (3.14), and b=0. In the first case, one has $q(x,y)=(y-y_1)(b(y-y_1)+c(x-x_1))$, and in the second one $q(x,y)=(x-x_1)(a(x-x_1)+c(y-y_1))$. In both cases the quadratic form is indefinit, which implies that the majorant constraint q(x,y)≥0 on H_1 or H_2 cannot be fulfilled. The only way to get a quadratic majorant is to disregard condition (C2), that is to set $p_{21}=0$, hence $q_1=p_1$ (no probability on the couple (x_2,y_1)). From (3.8) one obtains immediately ρ=1, which is complete dependence. To get a quadratic majorant one can set c=0 in (C6), (C7). Then one obtains

(3.15) $$q(x,y)=\frac{1}{2}\left\{\frac{(x-x_1)^2}{(x_2-x_1)}+\frac{(y-y_1)^2}{(y_2-y_1)}\right\}.$$

The discriminant of both $Q_i(x,y)$, $i=1,2$, equals

(3.16) $$\Delta=\frac{1}{(x_2-x_1)(y_2-y_1)}>0,$$

and furthermore

(3.17) $$\frac{\partial^2}{\partial x^2}Q_i(x,y)\Big|_{(x_i,y_i)}=\frac{1}{x_2-x_1}>0, \quad i=1,2.$$

By standard calculus one concludes that $Q_i(x,y)$ is positive definite, hence as required. Solving (3.14) using (3.8), one obtains the explicit maximizing Hoeffding-Fréchet bivariate diatomic couple (X,Y) summarized in (3.18). It remains to discuss the form (3.3) of the Hoeffding-Fréchet extremal diatomic distribution. Replacing in the above proof the couple (x_2,y_1) by (x_1,y_2), one obtains similarly that condition (C2) must be disregarded, hence $p_{12}=0$, $p_1=q_1$, and thus $\rho=1$. The same maximizing couple follows. In fact the applied bivariate quadratic majorant method shows the following stronger result.

Theorem 3.1. (*Characterization of the bivarite stop-loss inequality*) The bivariate quadratic majorant stop-loss sum problem over $BD_3^{(2)}=BD(\mu_X,\mu_Y,\sigma_X,\sigma_Y,\rho)$, $\rho>0$, is solvable if and only if $\rho=1$. The atoms and probabilities of the maximizing diatomic couple are given by (set $\sigma=\sigma_X+\sigma_Y$, $\mu=\mu_X+\mu_Y$)

(3.18) $$\begin{aligned} &x_1=-\frac{\sigma_X}{\sigma}\left\{\sqrt{(D-\mu)^2+\sigma^2}-(D-\mu)\right\}, \quad x_2=\frac{\sigma_X}{\sigma}\left\{\sqrt{(D-\mu)^2+\sigma^2}+(D-\mu)\right\}\\ &y_1=-\frac{\sigma_Y}{\sigma}\left\{\sqrt{(D-\mu)^2+\sigma^2}-(D-\mu)\right\}, \quad y_2=\frac{\sigma_Y}{\sigma}\left\{\sqrt{(D-\mu)^2+\sigma^2}+(D-\mu)\right\}\\ &p_{11}=\frac{1}{2}\Big(1+\frac{D-\mu}{\sqrt{(T-\mu)^2+\sigma^2}}\Big), \quad p_{22}=1-p_{11}, p_{12}=p_{21}=0,\end{aligned}$$

and the maximal bivariate stop-loss transform of a couple (X,Y) equals

(3.19) $$\frac{1}{2}\left\{\sqrt{(D-\mu)^2+\sigma^2}-(D-\mu)\right\}.$$

Proof. The formulas (3.18) follow from (3.14) as explained in the text, while (3.19) follows from (3.15) by noting that $E[q(X,Y)]=\max\{E[(X+Y-D)_+]\}$. The elementary calculations are left to the reader. ◊

3.2. Best lower bounds for bivariate stop-loss sums.

We proceed as in Subsection 3.1 with the difference that $q(x,y) \le f(x,y)$ and the fact that the minimum of $E[f(X,Y)]$ should be attained at the Hoeffding-Fréchet extremal lower bound distribution $H_*(x,y) = \max\{F(x)+G(y)-1, 0\}$. For diatomic couples with negative correlation coefficient $\rho<0$, two cases are possible (derivation is immediate) :

Case 1 : $p_1 + q_1 \le 1$

(3.20) $$p_{11} = 0, \quad p_{12} = p_1, \quad p_{21} = q_1 - p_1, \quad p_{22} = 1 - q_1$$

Case 2 : $p_1 + q_1 > 1$

(3.21) $$p_{11} = p_1 + q_1 - 1, \quad p_{12} = 1 - q_1, \quad p_{21} = q_1 - p_1, \quad p_{22} = 1 - q_1$$

Taking into account (2.13), the form of p_{11} implies the following relations :

(3.22) $$y_1 = \frac{1}{\rho}\frac{\sigma_Y}{\sigma_X} x_2 \text{ in Case 1, } y_2 = \frac{1}{\rho}\frac{\sigma_Y}{\sigma_X} x_1 \text{ in Case 2.}$$

Clearly (3.7) also holds. We show first that there cannot exist a bivariate quadratic minorant with non-zero quadratic coefficients a, b, c. Therefore, the minimum, if it exists, must be attained at a bivariate linear minorant. Since possibly $p_{11} = 0$ (as in Case 1), the non-trivial situation to consider is $x_1 + y_2 \le D < x_2 + y_2$. Proceed now as in Subsection 3.1. The simplest $q(x,y)$ takes the form

(3.23) $$\begin{aligned} &q(x,y) \\ &= a(x-x_1)^2 + b(y-y_2)^2 + c(x-x_1)(y-y_2) + d(x-x_1) + e(y-y_2) + f \end{aligned}$$

The partial derivatives are

(3.24) $$\begin{aligned} q_x(x,y) &= 2a(x-x_1) + c(y-y_2) + d \\ q_y(x,y) &= 2b(y-y_2) + c(x-x_1) + e \end{aligned}$$

The following 8 conditions must hold (up to cases where some probabilities vanish) :

(C1) $$q(x_1,y_1) = a(y_2-y_1)^2 + e(y_1-y_2) + f = 0$$

(C2) $$q(x_1,y_2) = f = 0$$

(C3) $$\begin{aligned} q(x_2,y_1) &= a(x_2-x_1)^2 + b(y_2-y_1)^2 + c(x_2-x_1)(y_1-y_2) \\ &+ d(x_2-x_1) + e(y_1-y_2) = (x_2+y_1-T)_+ \end{aligned}$$

(C4) $$q(x_2,y_2) = a(x_2-x_1)^2 + d(x_2-x_1) = x_2 + y_2 - T$$

(C5) $$q_x(x_1,y_2) = d = 0$$

(C6) $q_y(x_1, y_2) = e = 0$

(C7) $q_x(x_2, y_2) = 2a(x_2 - x_1) = 1$

(C8) $q_y(x_2, y_2) = (x_2 - x_1) = 1$

In particular one has d=e=f=0. By standard calculus, in order that $Q_i(x,y) \le 0$ on H_i, i=1,2, the quadratic form $Q_i(x,y)$ must be negative definite. Therefore its discriminant, which is $\Delta = 4ab - c^2$ for both i=1,2, must be positive, and a<0. But by (C7) one has a>0, which shows that no such q(x,y) can actually be found. Therefore the minimum must be attained for a bivariate linear form. Similarly to the univariate case, the candidates for a linear minorant are $\ell(x,y) = x + y - D$ if D≤0 and $\ell(x,y) \equiv 0$ if D>0.

<u>Case (I)</u> : D≤0, $\ell(x,y) = x + y - D$

Let us construct a diatomic couple with probabilities (2.20) such that

(3.25) either $x_1 + y_2 = D \le x_2 + y_1 \le x_2 + y_2$,
or $x_2 + y_1 = D \le x_1 + y_2 \le x_2 + y_2$.

Then one has $\Pr(\ell(X,Y) = (X + Y - D)_+) = 1$, $\ell(x,y) \le 0 = (x + y - D)_+$ on H_1, and $\ell(x,y) = x + y - D = (x + y - D)_+$ on H_2. Together this implies that $\min\{E[(X+Y-D)_+]\} = E[\ell(X,Y)] = -D$, as desired. Let us solve (3.25) using (3.23). Three subcases are distinguished :

(A) $\sigma_Y < \left(\frac{-1}{\rho}\right)\sigma_X$ (hence $\sigma_X + \rho\sigma_Y > 0$)

Since $y_2 = \rho\frac{\sigma_Y}{\sigma_X}x_1$ by (3.23), the equation $x_1 + y_2 = D$ has the solution

(3.26)
$$x_1 = \left(\frac{\sigma_X}{\sigma_X + \rho\sigma_Y}\right)\cdot T, \quad x_2 = -\frac{\sigma_X(\sigma_X + \rho\sigma_Y)}{D},$$
$$y_1 = -\frac{\sigma_Y(\sigma_X + \rho\sigma_Y)}{D}, \quad y_2 = \left(\frac{\sigma_Y}{\sigma_X + \rho\sigma_Y}\right)\cdot D.$$

One checks that $D \le x_2 + y_1 \le x_2 + y_2$ and that $p_1 + q_1 \le 1$ (condition for $p_{11} = 0$)

(B) $\sigma_Y > \left(\frac{-1}{\rho}\right)\sigma_X$

Exchange X and Y such that $\sigma_X > \left(\frac{-1}{\rho}\right)\sigma_Y$. Since $\rho^2 \le 1$ one gets $\sigma_Y < (-\rho)\sigma_X \le \frac{1}{\rho^2}(-\rho)\sigma_X = \left(\frac{-1}{\rho}\right)\sigma_X$, and one concludes as in Subcase (A).

(C) $\sigma_X + \rho\sigma_Y = 0$

Using (3.23) one gets the relations $y_1 = \frac{-1}{\rho^2} x_2$, $y_2 = -x_1$. Setting $x_2 = \left(\frac{\rho^2}{\rho^2 - 1}\right) T$ one obtains $x_2 + y_1 = D \le 0 = x_1 + y_2 \le x_2 + y_2$, which yields a couple with the property (3.25).

Case (II) : D>0, $\ell(x,y) = 0$

One must construct a diatomic couple with probabilities (3.20) such that $x_2 + y_2 \le D$. Then all mass couples belong to H_1, which implies that $\Pr(\ell(X,Y) = (X+Y-D)_+) = 1$. It follows that $\min\{E[(X+Y-D)_+]\} = 0$. Using that $y_2 = \rho \frac{\sigma_Y}{\sigma_X} x_1$, the equation $x_2 + y_2 = D$ has the solution

$$x_2 = \frac{1}{2}\left(D + \sqrt{D^2 - 4(-\rho)\sigma_X \sigma_Y}\right) \qquad (3.27)$$

provided $D \ge 2\sqrt{(-\rho)\sigma_X\sigma_Y}$. Removing the assumption $\mu_X = \mu_Y = 0$ (translation of X and Y), one obtains the following bivariate extension of the corresponding univariate result.

Theorem 3.2. The minimal bivariate stop-loss transform of a bivariate couple (X,Y) with marginal means μ_X, μ_Y, variances σ_X, σ_Y and negative correlation ρ<0 equals $(\mu_X + \mu_Y - D)_+$ provided $D \le \mu_X + \mu_Y$ or $D \ge 2\sqrt{(-\rho)\sigma_X\sigma_Y}$. It is attained by a diatomic couple with atoms as constructed above in Case (I) and Case (II).

4. A combined Hoeffding-Fréchet upper bound for expected positive differences.

To simplify the subsequent analysis and presentation, it is necessary to introduce a considerable amount of notations, conventions, and assumptions.

Let X, Y be random variables with distributions F(x), G(x), and let $[A_X, B_X]$, $[A_Y, B_Y]$ be the smallest closed intervals containing the supports of X, Y, which are defined by $A_X = \inf\{x : F(x) > 0\}$, $B_X = \sup\{x : F(x) < 1\}$, $-\infty \le A_X < B_X \le \infty$, and similar expressions for A_Y, B_Y. By convention one sets F(x)=0 if $x < A_X$, F(x)=1 if $x > B_X$, and G(x) is similarly extended to the whole real line. The notations $\underline{A} = \min\{A_X, A_Y\}$, $\overline{A} = \max\{A_X, A_Y\}$, $\underline{B} = \min\{B_X, B_Y\}$, $\overline{B} = \max\{B_X, B_Y\}$ will be used throughout. With the made conventions, the interval $[\underline{A}, \overline{B}]$, which can be viewed as a smallest common "extended" support of X and Y, turns out to be relevant for the present problem. The following *regularity assumption*, which is required in the proof of our results and is often fulfilled in concrete examples, is made :

(RA) $F(x) + G(x)$ is strictly increasing in x on the open interval $(\underline{A}, \overline{B})$

The joint probability function of the pair (X,Y) is denoted by H(x,y). Survival functions are denoted by $\overline{F}(x), \overline{G}(x), \overline{H}(x,y)$. One assumes that the means μ_X, μ_Y exist and are finite. The stop-loss transform of X is defined and denoted by $\pi_X(x) = E[(X-x)_+]$. Using partial integration, one shows that

(4.1) $$\pi_X(x) = \begin{cases} \mu_X - x, \quad if\, x \le A_X, \\ \int_x^{B_X} \overline{F}(u)du, \quad if\, A_X \le x \le B_X, \\ 0, \quad if\, x \ge B_X. \end{cases}$$

Finally, the indicator function of a set $\{\cdot\}$ is denoted by $I\{\cdot\}$.

It will be explained how bounds for the expected positive difference $E[(X-Y)_+]$ can be obtained. First of all, the symmetry relation

(4.2) $$E[(X-Y)_+] = \mu_X - \mu_Y + E[(Y-X)_+]$$

shows that in general a bound must be constructed by combining bounds for the left and right hand side in (4.2). For example, let $M(X,Y)$ be an upper bound for $E[(X-Y)_+]$, and let $M(Y,X)$ be an upper bound for $E[(Y-X)_+]$. Then a combined upper bound is

(4.3) $$M = \max\{M(X,Y), \mu_X - \mu_Y + M(Y,X)\}.$$

Without loss of generality, one can assume that $A_Y < B_X$ and $A_X < B_Y$. Otherwise the random variable $(X-Y)_+$ or $(Y-X)_+$ is identically zero, and the calculation is trivial. From the identity

(4.4) $$(X-Y)_+ = \int_{A_Y}^{B_X} I\{X \ge u, Y \le u\}du = \int_{A_Y}^{B_X} \left(I\{Y \le u\} - I\{X \le u, Y \le u\}\right)du,$$

one derives, taking expectations, the formula

(4.5) $$E[(X-Y)_+] = \int_{A_Y}^{B_X} \overline{H}(u,u)du - \int_{A_Y}^{B_X} \overline{G}(u)du,$$

and by symmetry, one has

(4.6) $$E[(Y-X)_+] = \int_{A_X}^{B_Y} \overline{H}(u,u)du - \int_{A_X}^{B_Y} \overline{F}(u)du.$$

Consider the extremal distributions

(4.7) $$H_*(x,y) = (F(x)+G(y)-1)_+ \le H(x,y) \le H^*(x,y) = \min\{F(x),G(y)\},$$

which provide the extremal bounds for a bivariate distribution over the space $BD(F,G)$ of all bivariate random pairs (X,Y) with given marginals $F(x)$ and $G(x)$, and which have been introduced by Hoeffding(1940) and Fréchet(1951). It follows that the survival function $\overline{H}(x,x)$ satisfies the bounds

(4.8) $$\overline{H}^*(x,x) = \max\{\overline{F}(x),\overline{G}(x)\} \le \overline{H}(x,x) \le \overline{H}_*(x,x) = \min\{\overline{F}(x)+\overline{G}(x),1\},$$

from which bounds for the expected positive difference can be constructed combining (4.2), (4.5) and (4.6). In fact, it is possible to determine bounds for expected values of the form $E[f(X-Y)]$, where $f(x)$ is any convex non-negative function, as observed by Tchen(1980), Corollary 2.3. As a general result, the same method allows to determine, under some regularity assumptions, bounds for expected values of the type $E[f(X,Y)]$, where f(x,y) is either a quasi-monotone (sometimes called superadditive) or a quasi-antitone right-continuous function (note that $f(x,y)=(x-y)_+$ is quasi-antitone). For this, consult the papers mentioned in Section 1, especially Cambanis et al.(1976).

Denote by $M_{HF}(X,Y), M_{HF}(Y,X)$ the upper bounds for $E[(X-Y)_+]$ obtained by inserting the Hoeffding-Fréchet extremal bound $\overline{H}_*(x,x)$ into (4.5), (4.6). A detailed calculation of the combined upper bound $M_{HF}=\max\{M_{HF}(X,Y), \mu_X-\mu_Y+M_{HF}(Y,X)\}$ yields the following result.

Theorem 4.1. Let $(X,Y)\in BD(F,G)$ be a bivariate random variable with marginal supports $[A_X,B_X]$, $[A_Y,B_Y]$, and finite marginal means μ_X,μ_Y. Suppose the regularity assumption (RA) holds. Then the combined Hoeffding-Fréchet upper bound is determined as follows :

Case (I) : $\overline{F}(x)+\overline{G}(x)\le 1$ for all $x\in(\underline{A},\overline{B})$

$$M_{HF}=\mu_X-\underline{A}$$

Case (II) : There exists a unique $x_0\in(\underline{A},\overline{B})$ such that $\overline{F}(x)+\overline{G}(x)\ge 1$ for $x\le x_0$ and $\overline{F}(x)+\overline{G}(x)\le 1$ for $x\ge x_0$

$$M_{HF}=\begin{cases} \overline{A}-\mu_Y+\pi_X(\overline{A})+\pi_Y(\overline{A}), if\ x_0\in(\underline{A},\overline{A}], \\ x_0-\mu_Y+\pi_X(x_0)+\pi_Y(x_0), \quad if\ x_0\in(\overline{A},\underline{B}), \\ \underline{B}-\mu_Y+\pi_X(\underline{B})+\pi_Y(\underline{B}), \quad if\ x_0\in[\underline{B},\overline{B}). \end{cases}$$

Case (III) : $\overline{F}(x)+\overline{G}(x)\ge 1$ for all $x\in(\underline{A},\overline{B})$

$$M_{HF}=\overline{B}-\mu_Y$$

Proof. This follows from a case by case calculation. The lower index in the M_{HF}'s is omitted.

Case (I) :

Inserting $\overline{H}_*(x,x)$ into (4.5) one has

$$M(X,Y)=\int_{A_Y}^{B_X}\{\overline{F}(x)+\overline{G}(x)\}dx-\int_{A_Y}^{B_X}\overline{G}(x)dx=\pi_X(A_Y), \tag{4.9}$$

and similarly $M(Y,X)=\pi_Y(A_X)$ by (4.6). Now make use of (4.1) and the monotone decreasing property of the stop-loss transform. If $A_X\le A_Y$ then one has $\mu_X-\mu_Y+\pi_Y(A_X)=\pi_X(A_X)\ge\pi_X(A_Y)$, and if $A_X\ge A_Y$ one has $\mu_X-\mu_Y+\pi_Y(A_X)\le\mu_X-\mu_Y+\pi_Y(A_Y)\le\pi_X(A_Y)$. Together this shows that $M=\pi_X(\underline{A})=\mu_X-\underline{A}$.

Case (II) :

To evaluate M(X,Y) from (4.5) three subcases are distinguished.

(IIa) $A_Y < x_0 < B_X$

By assumption one has

$$M(X,Y) = \int_{A_Y}^{x_0} dx + \int_{x_0}^{B_X} \{\overline{F}(x) + \overline{G}(x)\} du - \int_{A_Y}^{B_X} \overline{G}(x)dx \tag{4.10}$$
$$= x_0 - A_Y + \pi_X(x_0) - \int_{A_Y}^{x_0} \overline{G}(x)dx.$$

Furthermore, if $x_0 \le B_Y$ one has $\int_{A_Y}^{x_0} \overline{G}(x)dx. = \pi_Y(A_Y) - \pi_Y(x_0)$, hence $M(X,Y) = x_0 - \mu_Y + \pi_X(x_0) + \pi_Y(x_0)$. If $x_0 \ge B_Y$ one gets $M(X,Y) = x_0 - \mu_Y + \pi_X(x_0) = x_0 - \mu_Y + \pi_X(x_0) + \pi_Y(x_0)$, the last equality because $\pi_Y(x_0) = 0$.

(IIb) $A_X \le x_0 \le A_Y$

As in case (I) one obtains $M(X,Y) = \pi_X(A_Y) = A_Y - \mu_Y + \pi_X(A_Y) + \pi_Y(A_Y)$.

(IIc) $B_X \le x_0 \le B_Y$

One obtains successively

$$M(X,Y) = \int_{A_Y}^{B_X} dx - \int_{A_Y}^{B_X} \overline{G}(x)dx = B_X - A_Y - (\pi_Y(A_Y) - \pi_Y(B_X))$$
$$= B_X - \mu_Y + \pi_Y(B_X) = B_X - \mu_Y + \pi_X(B_X) + \pi_Y(B_X).$$

By symmetry M(Y,X) is obtained similarly. The three subcases are :

(IIa') $A_X < x_0 < B_Y$: $M(Y,X) = x_0 - \mu_X + \pi_X(x_0) + \pi_Y(x_0)$
(IIb') $A_Y \le x_0 \le A_X$: $M(Y,X) = A_X - \mu_X + \pi_X(A_X) + \pi_Y(A_X)$
(IIc') $B_Y \le x_0 \le B_X$: $M(Y,X) = B_Y - \mu_X + \pi_X(B_Y) + \pi_Y(B_Y)$.

Furthermore, if $x_0 \in (\underline{A}, \overline{B})$ then either $x_0 \in (A_Y, B_X)$ and/or $x_0 \in (A_X, B_Y)$ holds. Combining the above six subcases using that the univariate function $x + \pi_X(x) + \pi_Y(x)$ is decreasing for $x \le x_0$ and increasing for $x \ge x_0$, one sees that M takes the following values

(II1) If $x_0 \in (A_Y, B_X)$ and $x_0 \in (A_X, B_Y)$ then $M = x_0 - \mu_X + \pi_X(x_0) + \pi_Y(x_0)$.
(II2) If $x_0 \in (A_Y, B_X)$ and $x_0 \in (A_Y, A_X]$ then $M = A_X - \mu_Y + \pi_X(A_X) + \pi_Y(A_X)$.
(II3) If $x_0 \in (A_Y, B_X)$ and $x_0 \in [B_Y, B_X)$ then $M = B_Y - \mu_Y + \pi_X(B_Y) + \pi_Y(B_Y)$.
(II4) If $x_0 \in (A_X, B_Y)$ and $x_0 \in (A_X, A_Y]$ then $M = A_Y - \mu_Y + \pi_X(A_Y) + \pi_Y(A_Y)$
(II5) If $x_0 \in (A_X, B_Y)$ and $x_0 \in [B_X, B_Y)$ then $M = B_X - \mu_Y + \pi_X(B_X) + \pi_Y(B_X)$.

Rewritten in a more compact form, this is the desired result.

Case (III) :

If $B_Y \le B_X$ one obtains

$$M(X,Y) = \int_{A_Y}^{B_X} dx - \int_{A_Y}^{B_X} \overline{G}(x)dx = B_X - A_Y - \pi_Y(A_Y) = B_X - \mu_Y + \pi_Y(B_X).$$

If $B_Y \ge B_X$ one obtains the same expression from

$$M(X,Y) = \int_{A_Y}^{B_X} dx - \int_{A_Y}^{B_X} \overline{G}(x)dx = B_X - A_Y - (\pi_Y(A_Y) - \pi_Y(B_X)) = B_X - \mu_Y + \pi_Y(B_X).$$

By symmetry one obtains similarly $M(Y,X) = B_Y - \mu_X + \pi_X(B_Y)$. Now, use that the function $x + \pi_Y(x)$ is monotone increasing. If $B_X \le B_Y$ then $M(X,Y) \le B_Y - \mu_Y + \pi_Y(B_Y) = B_Y - \mu_Y = \mu_X - \mu_Y + M(Y,X)$, hence $M = B_Y - \mu_Y$. Similarly, if $B_X \ge B_Y$ then $M = B_X - \mu_Y$. Together this shows that $M = \overline{B} - \mu_Y$ as desired. The proof is complete. ◊

5. A minimax property of the upper bound.

There is an alternative way to derive an upper bound for expected positive differences, which a priori does not depend on the Hoeffding-Fréchet extremal distribution. It is based on the following simple positive difference inequality.

Lemma 5.1. For all real numbers α one has

$$(X-Y)_+ \le (X-\alpha)_+ + (\alpha - Y)_+ = \alpha - Y + (X-\alpha)_+ + (Y-\alpha)_+ . \tag{5.1}$$

Proof. It suffices to consider the case $X \ge Y$. It is immediate to check that the inequality holds in all of the three possible subcases $X \ge Y \ge \alpha$, $X \ge \alpha \ge Y$, $\alpha \ge X \ge Y$. ◊

One observes that the application of Lemma 5.1 to the symmetry relation $(X-Y)_+ = X - Y + (Y-X)_+$ does not lead to a new inequality. Therefore an alternative upper bound for the expected positive difference is obtained from the minimization problem

$$E\left[(X-Y)_+\right] \le \min_\alpha \{\alpha - \mu_Y + \pi_X(\alpha) + \pi_Y(\alpha)\}. \tag{5.2}$$

It is remarkable that both upper bounds are identical.

Theorem 5.1. (*Minimax property of the combined Hoeffding-Fréchet upper bound*) Let $(X,Y) \in BD(F,G)$ be a bivariate random variable with marginal supports $[A_X, B_X]$, $[A_Y, B_Y]$, and finite marginal means μ_X, μ_Y. Suppose the regularity assumption (RA) holds. Then the following property holds :

$$\begin{aligned}&\max\left\{ \max_{(X,Y)\in BD(F,G)} \{E[(X-Y)_+]\}, \mu_X - \mu_Y + \max_{(X,Y)\in BD(F,G)} \{E[(Y-X)_+]\}\right\} \\ &= \min_\alpha \{\alpha - \mu_Y + \pi_X(\alpha) + \pi_Y(\alpha)\}.\end{aligned} \tag{5.3}$$

Proof. Set $\varphi(\alpha) = \alpha - \mu_Y + \pi_X(\alpha) + \pi_Y(\alpha)$. For $\alpha < \underline{A}$ one has $\varphi(\alpha) = \mu_X - \alpha > \varphi(\underline{A})$, and for $\alpha > \overline{B}$ one has $\varphi(\alpha) = \alpha - \mu_Y > \varphi(\overline{B})$. Therefore it suffices to consider the minimum over the interval $[\underline{A},\overline{B}]$. The result depends upon the sign change of $\varphi'(\alpha)=1-\overline{F}(\alpha)-\overline{G}(\alpha)$.

Case (I) : $\overline{F}(x)+\overline{G}(x)\le 1$ for all $x\in(\underline{A},\overline{B})$

Since $\varphi(\alpha)$ is increasing on $(\underline{A},\overline{B})$, the minimum of $\varphi(\alpha)$ is attained at $\alpha=\underline{A}$.

Case (II) : There exists a unique $x_0\in(\underline{A},\overline{B})$ such that $\overline{F}(x)+\overline{G}(x)\ge 1$ for $x\le x_0$ and $\overline{F}(x)+\overline{G}(x)\le 1$ for $x\ge x_0$

We distinguish between three subcases.

(IIa) $x_0\in(\underline{A},\overline{A}]$

Since $\varphi(\alpha)$ is increasing on $(\overline{A},\overline{B})$, the minimum of $\varphi(\alpha)$ is attained over $[\underline{A},\overline{A}]$. One has for $\alpha\in[\underline{A},\overline{A}]$:

$$\varphi(\alpha)=\begin{cases}\pi_X(\alpha), & \text{if } \underline{A}=A_X,\\ \mu_X-\mu_Y+\pi_Y(\alpha), & \text{if } \underline{A}=A_Y.\end{cases}$$

In each case $\varphi(\alpha)$ is decreasing on $[\underline{A},\overline{A}]$, and the minimum is attained at $\alpha=\overline{A}$.

(IIb) $x_0\in[\underline{B},\overline{B})$

Since $\varphi(\alpha)$ is decreasing on $(\underline{A},\underline{B})$, the minimum of $\varphi(\alpha)$ is attained over $[\underline{B},\overline{B}]$. One has for $\alpha\in[\underline{B},\overline{B}]$:

$$\varphi(\alpha)=\begin{cases}\alpha-\mu_Y+\pi_Y(\alpha), & \text{if } \underline{B}=B_X,\\ \alpha-\mu_Y+\pi_X(\alpha), & \text{if } \underline{B}=B_Y.\end{cases}$$

In each case $\varphi(\alpha)$ is increasing on $[\underline{B},\overline{B}]$, and the minimum of $\varphi(\alpha)$ is attained at $\alpha=\underline{B}$.

(IIc) $x_0\in(\overline{A},\underline{B})$

By (IIa) the minimum over $[\underline{A},\overline{A}]$ is $\varphi(\overline{A})$, and by (IIb) it is $\varphi(\underline{B})$ over $[\underline{B},\overline{B}]$. Since $\varphi(\alpha)$ is decreasing on $(\overline{A},x_0)$ and increasing on $(x_0,\overline{B})$, the minimum is attained at $\alpha=x_0$.

Case (III) : $\overline{F}(x)+\overline{G}(x)\ge 1$ for all $x\in(\underline{A},\overline{B})$

Since $\varphi(\alpha)$ is decreasing on $(\underline{A},\overline{B})$, the minimum of $\varphi(\alpha)$ is attained at $\alpha=\overline{B}$.

In all cases, the minimum coincides with the corresponding maximum in Theorem 4.1. ◊

In the following two often encountered special cases, the determination of the upper bound simplifies considerably.

Example 5.1 : X, Y defined on $(-\infty,\infty)$ satisfying (RA)

One obtains that $\max\limits_{(X,Y)\in BD(F,G)}\{E[(X-Y)_+]\} = x_0 - \mu_Y + \pi_X(x_0) + \pi_Y(x_0)$, where x_0 is the unique solution of the equation $\overline{F}(x) + \overline{G}(x) = 1$.

Example 5.2 : X, Y defined on $[0,\infty)$ satisfying (RA)

One obtains that

$$\max_{(X,Y)\in BD(F,G)}\{E[(X-Y)_+]\} = \begin{cases} \mu_X, & \text{if } \overline{F}(x)+\overline{G}(x)\le 1 \quad \text{for} \quad x\ge 0 \\ x_0-\mu_Y+\pi_X(x_0)+\pi_Y(x_0), & \text{otherwise}, \end{cases}$$

where $x_0 \in (0,\infty)$ is the unique solution of the equation $\overline{F}(x) + \overline{G}(x) = 1$.

6. The upper bound by given ranges, means and variances of the marginals.

The best bound for the expected positive value of a random variable X, namely $E[X_+] = \frac{1}{2}\left(\sqrt{\sigma^2+\mu^2}+\mu\right)$, by given double-sided infinite range $(-\infty,\infty)$, mean μ and standard deviation σ, has been obtained by Bowers(1969). Its extension to an arbitrary range $[A,B]$, $-\infty \le A < B \le \infty$, has been first obtained by De Vylder and Goovaerts(1982) (see also Goovaerts et al.(1984), Jansen et al.(1986)). The best bound for the expected positive difference $E[(X-Y)_+] = \frac{1}{2}\left(\sqrt{\sigma^2+\mu^2}+\mu\right), \mu = \mu_X - \mu_Y, \sigma = \sigma_X + \sigma_Y$, by given marginal ranges $(-\infty,\infty)$, means μ_X, μ_Y and standard deviations σ_X, σ_Y of X, Y, is a consequence of Theorem 2 in Hürlimann(1993c). Its non-trivial extension to arbitrary marginal ranges is our main Theorem 6.1, which is a distribution-free version of the combined Hoeffding-Fréchet upper bound for expected positive differences presented in Theorems 4.1 and 5.1. The considerable simplification of the general result (solution to an extremal moment problem of so-called Hausdorff type) for the ranges $[0,\infty)$ (Stieltjes type) and $(-\infty,\infty)$ (Hamburger type) is formulated in Table 6.2 and Theorem 6.2. In the latter situation, sharpness of the upper bound is described in details.

Let $D_X = D([A_X,B_X];\mu_X,\sigma_X)$, $D_Y = D([A_Y,B_Y];\mu_Y,\sigma_Y)$ be the sets of all random variables X, Y with ranges $[A_X,B_X]$, $[A_Y,B_Y]$, finite means μ_X, μ_Y, and standard deviations σ_X, σ_Y. The coefficients of variation are denoted by $k_X = \frac{\sigma_X}{\mu_X}$, $k_Y = \frac{\sigma_Y}{\mu_Y}$. The set of all bivariate pairs (X,Y) such that $X \in D_X, Y \in D_Y$ is denoted by $BD = BD([A_X,B_X]x[A_Y,B_Y];\mu_X,\sigma_X,\mu_Y,\sigma_Y)$. The maximal stop-loss transforms over D_X, D_Y are denoted by $\pi_X^*(\alpha) = \max\limits_{X\in D_X}\{\pi_X(\alpha)\}$, $\pi_Y^*(\alpha) = \max\limits_{Y\in D_Y}\{\pi_Y(\alpha)\}$, α an arbitrary real number. Let X^*, Y^* be the stop-loss ordered maximal random variables such that $\pi_{X^*}(\alpha) = \pi_X^*(\alpha)$, $\pi_{Y^*}(\alpha) = \pi_Y^*(\alpha)$ for all α. Their survival functions are obtained from the derivatives of the maximal stop-loss transforms as $\overline{F}^*(x) = -\frac{d}{dx}\pi_X^*(x)$, $\overline{G}^*(x) = -\frac{d}{dx}\pi_Y^*(x)$. Setting $\varphi(\alpha; X, Y) = \alpha - \mu_Y + \pi_X(\alpha) + \pi_Y(\alpha)$, it follows that

(6.1) $$\varphi(\alpha; X, Y) \le \varphi(\alpha; X^*, Y^*) \text{ uniformly for all } \alpha, \text{ all } X \in D_X, Y \in D_Y.$$

This uniform property implies that

(6.2) $$\min_\alpha \varphi(\alpha; X, Y) \le \min_\alpha \varphi(\alpha; X^*, Y^*) \text{ uniformly for all } X \in D_X, Y \in D_Y.$$

By the minimax Theorem 5.1, the following distribution-free upper bound has been found :

(6.3) $$\begin{aligned} &\max_{(X,Y)\in BD}\left\{E\left[(X-Y)_+\right]\right\} \\ &\le M^* := \max\left\{\max_{(X,Y)\in BD(F^*,G^*)}\left\{E\left[(X-Y)_+\right]\right\}, \mu_X - \mu_Y + \max_{(X,Y)\in BD(F^*,G^*)}\left\{E\left[(Y-X)_+\right]\right\}\right\} \\ &= \min_\alpha\left\{\alpha - \mu_Y + \pi_X^*(\alpha) + \pi_Y^*(\alpha)\right\} \end{aligned}$$

Once the right-hand side has been determined, it remains, in order to obtain possibly a *best* upper bound, to analyze under which conditions the equality is attained. By construction, one knows that it is attained for the Hoeffding-Fréchet extremal survival function $\overline{H}_*^*(x,y) = \min\left\{\overline{F}^*(x) + \overline{G}^*(y), 1\right\}$ associated to $(X^*, Y^*) \in BD(F^*, G^*)$. However, since the standard deviations of X^*, Y^* are greater than σ_X, σ_Y, this does not guarantee the upper bound is attained for some $(X,Y) \in BD$. One knows that in some cases the upper bound is actually attained by a bivariate diatomic random variable, as in Theorem 6.2.

To state the main result, some simplifying notations will be convenient. Since the derivation is done in terms of standard random variables, one sets

$$a_X = \frac{A_X - \mu_X}{\sigma_X},\ b_X = \frac{B_X - \mu_X}{\sigma_X},\ a_Y = \frac{A_Y - \mu_Y}{\sigma_Y},\ b_Y = \frac{B_Y - \mu_Y}{\sigma_Y}.$$

The negative inverse of a non-zero number x is denoted by $\overline{x} = -x^{-1}$, which defines an involution mapping whose square is, by definition, the identity. A further notation is

$$\lambda_{X,Y} = \frac{\mu_X - \mu_Y}{\sigma_X + \sigma_Y}.$$

Theorem 6.1. (*Distribution-free Hoeffding-Fréchet combined upper bound for expected positive differences*)

Let $(X,Y) \in BD = BD([A_X, B_X]x[A_Y, B_Y]; \mu_X, \sigma_X, \mu_Y, \sigma_Y)$ be a bivariate pair of random variables with the given marginal ranges, means, and standard deviations. Then the distribution-free upper bound M^* in (6.3) for the expected positive difference $E\left[(X-Y)_+\right]$, $(X,Y) \in BD$, is determined by Table 6.1.

Table 6.1 : Distribution-free upper bound for expected positive differences by given ranges, means and variances of the marginals

case	conditions	parameter α_0^i	upper bound
(I)	$a_X a_Y \le 1,\ b_X b_Y \ge 1$		$\mu_X - \underline{A}$
(III)	$a_X a_Y \ge 1,\ b_X b_Y \le 1$		$\overline{B} - \mu_Y$
(II)	$a_X a_Y \ge 1,\ b_X b_Y \ge 1$		
(IIc)	$\alpha_o^i \in (\overline{A}, \underline{B})$		
(1)	$a_X + \overline{a}_X + b_Y + \overline{b}_Y \ge 0$ $\lambda_{X,Y} \le \frac{1}{2}(a_X + \overline{a}_X)$	$\alpha_0^1 =$ $\mu_Y - \frac{1}{2}\sigma_Y(a_X + \overline{a}_X)$	$\dfrac{\sigma_X + \sigma_Y - a_X(\mu_X - \mu_Y)}{\overline{a}_X - a_X}$
(2)	$a_Y + \overline{a}_Y + b_X + \overline{b}_X \ge 0$ $\lambda_{X,Y} \le \frac{1}{2}(a_Y + \overline{a}_Y)$	$\alpha_0^2 =$ $\mu_X - \frac{1}{2}\sigma_X(a_Y + \overline{a}_Y)$	$\dfrac{\sigma_X + \sigma_Y + \overline{a}_Y(\mu_X - \mu_Y)}{\overline{a}_Y - a_Y}$
(3)	$\frac{1}{2}(a_X + \overline{a}_X) \le -\lambda_{X,Y} \le \frac{1}{2}(b_X + \overline{b}_X)$ $\frac{1}{2}(a_Y + \overline{a}_Y) \le \lambda_{X,Y} \le \frac{1}{2}(b_Y + \overline{b}_Y)$	$\alpha_0^3 = \dfrac{\mu_X\sigma_Y + \mu_Y\sigma_X}{\sigma_X + \sigma_Y}$	$\frac{1}{2}\left\{\sqrt{(\sigma_X + \sigma_Y)^2 + (\mu_X - \mu_Y)^2} + (\mu_X - \mu_Y)\right\}$
(4)	$a_X + \overline{a}_X + b_Y + \overline{b}_Y \le 0$ $\lambda_{X,Y} \ge \frac{1}{2}(b_Y + \overline{b}_Y)$	$\alpha_0^4 =$ $\mu_X - \frac{1}{2}\sigma_X(b_Y + \overline{b}_Y)$	$\dfrac{\sigma_X + \sigma_Y + b_Y(\mu_X - \mu_Y)}{b_Y - \overline{b}_Y}$
(5)	$a_Y + \overline{a}_Y + b_X + \overline{b}_X \le 0$ $\lambda_{X,Y} \ge \frac{1}{2}(b_X + \overline{b}_X)$	$\alpha_0^5 =$ $\mu_Y - \frac{1}{2}\sigma_Y(b_X + \overline{b}_X)$	$\dfrac{\sigma_X + \sigma_Y - \overline{b}_X(\mu_X - \mu_Y)}{b_X - \overline{b}_X}$
(IIa)	$\alpha_o^i \in (\underline{A}, \overline{A}]$		$\overline{A} - \mu_Y + \pi_X^*(\overline{A}) + \pi_Y^*(\overline{A})$
(IIb)	$\alpha_o^i \in [\underline{B}, \overline{B})$		$\underline{B} - \mu_Y + \pi_X^*(\underline{B}) + \pi_Y^*(\underline{B})$

Proof. To obtain M^* one simplifies calculation by *reduction* to the case of stop-loss ordered *standard* maxima $Z(X^*) = \dfrac{X^* - \mu_X}{\sigma_X}$, $Z(Y^*) = \dfrac{Y^* - \mu_Y}{\sigma_Y}$ with distributions F(x), G(x), and ranges $[a_X, b_X], [a_Y, b_Y]$. Then one has the relation

$$\varphi(\alpha; X^*, Y^*) = \sigma_X \cdot \pi_{Z(X^*)}\left(\frac{\alpha - \mu_X}{\sigma_X}\right) + \sigma_Y \cdot \left\{\frac{\alpha - \mu_Y}{\sigma_Y} + \pi_{Z(Y^*)}\left(\frac{\alpha - \mu_Y}{\sigma_Y}\right)\right\}. \tag{6.4}$$

As seen in Section 5, the value of M^* depends upon the sign change of

$$\overline{F}^*(\alpha) + \overline{G}^*(\alpha) - 1 = \overline{F}\left(\frac{\alpha - \mu_X}{\sigma_X}\right) + \overline{G}\left(\frac{\alpha - \mu_Y}{\sigma_Y}\right) - 1.$$

For mathematical convenience set $\alpha_X = \dfrac{\alpha - \mu_X}{\sigma_X}$, $\alpha_Y = \dfrac{\alpha - \mu_Y}{\sigma_Y}$, where both quantities are related by $\mu_Y - \mu_X = \alpha_X\sigma_X - \alpha_Y\sigma_Y$. From Table IV.2.2 one obtains that $\overline{F}(x)$ consists of five pieces described as follows :

(6.5)
$$\overline{F}(x) = \begin{cases} \overline{F}_0(x) = 1, \quad x < a_X, \\ \overline{F}_1(x) = \dfrac{a_X^2}{1+a_X^2}, \quad a_X \le x \le \frac{1}{2}(a_X + \overline{a}_X), \\ \overline{F}_2(x) = \dfrac{\psi(x)^2}{1+\psi(x)^2}, \quad \frac{1}{2}(a_X + \overline{a}_X) \le x \le \frac{1}{2}(b_X + \overline{b}_X), \\ \overline{F}_3(x) = \dfrac{1}{1+b_X^2}, \quad \frac{1}{2}(b_X + \overline{b}_X) \le x < b_X, \\ \overline{F}_4(x) = 0, \quad x \ge b_X. \end{cases}$$

In this formula the function $\psi(x) = x - \sqrt{1+x^2} < 0$ is the inverse of the function $\omega(x) = \frac{1}{2}(x + \overline{x}), \overline{x} = -x^{-1}$. The distribution $\overline{G}(x)$ is defined similarly. We distinguish between several cases as in the proof of Theorem 5.1.

<u>Case (I)</u> : $\overline{F}^*(\alpha) + \overline{G}^*(\alpha) \le 1$ for all $\alpha \in (\underline{A}, \overline{B})$

This occurs when $\overline{F}_1(\alpha_X) + \overline{G}_1(\alpha_Y) \le 1$, that is $a_X a_Y \le 1$, and implies automatically $b_X b_Y \ge 1$. The upper bound is $M^* = \mu_X - \underline{A}$.

<u>Case (III)</u> : $\overline{F}^*(\alpha) + \overline{G}^*(\alpha) \ge 1$ for all $\alpha \in (\underline{A}, \overline{B})$

This occurs when $\overline{F}_3(\alpha_X) + \overline{G}_3(\alpha_Y) \ge 1$, that is $b_X b_Y \le 1$, and implies automatically $a_X a_Y \ge 1$. The upper bound is $M^* = \overline{B} - \mu_Y$.

<u>Case (II)</u> : there exists a unique $\alpha_0 \in (\underline{A}, \overline{B})$ such that $\overline{F}^*(\alpha) + \overline{G}^*(\alpha) \ge 1$ for $\alpha \le \alpha_0$ and $\overline{F}^*(\alpha) + \overline{G}^*(\alpha) \le 1$ for $\alpha \ge \alpha_0$

This can only occur provided $a_X a_Y \ge 1$ and $b_X b_Y \ge 1$. The equation $\overline{F}(\alpha_X) + \overline{G}(\alpha_Y) = 1$ consists of five pieces $\overline{F}_i(\alpha_X) + \overline{G}_j(\alpha_Y) = 1$, i,j=1,2,3, leading to five subcases.

(1) $\overline{F}_1(\alpha_X) + \overline{G}_2(\alpha_Y) = 1 \iff \psi(\alpha_Y) = -\overline{a}_X$

One has $\alpha_Y = (\omega \circ \psi)(\alpha_Y) = \omega(-\overline{a}_X) = -\frac{1}{2}(a_X + \overline{a}_X)$. Furthermore the constraints

$$\frac{1}{2}(a_Y + \overline{a}_Y) \le \alpha_Y \le \frac{1}{2}(b_Y + \overline{b}_Y),$$
$$a_X \le \alpha_X = \frac{\mu_Y - \mu_X}{\sigma_X} + \frac{\sigma_Y}{\sigma_X}\alpha_Y \le \frac{1}{2}(a_X + \overline{a}_X)$$

are equivalent to the conditions

$$a_X + \overline{a}_X + a_Y + \overline{a}_Y \le 0 \le a_X + \overline{a}_X + b_Y + \overline{b}_Y,$$
$$\lambda_{X,Y} \le \frac{1}{2}(a_X + \overline{a}_X),$$

from which the first inequality is always fulfilled because $a_X a_Y \ge 1$. The value of M^* depends on $\alpha_0 = \mu_X + \alpha_X \sigma_X = \mu_Y + \alpha_Y \sigma_Y$ as in Theorem 4.1, leading to three further subcases :

(1a) $\alpha_0 \in \left(\underline{A}, \overline{A}\right]$ $M^* = \overline{A} - \mu_Y + \pi_X^*(\overline{A}) + \pi_Y^*(\overline{A})$

(1b) $\alpha_0 \in \left[\underline{B}, \overline{B}\right)$ $M^* = \underline{B} - \mu_Y + \pi_X^*(\underline{B}) + \pi_Y^*(\underline{B})$

(1c) $\alpha_0 \in \left(\overline{A}, \underline{B}\right)$:

Applying the reduction step above and Table II.5.1 for the maximal stop-loss transform, one obtains after some straightforward algebra :

$$M^* = \varphi(\alpha_0, X^*, Y^*) = \sigma_X(-a_X)\frac{1 + a_X\alpha_X}{1 + a_X^2} + \frac{1}{2}\sigma_Y \cdot (\alpha_Y + \sqrt{1+\alpha_Y^2})$$
$$= \frac{\sigma_X + \sigma_Y - a_X(\mu_X - \mu_Y)}{\overline{a}_X - a_X}.$$

(2) $\overline{F}_2(\alpha_X) + \overline{G}_1(\alpha_Y) = 1 \quad \Leftrightarrow \quad \psi(\alpha_X) = -\overline{a}_Y$

By symmetry to the subcase (1) one gets $\alpha_X = -\frac{1}{2}(a_Y + \overline{a}_Y)$, and the constraints are equivalent to the conditions

$$a_X + \overline{a}_X + a_Y + \overline{a}_Y \le 0 \le a_Y + \overline{a}_Y + b_X + \overline{b}_X, \quad \lambda_{X,Y} \le \tfrac{1}{2}(a_Y + \overline{a}_Y),$$

from which the first inequality is always fulfilled because $a_X a_Y \ge 1$. The subcases (2a), (2b) are the same as (1a), (1b). For (2c) one has, by the symmetry relation $(X-Y)_+ = X - Y + (Y-X)_+$, and using subcase (1c), that

$$M^* = \mu_X - \mu_Y + \frac{\sigma_X + \sigma_Y - a_Y(\mu_X - \mu_Y)}{\overline{a}_Y - a_Y} = \frac{\sigma_X + \sigma_Y + \overline{a}_Y(\mu_X - \mu_Y)}{\overline{a}_Y - a_Y}.$$

(3) $\overline{F}_2(\alpha_X) + \overline{G}_2(\alpha_Y) = 1 \quad \Leftrightarrow \quad \psi(\alpha_X)\psi(\alpha_Y) = 1$

Using that $\omega\left(\frac{1}{\psi(x)}\right) = -x$ one has $\alpha_X + \alpha_Y = 0$, hence $\alpha_X = -\lambda_{X,Y}$. The conditions under which (3) holds are

$$\tfrac{1}{2}(a_X + \overline{a}_X) \le -\lambda_{X,Y} \le \tfrac{1}{2}(b_X + \overline{b}_X),$$
$$\tfrac{1}{2}(a_Y + \overline{a}_Y) \le \lambda_{X,Y} \le \tfrac{1}{2}(b_Y + \overline{b}_Y).$$

The subcases (3a), (3b) are the same as (1a), (1b). For (3c) a calculation yields

$$M^* = \frac{1}{2}\sigma_X \cdot (\sqrt{1+\alpha_X^2} - \alpha_X) + \frac{1}{2}\sigma_Y \cdot (\sqrt{1+\alpha_Y^2} + \alpha_Y)$$
$$= \frac{1}{2} \cdot \left\{\sqrt{(\sigma_X + \sigma_Y)^2 + (\mu_X - \mu_Y)^2} + (\mu_X - \mu_Y)\right\}$$

(4) $\overline{F}_2(\alpha_X) + \overline{G}_3(\alpha_Y) = 1 \quad \Leftrightarrow \quad \psi(\alpha_Y) = -b_Y$

One has $\alpha_X = (\omega \circ \psi)(\alpha_X) = \omega(-b_Y) = -\frac{1}{2}(b_Y + \overline{b}_Y)$. Furthermore the constraints

$$\tfrac{1}{2}(a_X + \bar{a}_X) \le \alpha_X \le \tfrac{1}{2}(b_X + \bar{b}_X),$$
$$\tfrac{1}{2}(b_Y + \bar{b}_Y) \le \alpha_Y \le b_Y$$

are equivalent to the conditions

$$a_X + \bar{a}_X + b_Y + \bar{b}_Y \le 0 \le b_X + \bar{b}_X + b_Y + \bar{b}_Y,$$
$$\lambda_{X,Y} \ge \tfrac{1}{2}(b_Y + \bar{b}_Y),$$

from which the second inequality is always fulfilled because $b_X b_Y \ge 1$. The subcases (4a), (4b) are the same as (1a), (1b). For (4c) one obtains after some calculation

$$M^* = \frac{1}{2}\sigma_X \cdot (\sqrt{1+\alpha_X^2} - \alpha_X) + \sigma_Y \cdot (\alpha_Y + \frac{b_Y - \alpha_Y}{1+b_Y^2})$$
$$= \frac{\sigma_X + \sigma_Y - b_Y(\mu_X - \mu_Y)}{b_Y - \bar{b}_Y}.$$

(5) $\quad \bar{F}_3(\alpha_X) + \bar{G}_2(\alpha_Y) = 1 \quad \Leftrightarrow \quad \psi(\alpha_Y) = -b_X$

By symmetry to the subcase (4) one gets $\alpha_Y = -\tfrac{1}{2}(b_X + \bar{b}_X)$, and the constraints are equivalent to the conditions

$$a_Y + \bar{a}_Y + b_X + \bar{b}_X \le 0 \le b_X + \bar{b}_X + b_Y + \bar{b}_Y, \quad \lambda_{X,Y} \ge \tfrac{1}{2}(b_X + \bar{b}_X),$$

from which the second inequality is always fulfilled because $b_X b_Y \ge 1$. The subcases (5a), (5b) are the same as (1a), (1b). For (5c) one obtains by symmetry from case (4) :

$$M^* = \mu_X - \mu_Y + \frac{\sigma_X + \sigma_Y + b_X(\mu_X - \mu_Y)}{b_X - \bar{b}_X} = \frac{\sigma_X + \sigma_Y - \bar{b}_X(\mu_X - \mu_Y)}{b_X - \bar{b}_X}. \ \Diamond$$

In Table 6.1, if $A_X = A_Y$ and $B_X = B_Y$, then the cases (b) and (c) do not occur. Further simplifications take place for the important special cases $A_X = A_Y = 0$, $B_X = B_Y = \infty$ and $A_X = A_Y = -\infty$, $B_X = B_Y = \infty$. The results are reported in Table 6.2 and Theorem 6.2.

Theorem 6.2. Let $(X,Y) \in BD = BD((-\infty,\infty)x(-\infty,\infty); \mu_X, \sigma_X, \mu_Y, \sigma_Y)$ be a bivariate pair of random variables with the given marginal ranges, means, and standard deviations. Then the maximum expected positive difference is determined in three alternative ways as follows :

$$\max_{(X,Y)\in BD} \{E[(X-Y)_+]\} = \max_{(X,Y)\in BD(F^*,G^*)} \{E[(X-Y)_+]\}$$
$$= \min_{\alpha}\{\alpha - \mu_Y + \pi_X^*(\alpha) + \pi_Y^*(\alpha)\} = \frac{1}{2}\left\{\sqrt{(\sigma_X + \sigma_Y)^2 + (\mu_X - \mu_Y)^2} + (\mu_X - \mu_Y)\right\}$$

Moreover, the *first* maximum is attained for a bivariate diatomic random variable with support $\{x_1, x_2\}x\{y_1, y_2\} = \{\mu_X - \sigma_X z, \mu_X - \sigma_X \bar{z}\}x\{\mu_Y - \sigma_Y \bar{z}, \mu_Y + \sigma_Y z\}$ and joint probabilities determined by the 2x2-contingency table

	y_1	y_2
x_1	0	$\frac{1}{1+z^2}$
x_2	$\frac{z^2}{1+z^2}$	0

where $z=\gamma+\sqrt{1+\gamma^2}$, $\gamma=\frac{\mu_X-\mu_Y}{\sigma_X+\sigma_Y}$. The *second* maximum is attained by the Hoeffding-Fréchet lower bound bivariate distribution with joint survival function

$$\overline{H}_*^*(x,y)=\min\left\{\overline{F}^*(x)+\overline{G}^*(y),1\right\},$$

where the stop-loss ordered maximal marginals are given by

$$\overline{F}^*(x)=\frac{1}{2}\left\{1-\frac{x-\mu_X}{\sqrt{\sigma_X^2+(x-\mu_X)^2}}\right\},\overline{G}^*(x)=\frac{1}{2}\left\{1-\frac{x-\mu_Y}{\sqrt{\sigma_Y^2+(x-\mu_Y)^2}}\right\}.$$

Finally, the *third* minimum-maximum is attained at $\alpha_0=\frac{\mu_X\sigma_Y+\mu_Y\sigma_X}{\sigma_X+\sigma_Y}$.

Table 6.2 : Distribution-free upper bound for expected positive differences by given ranges $[0,\infty)$, means and variances of the marginals

case	conditions	Hoeffding-Fréchet upper bound
(I)	$k_Xk_Y\ge 1$	μ_X
(II)	$k_Xk_Y\le 1$	
(1)	$\frac{\mu_Y-\mu_X}{\sigma_X+\sigma_Y}\le\frac{1}{2}(\frac{k_X^2-1}{k_X})$	$\mu_X-\left(\frac{1-k_Xk_Y}{1+k_X^2}\right)\mu_Y$
(2)	$\frac{\mu_X-\mu_Y}{\sigma_X+\sigma_Y}\le\frac{1}{2}(\frac{k_Y^2-1}{k_Y})$	$\mu_X-\left(\frac{1-k_Xk_Y}{1+k_Y^2}\right)\mu_X$
(3)	$\frac{\mu_Y-\mu_X}{\sigma_X+\sigma_Y}\ge\frac{1}{2}(\frac{k_X^2-1}{k_X})$ $\frac{\mu_X-\mu_Y}{\sigma_X+\sigma_Y}\ge\frac{1}{2}(\frac{k_Y^2-1}{k_Y})$	$\frac{1}{2}\cdot\left\{\sqrt{(k_X\mu_X+k_Y\mu_Y)^2+(\mu_X-\mu_Y)^2}+(\mu_X-\mu_Y)\right\}$

Proof of Theorem 6.2. The last two representations follow from the minimax Theorem 5.1 and Table 6.1 by observing that in case of infinite ranges $(-\infty,\infty)$ only the subcase (3) of case (II) can occur. The fact that the first equality holds for the displayed bivariate diatomic random variable can be checked without difficulty. Alternatively, one may invoke that the first maximum is equivalent with the bivariate version of the inequality of Bowers(1969) obtained by replacing Y by D–Y, D an arbitrary constant, for which the result is in Hürlimann(1993c), Theorem 2. ◊

7. Notes.

A derivation of the Hoeffding-Fréchet extremal distributions can be found in many statistical textbooks, for example Mardia(1970), p. 31. A generalization of the Fréchet bounds, together with the verification of the sharpness of these bounds, is found in Rüschendorf(1981). Of some interest are also the bounds by Smith(1983) for the situation of stochastically ordered marginals.

Bivariate and multivariate versions of Chebyshev type inequalities are numerous in the statistical literature. Good sources of material are in Karlin and Studden(1966) and valuable references are in Godwin(1955). For further results consult the papers by Pearson(1919), Leser(1942), Lal(1955), Olkin and Pratt(1958), Whittle(1958a/b), Marshall and Olkin(1960a/b), Mudholkar and Rao(1967) and Arharov(1971).

Section 3 is also part of the paper Hürlimann(1998a) while Section 4 to 6 closely follows Hürlimann(1997j). Theorems 3.1 and 6.2 provide further proofs of the bivariate version of the inequality of Bowers(1969) given in Hürlimann(1993c). Unfortunately, the problem of finding a best upper stop-loss bound by fixed positive correlation coefficient remains unsolved (possibly a solution does not exist at all), a question raised by Gerber at the XXII-th ASTIN Colloquium in Montreux, 1990 (comment after Theorem 2 in Hürlimann(1993c)).

Theorem 5.1 is a so-called "separation" property. For a general separation theorem one may consult Rüschendorf(1981). Separation is also exploited to derive bounds for the expected maximum of linear combinations of random variables, as shown by Meilijson and Nàdas(1979) and Meilijson(1991).

CHAPTER VI

APPLICATIONS IN ACTUARIAL SCIENCE

1. The impact of skewness and kurtosis on actuarial calculations.

In view of the quite advanced mathematics and the numerous calculations required to clarify the analytical structure of the Chebyshev-Markov extremal distributions, it is of primordial importance to demonstrate the necessity of considering higher moments at some significant applications. We illustrate this fact at the stable pricing method, which consists to set prices for a risky line of business by fixing in advance a small probability of loss.

1.1. Stable prices and solvability.

Let X be the claim size of a line of insurance business over some fixed period, usually called *risk* in Actuarial Science, which is described by a random variable with known moments up to the n-th order, n=2,3,4,..., that is $X \in D_n$. By assumption, at least the mean μ and standard deviation σ are known. Let $P = H[X]$ be the *price* of the risk, where $H[\cdot]$ is a real probability functional from D_n to R_+. The possible *loss* over the fixed period is described by the random variable $L = X - P$. It is often reasonable to set prices according to the *stability criterion* $\Pr(L > 0) = \overline{F}_X(P) \le \varepsilon$, where ε is a small prescribed positive number. It says that the *probability of insolvability* in a long position of this line of business is less than ε. In a situation of *incomplete information* like $X \in D_n$, stability is achieved provided

$$\max_{X \in D_n} \left\{ \overline{F}_X(P) \right\} = \varepsilon \,, \tag{1.1}$$

which can be called *distribution-free stability criterion.* The solution P to (1.1) will be called *stable price*. If $F_\ell^{(n)}(x)$ denotes the standard Chebyshev-Markov minimal distribution, then (1.1) says that the stable price is set according to a so-called *standard deviation principle* $P = H_\varepsilon^{(n)}[X] = \mu + \theta_\varepsilon^{(n)} \cdot \sigma$, where the *loading factor* $\theta_\varepsilon^{(n)}$ is the ε-percentile obtained from

$$\overline{F}_\ell^{(n)}(\theta_\varepsilon^{(n)}) = \varepsilon \,. \tag{1.2}$$

The dependence upon the portfolio size is very simple. For a portfolio of N independent and identically distributed risks X_i with aggregate portfolio risk $X = \Sigma\, X_i$, the loading factor can be reduced by a factor of $\left(\sqrt{N}\right)^{-1}$. Indeed, let $P_i = \mu + \theta_{\varepsilon,i}^{(n)} \cdot \sigma$ be the stable price of an individual risk X_i with mean μ and standard deviation σ, where $\theta_{\varepsilon,i}^{(n)}$ is an *individual loading factor*. Then the portfolio risk $X = \Sigma\, X_i$ has mean $N \cdot \mu$ and standard deviation $\sqrt{N} \cdot \sigma$. Let $P = N \cdot \mu + \theta_\varepsilon^{(n)} \cdot \sqrt{N} \cdot \sigma$ be the portfolio stable price, where the *portfolio loading factor* $\theta_\varepsilon^{(n)}$ is determined by (1.2). Since $P = \Sigma\, P_i$ one must have $\theta_{\varepsilon,i}^{(n)} = \left(\sqrt{N}\right)^{-1} \cdot \theta_\varepsilon^{(n)}$, which determines the individual loading factor in terms of the portfolio loading factor. In view of the general inequalities $F_{\ell}^{(n+1)}(x) \le F_{\ell}^{(n)}(x)$, n=2,3,4,..., stated in Theorem IV.2.1, the loading factor $\theta_{\varepsilon}^{(n)}$ is a decreasing function of n, that is the stable price decreases when more and more information is available about the risk. Since $F_{\ell}^{(n)}(x)$ has been completely described for n=2,3,4 in Section III.4, the corresponding stable prices can be determined, at least numerically, using a computer algebra system.

To illustrate analytically the impact of skewness and kurtosis on solvability, we will consider the risk associated to daily returns in financial markets, which often turns out to be adequately represented by a symmetric random variable (cf. Taylor(1992), Section 2.8). For simplicity, we assume a double-sided infinite range $(-\infty,\infty)$, and suppose the skewness $\gamma = 0$ and the kurtosis $\gamma_2 = \delta - 3$ are known. Relevant are the following formulas :

$$\overline{F}_{\ell}^{(2)}(x) = \frac{1}{1+x^2}, x \ge 0, \tag{1.3}$$

$$\overline{F}_{\ell}^{(4)}(x) = \frac{\delta - 1}{(x^2-1)^2 + (\delta-1)\cdot(1+x^2)}, \quad x \ge 1. \tag{1.4}$$

The loading factors are obtained by solving quadratic equations. One obtains the solutions

$$\theta_{\varepsilon}^{(2)} = \sqrt{\frac{1-\varepsilon}{\varepsilon}}, \tag{1.5}$$

$$\theta_{\varepsilon}^{(4)}(\delta) = \frac{\sqrt{2}}{2}\sqrt{\sqrt{(\delta-3)^2 + 4\delta\left(\frac{1-\varepsilon}{\varepsilon}\right) - \frac{4}{\varepsilon}} - (\delta-3)}. \tag{1.6}$$

In the last formula one must assume $0 < \varepsilon \le 1 - \frac{1}{\delta}, \delta > 1, \theta_{\varepsilon}^{(4)}(\delta) \ge 1$. A similar formula for $0 < \theta_{\varepsilon}^{(4)}(\delta) \le 1$ can also be written down. For $\delta = 3$, which corresponds to a standard normal distribution, one has further

$$\theta_{\varepsilon}^{(4)}(\delta = 3) = \frac{\sqrt{2}}{2}\sqrt[4]{12\left(\frac{1-\varepsilon}{\varepsilon}\right) - \frac{4}{\varepsilon}}. \tag{1.7}$$

However, in financial markets one observes often $\delta \ge 6$. For this one has

$$\theta_{\varepsilon}^{(4)}(\delta = 6) = \frac{\sqrt{2}}{2}\sqrt{\sqrt{9 + 24\left(\frac{1-\varepsilon}{\varepsilon}\right) - \frac{4}{\varepsilon}} - 3}. \tag{1.8}$$

For the sake of comparison, if $\varepsilon = 0.01$ one has

$$\begin{aligned} &\theta_{\varepsilon}^{(2)} = \sqrt{99} = 9.95, \\ &\theta_{\varepsilon}^{(4)}(\delta = 3) = \frac{\sqrt{2}}{2}\sqrt[4]{788} = 3.75, \\ &\theta_{\varepsilon}^{(4)}(\delta = 6) = \frac{\sqrt{2}}{2}\sqrt{\sqrt{1985} - 3} = 4.56, \end{aligned} \tag{1.9}$$

and if $\varepsilon = 0.05$ one has

$$\theta_\varepsilon^{(2)} = \sqrt{19} = 4.36,$$

(1.10)
$$\theta_\varepsilon^{(4)}(\delta = 3) = \frac{\sqrt{2}}{2}\sqrt[4]{148} = 2.47,$$

$$\theta_\varepsilon^{(4)}(\delta = 6) = \frac{\sqrt{2}}{2}\sqrt{\sqrt{385} - 3} = 2.88.$$

Taking into account only the mean and variance, stable prices are too crude to be applicable. Furthermore, the kurtosis of a standard normal distribution underestimates stable prices in financial markets. This well-known empirical fact finds herewith a simple theoretical explanation. Though many theoretical pricing principles have been developed in recent years, the standard deviation principle remains, besides the expected value principle $H[X] = (1+\theta)\cdot E[X]$, of main practical importance. In view of this fact, the evaluation of stable prices based on the knowledge of the skewness and kurtosis of risks is a valuable and adequate method to determine the unknown loading factor of the standard deviation principle.

1.2. Stable stop-loss prices.

The stable price method, developed in Section 1.1, can be applied to transformed risks of a line of business. To illustrate, let us calculate the stable stop-loss prices of a stop-loss risk $X(d) = (X-d)_+$ with deductible d for a random variable $X \in D_n$ with known moments up to the n-th order, n=2,3,4. Denote by $\pi_\varepsilon^{(n)}(d)$ the stable stop-loss price. The possible loss over a fixed period is described by the random variable $L = X - d - \pi_\varepsilon^{(n)}(d)$. Then the distribution-free stability criterion

(1.11)
$$\max_{X \in D_n}\left\{\overline{F}_X(d + \pi_\varepsilon^{(n)}(d))\right\} = \varepsilon$$

implies a stable stop-loss price of amount

(1.12)
$$\pi_\varepsilon^{(n)}(d) = \theta_\varepsilon^{(n)} \cdot \sigma - (d - \mu).$$

Suppose now that stop-loss prices are set according to the standard deviation principle

(1.13)
$$H[X(d)] = \pi(d) + \theta \cdot \sigma(d),$$

where $\pi(d) = E[X(d)]$, $\sigma(d) = \sqrt{\mathrm{Var}[X(d)]}$ are the expected value and the standard deviation of the stop-loss risk. Comparing (1.12) and (1.13) the standard deviation loading must be equal to

(1.14)
$$\theta \cdot \sigma(d) = \theta_\varepsilon^{(n)} \cdot \sigma - (d - \mu) - \pi(d).$$

For a portfolio of N independent and identically distributed stop-loss risks, the individual loading factor can be reduced by a factor of $\left(\sqrt{N}\right)^{-1}$ as explained after formula (1.2). Therefore, it is reasonable to set stable stop-loss prices at the level

(1.15)
$$H_N[X(d)] = \pi(d) + l_N(d),$$

where the loading equals

(1.16) $$\ell_N(d)=\frac{1}{\sqrt{N}}\cdot\left(\theta_\varepsilon^{(n)}\cdot\sigma-(d-\mu)-\pi(d)\right).$$

A concrete numerical illustration follows for the important special case $d=\mu$, which is known to be the optimal deductible of the mean self-financing stop-loss strategy, which will be considered in Section 2.

For the sake of comparisons, we set $\theta_\varepsilon^{(n)}=5$, where now the probability of insolvability $\varepsilon=\varepsilon(n)$ depends on the amount of available information via n and decreases with increasing n. Furthermore, we suppose that $X\sim\ln N(\nu,\tau)$ is lognormally distributed with parameters ν,τ. Leaving the details of elementary calculation to the reader, one obtains the following formulas (N(x) denotes the standard normal distribution) :

(1.17) $$\pi(\mu)=\left\{2\cdot N(\tfrac{1}{2}\tau)-1\right\}\cdot\mu,$$

$$k^2=\left(\frac{\sigma}{\mu}\right)^2=\exp(\tau^2)-1, \text{ the squared coefficient of variation,}$$

(1.18) $$\gamma=k\cdot(3+k^2), \text{ the skewness,}$$
$$\gamma_2=\delta-3=k^2\cdot(16+15k^2+6k^4+k^6), \text{ the kurtosis.}$$

For example, if k=0.2 the loading equals

(1.19) $$\ell_N(\mu)=\frac{1}{\sqrt{N}}\cdot\left(5\cdot k-2\cdot N(\tfrac{1}{2}k)+1\right)\cdot\mu=\frac{0.92}{\sqrt{N}}\cdot\mu.$$

What is now a "realistic" probability of insolvability corresponding to the choice $\theta_\varepsilon^{(n)}=5$ in case $k=0.2, \gamma=0.608, \delta=3.664$ are the values obtained from (1.18) ? The further relevant quantities for evaluation of the Chebyshev-Markov bounds are by Sections I.5 and III.4 the standardized range $[a,\infty)=\left[-\frac{1}{k},\infty\right)=[-5,\infty)$, $\bar{a}=k=0.2$, $\bar{c}=\frac{1}{2}\left(\gamma+\sqrt{4+\gamma^2}\right)=1.349$, $\Delta=\delta-(\gamma^2+1)=2.294$, $q(a)=1+\gamma a-a^2=-27.04$, $C(a)=\gamma q(a)+\Delta a=-27.91$, $D(a)=\Delta+q(a)=-27.746$, and

(1.20) $$a^*=\frac{1}{2}\left\{\frac{C(a)-\sqrt{C(a)^2+4q(a)D(a)}}{q(a)}\right\}=1.603.$$

Then the Chebyshev-Markov probabilities of insolvability are given by

(1.21) $$\overline{F}_\ell^{(2)}(x=5)=\frac{1}{1+x^2}=0.0385, \quad x=5\geq\bar{a}=0.2,$$

(1.22) $$\overline{F}_\ell^{(3)}(x=5)=\frac{1+\gamma a-a^2}{(x-a)(2x-\gamma+(1+x^2)a)}=0.0224, \quad x=5\geq\bar{c}=1.349,$$

(1.23) $$\overline{F}_\ell^{(4)}(x=5)=\frac{\Delta}{(1+\gamma x-x^2)^2+\Delta(1+x^2)}=0.0046, \quad x=5\geq a^*=1.603.$$

On the other side, the probability of loss from a lognormally distributed random variable X with coefficient of variation k=0.2, that is volatility parameter τ=0.198, equals

$$(1.24)\quad \begin{aligned}\overline{F}_X(\mu+5\sigma)&=\overline{F}_X((1+5k)\cdot\mu)=\overline{N}\left(\frac{\ln\{\mu\}+\ln\{1+5k^2\}-\nu}{\tau}\right)\\&=\overline{N}\left(\tfrac{1}{2}\tau+\frac{\ln\{1+5k\}}{\tau}\right)=\overline{N}(3.6)=0.0002.\end{aligned}$$

Note that the value of the mean parameter μ must not be known (to evaluate loss probabilities). This fact reminds one of the similar property, characteristic of the evaluation of option prices, in the famous model by Black-Scholes(1973). However, there is a *model risk*. One will never be certain that the true distribution of the risk is actually a lognormal distribution. Our example suggests that stopping calculations by knowledge of the skewness and kurtosis yields a sufficiently low probability of insolvability for use in practical work.

2. Distribution-free prices for a mean self-financing portfolio insurance strategy.

Suppose the *claims* of a line of business are described by a random variable X with range $[0,B]$, finite mean μ and standard deviation σ (known values of the skewness and kurtosis may also be taken into account). The set of all claims with this property is denoted by $D_2 = D_2([0,B];\mu,\sigma)$. For a non-negative number d consider the "fundamental identity of portfolio insurance" (valid with probability one) :

$$(2.1)\qquad d+(X-d)_+ = X+(d-X)_+.$$

It states that the amount d plus the random claims outcome $X(d)=(X-d)_+$ of a *stop-loss* reinsurance with deductible d meets exactly the claims plus the random outcome $(d-X)_+$, which is interpreted as *surplus* of the portfolio insurance strategy defined by (2.1). The expected costs of this strategy are described by the *premium* functional

$$(2.2)\qquad P(d)=d+\pi(d)=\mu+\chi(d),$$

where the stop-loss transform $\pi(d)=E[X(d)]$ is the expected amount of the stop-loss reinsurance contract, and $\chi(d)=E[(d-X)_+]$ is the expected amount of the surplus.

For an optimal long-term portfolio insurance strategy, the following two properties are relevant :

(P1) In the long-run, the portfolio insurance strategy should be mean self-financing in the sense defined below.

(P2) The expected costs of the strategy should be minimized.

A time dependent mean self-financing infinite periodic dynamic portfolio insurance strategy is obtained as follows. At beginning of the first period, put aside the amounts d and $\pi(d)$ into two separate accounts, called first and second account, where in real-world applications the amount $\pi(d)$ for the second account will in general be adjusted by some security loading, for example as in Section 1.2. At the end of the first period, take the amounts d and $(X-d)_+$ from the first and second account. The total amount can be used to pay by (2.1) the claims of the line of business, and there remains the surplus $(d-X)_+$, which is put as gain into the second account. This financial transaction is completed by putting aside the amounts

d and $\pi(d)$ in their corresponding accounts at beginning of the second period. The same steps are repeated in each period ad infinitum. This dynamic strategy is called *mean self-financing* provided the expected outcome of the second account is at least equal to the expected cost $\pi(d)$ required to cover the stop-loss claims, that is $\chi(d) \ge \pi(d)$. Since $\chi(d) = d - \mu + \pi(d)$ this implies the restriction $d \ge \mu$ on the deductible. To satisfy property (P2), look at the derivative of the premium functional (2.2), which is $P'(d) = 1 + \pi'(d) = F(d) > 0$, where $F(x)$ is the distribution of X. Since $P(d)$ is monotone increasing and $d \ge \mu$, the premium functional is minimal at $d = \mu$. The corresponding minimal premium $P(\mu) = \mu + \pi(\mu)$ defines, in actuarial terminology, a *pricing principle* of the form

$$H[X] = \mu + E[(X-\mu)_+], \quad X \in D_2. \tag{2.3}$$

This particular principle belongs to the class of so-called Dutch pricing principles considered first by van Heerwaarden(1991). Let us name (2.3) *special Dutch pricing principle* (see the notes for this quite important special case). As a straightforward application of the theory of extremal stop-loss transforms, it is possible to determine the extremal Dutch prices. To illustrate, let us determine the extremal special Dutch prices by given range, mean and variance, which are defined by the optimization problems

$$H_*[X] = \min_{X \in D_2}\{H[X]\}, \quad H^*[X] = \max_{X \in D_2}\{H[X]\}. \tag{2.4}$$

Theorem 2.1. (*Extremal special Dutch prices*) For a claims random variable $X \in D_2([0,B];\mu,\sigma)$, with coefficient of variation $k = \frac{\sigma}{\mu}$, the extremal special Dutch prices to the optimal mean self-financing portfolio insurance strategy satisfying the properties (P1), (P2), are determined as follows :

$$H_*[X] = \left(1 + \frac{\mu}{B}\cdot k^2\right)\cdot \mu \tag{2.5}$$

$$H^*[X] = \begin{cases} (1 + \frac{k^2}{1+k^2})\cdot\mu, & \text{if} \quad k \ge 1, \\ (1 + \frac{1}{2}k)\cdot\mu, & \text{if} \quad k \le 1,\ B \ge (1+k)\cdot\mu, \\ (1 + \frac{b}{1+b^2}\cdot k)\cdot\mu, & \text{if} \quad k \le 1,\ B \le (1+k)\cdot\mu,\ b = \frac{B-\mu}{\sigma}. \end{cases} \tag{2.6}$$

Proof. From the Tables II.5.1 and II.5.2, one obtains the extremal stop-loss transform values for a standard random variable defined on $[\bar{k}, b]$, $\bar{k} = -k^{-1}$:

$$\pi_{*Z}(0) = \frac{1}{b - \bar{k}} = \frac{\mu}{B}\cdot k \tag{2.7}$$

$$\pi_Z^*(0) = \begin{cases} \frac{k}{1+k^2}, & \text{if} \quad \frac{1}{2}(k+\bar{k}) \ge 0, \\ \frac{1}{2}, & \text{if} \quad \frac{1}{2}(k+\bar{k}) \le 0 \le \frac{1}{2}(b+\bar{b}), \\ \frac{b}{1+b^2}, & \text{if} \quad \frac{1}{2}(b+\bar{b}) \le 0. \end{cases} \tag{2.8}$$

One concludes by using the relations $\pi_*(\mu) = \sigma \cdot \pi_{*Z}(0)$, $\pi^*(\mu) = \sigma \cdot \pi^*_Z(0)$, and by showing that the inequality conditions in (2.8) are equivalent to those in (2.6). ◊

Remark 2.1. The variance inequality $\sigma^2 \le \mu \cdot (B - \mu)$ (see Theorem I.4.1, (4.14)) implies that $B \ge (1 + k^2) \cdot \mu$, where equality is attained for a diatomic random variable with support $\{0, (1+k^2)\cdot\mu\}$ and probabilities $\left\{\frac{k^2}{1+k^2}, \frac{1}{1+k^2}\right\}$. In this extreme situation of maximal variance, one has $H_*[X] = H^*[X] = (1 + \frac{k^2}{1+k^2}) \cdot \mu$ for all values of k.

Similar results can be derived by additional knowledge of the skewness and kurtosis. A full analytical treatment is quite cumbersome, but a computer algebra system implementation is certainly feasible. As illustration, let us state without proof the next simplest result by a known value of the skewness.

Example 2.1 : Extremal special Dutch prices on $[0, \infty)$ by known mean, variance and skewness

For a claims random variable $X \in D_3([0,\infty); \mu, \sigma, \gamma)$, $\gamma \ge k + \overline{k}$, one has the extremal special Dutch prices

$$H_*[X] = \left(1 + \frac{k^2}{2 + \gamma k}\right) \cdot \mu \tag{2.9}$$

$$H^*[X] = \begin{cases} (1 + \frac{k^2}{1+k^2}) \cdot \mu, & \text{if} \quad k \ge 1, \\ (1 + \frac{1}{2}k) \cdot \mu, & \text{if} \quad k \le 1, \gamma \ge 0, \\ (1 + \frac{(-c)}{1+c^2} \cdot k) \cdot \mu, & \text{if} \quad (-\frac{1}{2}\gamma)c^2 \le k \le 1, \gamma \le 0, c = \frac{1}{2}(\gamma + \sqrt{4+\gamma^2}). \end{cases} \tag{2.10}$$

If $k \le (-\frac{1}{2}\gamma)c^2$, $\gamma \le 0$, the maximal price depends on the solution of a cubic equation.

3. Analytical bounds for risk theoretical quantities.

A main quantity of interest in Risk Theory is the *aggregate claims* random variable, which is described by a stochastic process of the type

$$S(t) = X_1 + \ldots + X_{N(t)}, \tag{3.1}$$

where $t \ge 0$ is the *time* parameter, the *claim number* process $\{N(t)\}, t \ge 0$, is a fixed counting process, and the (non-negative) *claim sizes* X_i are independent identically distributed, say $X_i \sim X$, and independent from the process $\{N(t)\}$. Important related quantities are the (net) *stop-loss premium* to the *deductible* d, represented by the stop-loss transform

$$\pi_S(d) = E[(S(t) - d)_+], \quad d \ge 0, \tag{3.2}$$

and the *probability of ruin*, defined by

$$\psi_S(u) = \Pr(\inf\{t : u + ct - S(t) < 0\} < \infty), \quad u \geq 0, \tag{3.3}$$

where u is the *initial reserve* and c is the constant *premium rate* received continuously per unit of time.

The classical actuarial risk model assumes that $\{N(t)\}$ is a Poisson process with *intensity* λ, that is $S(t)$ is a so-called *compound Poisson* stochastic process, and that $c = \lambda\mu(1+\theta)$, $\mu = E[X]$ the mean claim size and $\theta > 0$ the *security loading*. In this situation one has (see any recent book on Risk Theory) :

$$\pi_S(d) = \sum_{n=0}^{\infty} \frac{(\lambda t)^n}{n!} \cdot e^{-\lambda t} \cdot E[(X_1 + \ldots + X_n - d)_+], \tag{3.4}$$

and

$$\psi_S(u) = \overline{F}_L(u), \tag{3.5}$$

where $L = \max_{t \geq 0}\{S(t) - ct\}$ is the maximal aggregate loss. If $L_1, L_2, \ldots$ are the amounts by which record lows of the surplus process $\{u + ct - S(t)\}$ are broken, and M counts the number of record lows, then

$$L = \sum_{i=1}^{M} L_i, \tag{3.6}$$

where M has a geometric distribution with parameter $\psi(0) = (1+\theta)^{-1}$, and the distribution function of L_i equals

$$F_{L_i}(x) = 1 - \frac{\pi_X(x)}{\mu}, \quad x \geq 0. \tag{3.7}$$

One obtains the so-called *Beekman convolution formula* for the ultimate ruin probability

$$\psi_S(u) = \frac{\theta}{1+\theta} \cdot \sum_{n=0}^{\infty} \left(\frac{1}{1+\theta}\right)^n \cdot \Pr(L_1 + \ldots + L_n > u). \tag{3.8}$$

3.1. Inequalities for stop-loss premiums and ruin probabilities.

Under incomplete information, the claim size random variable X will not be known with certainty. Suppose $X \in D_n = D_n([0,B]; \mu_1, \ldots, \mu_n)$, that is the range $[0,B]$ and the first n moments are given. For $n=2,3,4,\ldots$, let $X_*^{(n)}, X^{*(n)}$ be the stop-loss ordered extremal random variables for the sets D_n considered in Chapter IV. Denote by $\{S_*^{(n)}(t)\}, \{S^{*(n)}(t)\}$ the compound Poisson processes of the type (3.1) obtained when replacing X by $X_*^{(n)}, X^{*(n)}$ respectively. As a consequence of the formulas (3.4), (3.7) and (3.8), and stop-loss ordering, one obtains the following inequalities between stop-loss premiums and ruin probabilities :

(3.9) $\pi_{S_*^{(n-1)}}(d) \le \pi_{S_*^{(n)}}(d) \le \pi_S(d) \le \pi_{S^{*(n)}}(d) \le \pi_{S^{*(n-1)}}(d)$,
uniformly for all deductibles $d \ge 0$, all $X \in D_n$, $n = 3,4,\ldots$,

(3.10) $\psi_{S_*^{(n-1)}}(u) \le \psi_{S_*^{(n)}}(u) \le \psi_S(u) \le \psi_{S^{*(n)}}(u) \le \psi_{S^{*(n-1)}}(u)$,
uniformly for all initial reserves $u \ge 0$, all $X \in D_n$, $n = 3,4,\ldots$

The inequalities are shown as in Kaas(1991) (see also Kaas et al.(1994), chap. XI) under application of Theorem IV.2.2.

3.2. Ordered discrete approximations.

Of main practical importance is $X \in D_2 = D_2([0,B];\mu,\sigma)$, that is the range $[0,B]$, the mean μ and the standard deviation σ of the claim size are known. Let X_*, X^* be the corresponding stop-loss ordered extremal random variables. It is convenient to express results in terms of the following parameters :

$\upsilon = k^2 = \left(\frac{\sigma}{\mu}\right)^2$: relative variance of X, or squared coefficient of variation,

$\upsilon_0 = \frac{B-\mu}{\mu}$: maximal relative variance over $D_1([0,B];\mu)$
(consequence of variance inequality (I.4.14))

$\upsilon_r = \frac{\upsilon}{\upsilon_0}$: relative variance ratio

From Table IV.2.1 one sees that X_* is a diatomic random variable with support $\{(1-\upsilon_r)\cdot\mu, (1+\upsilon)\cdot\mu\}$ and probabilities $\left\{\frac{\upsilon_0}{1+\upsilon_0}, \frac{1}{1+\upsilon_0}\right\}$. For numerical calculations, it is appropriate to replace X^* by the slightly less tight finite atomic upper bound approximation $X_d^* \ge_{D,=} X^*$ obtained from Proposition IV.3.1 applying mass dispersion. One sees that X_d^* is a 4-atomic random variable with support $\{0, \frac{1}{2}(1+\upsilon)\cdot\mu, [1+\frac{1}{2}(\upsilon_0-\upsilon_r)]\cdot\mu, (1+\upsilon_0)\cdot\mu\}$ and probabilities $\left\{\frac{\upsilon}{1+\upsilon}, \frac{\upsilon_0-\upsilon}{(1+\upsilon)(1+\upsilon_0)}, \frac{\upsilon_0-\upsilon}{(\upsilon_r+\upsilon_0)(1+\upsilon_0)}, \frac{\upsilon_r}{\upsilon_r+\upsilon_0}\right\}$. In this sitution the uniform bounds $\pi_{S_*}(d) \le \pi_S(d) \le \pi_{S_d^*}(d)$ and $\psi_{S_*}(u) \le \psi_S(u) \le \psi_{S_d^*}(u)$ can be evaluated in an analytical way by means of the following general procedure.

Suppose the claim size random variable X is a finite (m+1)-atomic random variable with support $\{x_0 = 0, x_1, \ldots, x_m\}$ and probabilities $\{p_0, p_1, \ldots, p_m\}$. Then it is well-known that the compound Poisson random variable S(t) can be expressed as (see the notes)

(3.11) $$S(t) = \sum_{j=1}^{m} x_j \cdot N_j(t),$$

where the $\{N_j(t)\}s$, which count the number of occurences of claim size x_j, are independent Poisson processes with intensities λp_j. This representation implies the following analytical formulas :

(3.12) $$\pi_S(d)=\lambda t\mu-d+\exp\{-\lambda t(1-p_0)\}\cdot\sum_{n_1,\dots,n_m=0}^{\infty}(d-\sum_{j=1}^{m}n_jx_j)_+\cdot\prod_{j=1}^{m}\frac{(\lambda tp_j)^{n_j}}{n_j!},$$

(3.13) $$\psi_S(u)=1-\frac{\theta}{1+\theta}\cdot\sum_{n_1,\dots,n_m=0}^{\infty}\exp\{z(1-p_0)\}(-z)^{n_1+\dots+n_m}\cdot\prod_{j=1}^{m}\frac{p_j^{\,n_j}}{n_j!},\ \text{with}$$

$$z=\frac{\lambda}{c}(u-\sum_{j=1}^{m}n_jx_j)_+ .$$

Since summation occurs only for $\sum_{j=1}^{m}n_jx_j<d,u$, these infinite series representations are always finite. Using a computer algebra system, the desired uniform bounds can be evaluated without difficulty. A concrete numerical illustration, including a comparison with less tight bounds, is given in Hürlimann(1996a).

3.3. The upper bounds for small deductibles and initial reserves.

Suppose the deductible of a stop-loss reinsurance contract over a unit of time period $[0,t=1]$ is small in the sense that $d\le\frac{1}{2}(1+\upsilon)\cdot\mu$. Then the series representation (3.12) for $X=X_d^*$ shrinks to the only term $n_1=n_2=n_3=0$, and thus one has

(3.14) $$\pi_{S_d^*}(d)=\lambda\mu-d+\exp\{-\lambda(1-p_0)\}\cdot d .$$

In terms of the mean $\mu_S=\lambda\mu$ and the relative variance $\upsilon_S=(1+\upsilon)\cdot\lambda^{-1}$ of a compound Poisson random variable S with claim size $X\in D_2$, this can be rewritten as

(3.15) $$\pi_{S_d^*}(d)=\mu_S-d+\exp\left\{\frac{-1}{\upsilon_S}\right\}\cdot d, d\le\frac{1}{2}(1+\upsilon)\cdot\mu .$$

From Kaas(1991), p. 141, one knows that for small deductibles the maximizing claim size random variable for $X\in D_2$ is the diatomic random variable with support $\{0,(1+\upsilon)\cdot\mu\}$ and probabilities $\left\{\frac{\upsilon}{1+\upsilon},\frac{1}{1+\upsilon}\right\}$. It leads to the same compound Poisson stop-loss premium as (3.15). A similar results holds for the ultimate ruin probabilities. Therefore the analytical upper bounds obtained from the stop-loss ordered maximal random variable coincides in the special case of small deductibles and small initial reserves with the optimal, that is best upper bounds.

3.4. The upper bounds by given range and known mean.

Suppose $X\in D_1=D_1([0,B];\mu)$, that is only the range and mean of the claim size are known. In this situation, the relative variance is unknown and satisfies by (I.4.14) the inequality $0\le\upsilon\le\upsilon_0$. One shows that the "worst case" is obtained when $\upsilon=\upsilon_0$, for which both X_*, X_d^* of Section 3.2 go over to a diatomic random variable with support $\{0,B=(1+\upsilon_0)\cdot\mu\}$ with probabilities $\left\{\frac{\upsilon_0}{1+\upsilon_0},\frac{1}{1+\upsilon_0}\right\}$. One obtains the best upper bounds

(3.16) $$\pi_{S^*}(d)=\mu_S-d+\exp\left\{\frac{-\mu_S}{B}\right\}\cdot\sum_{n=0}^{n_d}\frac{(d-B\cdot n)}{n!}\cdot\left(\frac{\mu_S}{B}\right)^n,\ \text{with}\ n_d=\left[\frac{d}{B}\right]$$

($[x]$ the greatest integer less than x), (3.17) $$\psi_{S^*}(u) = 1 - \frac{\theta}{1+\theta} \cdot \sum_{n=0}^{n_u} \frac{\exp\left\{\frac{z \cdot \mu_S}{B}\right\}}{n!} \cdot \left(\frac{-z \cdot \mu_S}{B}\right)^n, \text{ with } n_u = \left[\frac{u}{B}\right],\ z = \frac{(u - B \cdot n)}{(1+\theta) \cdot \mu_S}.$$

The latter formula can be viewed as a positive answer to the following *modified Schmitter problem* (see the notes). Given that the claim size random variable has mean μ and maximal variance $\upsilon_0 \cdot \mu^2 = \mu \cdot (B - \mu)$ over the interval $[0, B]$, does a diatomic claim size random variable maximize the ultimate ruin probability ?

3.5. Conservative special Dutch price for the classical actuarial risk model.

Under the assumption $X \in D_1$ made in Section 3.4, it is interesting to look at the implied maximal special Dutch price

(3.18) $$H^*[S] = H[S^*] = \mu_S + \pi_{S^*}(\mu_S),$$

which has found motivation in Section 2. From (3.16) one gets

(3.19) $$\pi_{S^*}(\mu_S) = \mu_S \cdot \frac{\exp\{-\lambda_B + [\lambda_B] \cdot \ln\{\lambda_B\}\}}{[\lambda_B]!} \approx \mu_S \cdot \frac{\exp\{-\lambda_B + \lambda_B \cdot \ln\{\lambda_B\}\}}{\Gamma(\lambda_B + 1)}, \quad \lambda_B = \frac{\mu_S}{B}.$$

This is the special stop-loss premium of a compound Poisson(λ_B) random variable with individual claims of fixed size equal to the maximal amount B. It is approximately equal to the special stop-loss premium of a Gamma($\lambda_B, \frac{\lambda_B}{\mu_S}$) random variable. Applying Stirling's approximation formula for the Gamma function, one obtains

(3.20) $$\pi_{S^*}(\mu_S) \approx \mu_S \cdot (2\pi\lambda_B)^{-\frac{1}{2}} \cdot (1 + \frac{1}{12\lambda_B} + ...) < \mu_S \cdot \sqrt{\frac{B}{2\pi \cdot \mu_S}},$$

which yields the conservative special Dutch price

(3.21) $$H^*[S] < (1 + \sqrt{\frac{B}{2\pi \cdot \mu_S}}) \cdot \mu_S$$

4. Distribution-free excess-of-loss reserves by univariate modelling of the financial loss.

Most risks in Actuarial Science and Finance are built up from three main categories, namely life insurance risks, non-life insurance risks and financial risks. In each category of risks, it is possible to define mathematical objects, called *contracts*, which specify the risks covered either for an individual contract or for a portfolio of such contracts. Mathematically, an *individual contract* is a triplet $C = \{A, L, R\}$, which represents a security, and whose three components describe **A**ssets, **L**iabilities and **R**eserves. These quantities are modeled by stochastic processes $A = \{A(t)\}$, $L = \{L(t)\}$, $R = \{R(t)\}$, $t \geq 0$ the time parameter. In particular, at each time t the values $A(t)$, $L(t)$ and $R(t)$ are random variables. One supposes that there exists a sufficiently large *portfolio* of similar contracts, where no precise statement about this is required in the following.

One observes that at each future time the amount of the liability positions of an individual contract may exceed the amount of the asset positions, which results in a positive loss. To compensate a *positive loss* $V(t) = L(t) - A(t) > 0$ of an individual contract, a risk manager is supposed to accumulate at time t an amount $R(t)=R(t;A,L)$, called *excess-of-loss reserve*, to be determined, and which depends upon the past evolution of the assets and liabilities. It will be assumed that $R(t)$ can be funded by the *positive gains* $G(t) = A(t) - L(t) > 0$ of similar contracts in the considered portfolio. Clearly, an excess-of-loss reserve can be accumulated only if the *financial gain* $G(t) = A(t) - L(t)$ is positive, that is one has the constraint $0 \le R(t) \le G(t)_+$, $t > 0$, where $G(t)_+ = G(t)$ if $G(t) > 0$ and $G(t)_+ = 0$ else. The stochastic process $U = \{U(t)\}$, defined by $U(t) = U(t;A,L) = (R(t) - G(t))_+$, $t > 0$, is called *excess-of-loss*, and describes the possible loss incurred after deduction of the gain from the excess-of-loss reserve. The corresponding possible gain is described by a stochastic process $D = \{D(t)\}$, defined by $D(t) = D(t;A,L) = (G(t) - R(t))_+$, $t > 0$, and is called *excess-of-gain*, in particular contexts it is also named dividend or bonus. The *net outcome* of the holder of an individual contract after deduction of the excess-of-gain from the gain is modeled by the stochastic process $NO = \{NO(t)\}$ defined by $NO(t) = G(t) - D(t)$, $t > 0$.

The formal structure of an individual contract is determined by the following relationships between gain G, positive gain G_+, excess-of-gain D, excess-of-loss reserve R, excess-of-loss U and net outcome NO. The omission of indices, here and afterwards, supposes that the made statements are valid at each time of the evolution of a contract.

Theorem 4.1. (Hürlimann(1995d)) An individual contract $C = \{A, L, R\}$ satisfies the following structural relationships :

(R1) $0 \le R \le G_+$,
(R2) $U = V_+ = G_-$,
(R3) $NO = G - D = R - U = \min\{R, G\}$,
(R4) $D = (G - R)_+ = G_+ - R$,
(R5) $R = (G - D)_+ = G_+ - D$. ◊

The obvious symmetry of the relations (R1) to (R5) with respect to R and D shows that the transformation, which maps the excess-of-loss reserve R to the excess-of-gain $D = (G - R)_+$, has an inverse transformation, which maps the excess-of-gain D to the excess-of-loss reserve $R = (G - D)_+$.

An important question in the theory of excess-of-loss reserves is the appropriate choice of the formula, which determines the excess-of-loss reserve, or by symmetry the excess-of-gain. It is intuitively clear that the financial success on the insurance of finance market of such contracts depends upon the choice of the excess-of-loss reserve/excess-of-gain formula, a choice which may vary among different lines of business. A decision can only be taken provided the universe of feasible excess-of-loss reserve/excess-of-gain strategies is specified. To illustrate consider a simple popular strategy based on the *net outcome principle* $E[NO] = 0$, which can and has already been justified in different ways. Then the possible universe of excess-of-loss reserves is given by the set $S = \{R : 0 \le R \le G_+, E[NO] = 0\}$. If a decision-maker wants the least possible fluctuations of the excess-of-loss reserve, that is a minimal variance $Var[R] = \min.$, then the unique choice is determined as follows.

Theorem 4.2. (Hürlimann(1991b)) The optimal individual contract $C^* = \{A, L, R^*\}$, which solves the optimization problem $Var[R^*] = \min_{R\in S}\{Var[R]\}$, is given by the *stable excess-of-loss reserve*

$$R^* = \min\{B, G_+\}, \tag{4.1}$$

where the deterministic process $B = \{B(t)\}, t > 0$, is solution of the expected value equation

$$E[(G-B)_+] = E[G]. \ \lozenge \tag{4.2}$$

By symmetry, the excess-of-gain associated to the stable excess-of-loss reserve is $D^* = (G-R^*)_+ = (G-B)_+$. One may exchange the role of R and D, which defines an alternative *stable excess-of-gain* strategy $\overline{D}^* = \min\{B, G_+\}$ with associated excess-of-loss reserve $\overline{R}^* = (G-B)_+$.

A general mathematical and statistical problem is the evaluation and discussion of the properties of the deterministic process B provided the stochastic processes A and L modelling the assets and liabilities belong to some specific class of financial models. In the present and next Sections, some distribution-free upper bounds by given range(s), mean(s) and variance(s) of the *financial loss* $V = L - A$ are determined.

We begin with the univariate modelling of the financial loss. The *financial gain* of a line of business is described by a random variable G with finite mean $\mu = E[G] > 0$ and finite variance $\sigma^2 = Var[G]$. The range of G is an interval $[A,B]$, $-\infty \le A < B \le \infty$. The set of all such financial gains is denoted by $D = D([A,B];\mu,\sigma)$. Since $\mu > 0$ the coefficient of variation $k = \dfrac{\sigma}{\mu}$ exists and is finite. The *maximal excess-of-loss reserve* by given range, mean and variance, is denoted by $R^* = R^*([A,B],\mu,\sigma)$, and by Theorem 4.2 it is solution of the expected value equation

$$\max_{G\in D}\{E[(G-R^*)_+]\} = E[G]. \tag{4.3}$$

Theorem 4.3. The maximal excess-of-loss reserve associated to a financial gain with range $[A,B]$, positive mean $\mu \in (A,B)$ and variance $\sigma^2 \le (\mu - A)(B-\mu)$ is given by

$$R^* = \begin{cases} 0, \quad if \ 0 \le A < \mu \\ \left(\dfrac{\sigma}{\mu - A}\right)^2(-A), \quad if \ -\mu \le A < 0 \\ \dfrac{1}{4}k^2\mu, \quad if \ A \le -\mu, \ B \ge (1+\dfrac{1}{2}k^2)\mu \\ (B-\mu) - \left(\dfrac{B-\mu}{\sigma}\right)^2\mu, \quad if \ A < 0, \ \mu < B \le (1+\dfrac{1}{2}k^2)\mu \end{cases} \tag{4.4}$$

Proof. Let $Z = \dfrac{G-\mu}{\sigma}$ be the standard financial gain random variable with range $[a,b]$, $a = \dfrac{A-\mu}{\sigma}$, $b = \dfrac{B-\mu}{\sigma}$. The maximal stop-loss transform of a standard random variable with range $[a,b]$ is denoted by $\pi^*(z)$. Then the defining equation (4.3) reads

$$\sigma \cdot \pi^*\left(\frac{R^* - \mu}{\sigma}\right) = \mu . \tag{4.5}$$

To simplify calculations, it is convenient to use Table III.5.1. To show (4.4) several cases are distinguished.

<u>Case (I)</u> : $A \geq 0$

Since $G \geq 0$ one has $E[G_+] = E[G]$, hence R=0 for all G, a fortiori $R^* = 0$.

<u>Case (II)</u> : $A < 0$ (hence $-A = (-a)\sigma - \mu > 0$)

In view of Table III.5.1, it is appropriate to set $R^* = \mu + d_i(x)\sigma,\ i = 1,2,3.$

<u>Subcase (1)</u> : i=1

Using case (1) in Table III.5.1 one sees that (4.5) is equivalent with the condition

$$\frac{(-a)(\bar{a}-x)}{(\bar{a}-x)+(\bar{a}-a)} \cdot \sigma = \mu, \quad x \leq a . \tag{4.6}$$

The solution $x = \bar{a} + \dfrac{(\bar{a}-a)\mu}{\mu + a\sigma}$ satisfies the constraint $x \leq a$ exactly when $2\mu + a\sigma \geq 0$, that is $A = \mu + a\sigma \geq -\mu$. Straightforward algebra shows that

$$R^* = \mu + d_1(x)\sigma = \frac{\sigma - (\bar{a})\mu}{(-a)} = (-A)\left(\frac{\sigma}{\mu - A}\right)^2 . \tag{4.7}$$

<u>Subcase (2)</u> : i=2

One sees that (4.5) is equivalent with the condition

$$\frac{1}{2}(-x) \cdot \sigma = \mu, \ a \leq x \leq \bar{b} . \tag{4.8}$$

The constraint holds if and only if $A \leq -\mu,\ B \geq \mu + \frac{1}{2} \cdot \frac{\sigma^2}{\mu}$. Furthermore one gets

$$R^* = \mu + d_2(x)\sigma = \frac{1}{4} \cdot \frac{\sigma^2}{\mu} . \tag{4.9}$$

Subcase (3) : i=3

One obtains that (4.5) is equivalent with the condition

(4.10) $$\frac{(-\bar{b})(b-\bar{b})}{(b-\bar{b})+(x-\bar{b})}\cdot\sigma=\mu, \quad x\ge b.$$

The solution $x=\bar{b}-\frac{(b-\bar{b})(\mu+\bar{b}\sigma)}{\mu}$ satisfies $x\ge b$ if and only if $b\le\frac{1}{2}\left(\frac{\sigma}{\mu}\right)$, that is $B\le\mu+\frac{1}{2}\cdot\frac{\sigma^2}{\mu}$. Through calculation one gets

(4.11) $$R^*=\mu+d_3(x)\sigma=b(\sigma-b\mu)=(B-\mu)-\left(\frac{B-\mu}{\sigma}\right)^2\cdot\mu.$$

The determination of the maximal loss reserve is complete. ◊

For a finite range $[A,B]$, the most conservative excess-of-loss reserve, which depends only on the mean, is determined as follows.

Corollary 4.1. The maximal excess-of-loss reserve associated to a financial gain with finite range $[A,B]$ and positive mean $\mu\in(A,B)$ is given by

(4.12) $$R^*=\begin{cases}0, if\ 0\le A<\mu,\\ \left(\frac{B-\mu}{\mu-A}\right)(-A), if\ A<0.\end{cases}$$

Proof. One observes that R^* in (4.4) is a monotone increasing function of the variance, and thus it is maximal by maximal variance $\sigma^2=(\mu-A)(B-\mu)$. One shows that (4.4) simplifies to (4.12). ◊

By Section III.5 the maximal stop-loss transforms of a standard random variable on $[a,b]$ by additional knowledge of either the skewness or the skewness and the kurtosis can be structured similarly to Table III.5.1. One can take advantage of this to determine, similarly as in the proof of Theorem 4.3, maximal excess-of-loss reserves by given range and known moments up to the fourth order. However, the detailed analytical discussion is in general quite complex, and it seems preferable to determine them numerically using a computer algebra system.

A single application suffices to demonstrate the usefulness of the above results.

Example 4.1 : maximal financial risk premium for a guaranteed rate of return

Let R be the random accumulated rate of return of an investment portfolio, and suppose a fixed accumulated rate of return r_g should be guaranteed. Then the financial gain $G=R-r_g$ represents the excess return, which may take negative values. Let $r=E[R]$,

$\sigma^2 = Var[G] = Var[R]$, and suppose that $\mu = E[G] = r - r_g > 0$. In this situation, the "excess-of-loss reserve", which is now denoted by b, is solution of the expectation equation

$$E\left[(R - r_g - b)_+\right] = r - r_g. \tag{4.13}$$

As suggested in Hürlimann(1991c), the constant b can be interpreted as financial risk premium needed to cover the risk of a negative excess return. Suppose it is known that the expected return variies between the bounds $r_{min} \le r \le r_{max}$, which is a reasonable assumption, at least if R represents a random accumulated rate of interest. With $A = r_{min} - r_g$, $B = r_{max} - r_g$, the following maximal financial risk premium is obtained from (4.4) :

$$(4.14) \qquad b^* = \begin{cases} 0, & if\, r_g \le r_{\min} \\ (r_g - r_{\min}) \cdot \left(\dfrac{\sigma}{r - r_{\min}}\right)^2, & if\;\; r_{\min} \le r_g \le \dfrac{1}{2}(r + r_{\min}) \\ \dfrac{1}{4} \cdot \dfrac{\sigma^2}{(r - r_g)}, & if\;\; \dfrac{1}{2}(r + r_g) \le r_g \le r - \dfrac{1}{2}\dfrac{\sigma^2}{(r_{\max} - r)} \\ (r_{\max} - r) - (r - r_g) \cdot \left(\dfrac{r_{\max} - r}{\sigma}\right)^2, & if\;\; r - \dfrac{1}{2}\dfrac{\sigma^2}{(r_{\max} - r)} \le r_g < r \le r_{\max} \end{cases}$$

A most conservative upper bound, which turns out to be "volatility" independent, is obtained from (4.12) :

$$(4.15) \qquad b^* = \begin{cases} 0, & \text{if} \quad r_g \le r_{min} \\ \dfrac{(r_g - r_{min})(r_{max} - r)}{(r - r_{min})}, & \text{if} \quad r_{min} \le r_g < r \le r_{max} \end{cases}$$

Note that by nearly maximal volatility $\sigma^2 \approx (r - r_{min})(r_{max} - r)$, the latter upper bound is adequate. To illustrate the differences numerically, let $r_{min} = 1.03$, $r = 1.05$, $r_{max} = 1.07$, $r_g = 1.04$, $\sigma = 0.01$. Then one has $b^* = 0.0025$ by (4.14) and $b^* = 0.01$ by (4.15). If $\sigma = 0.02$ then both bounds are equal.

5. Distribution-free excess-of-loss reserves by bivariate modelling of the financial loss.

It is often more realistic to think of the financial gain as a difference between assets and liabilities, and to model it as a difference $G = A - L$ of two random variables A and L. We suppose that A and L are taken from the sets $D_A = D([A_m, A_M]; \mu_A, \sigma_A)$, $D_L = D([L_m, L_M]; \mu_L, \sigma_L)$ of all random variables with given ranges, means and variances. A dependence structure between A and L is not assumed to be known. Thus the random pair (A,L) is taken from the set

$$BD = BD([A_m, A_M]x[L_m, L_M]; \mu_A, \mu_L, \sigma_A, \sigma_L) = \{(A, L) : A \in D_A, L \in D_L\}$$

of all bivariate random variables by given ranges and known marginal means and variances.

Under these assumptions a maximal excess-of-loss reserve

$$R^* = R^*([A_m, A_M]x[L_m, L_M]; \mu_A, \mu_L, \sigma_A, \sigma_L)$$

could be defined as solution of the expected value equation

$$\max_{(A,L)\in BD}\{E[(A-L-R^*)_+]\} = E[A-L]. \tag{5.1}$$

In the special case of infinite ranges $(-\infty,\infty)$, the bivariate version of the inequality of Bowers(1969), shown in Hürlimann(1993c), yields the solution

$$R^* = \frac{1}{4}k^2\mu, \quad k = \frac{\sigma_A + \sigma_L}{\mu_A - \mu_L}, \quad \mu = \mu_A - \mu_L > 0, \tag{5.2}$$

which generalizes the corresponding univariate result in Theorem 4.3. Since the maximum in (5.1) is attained when A and L are completely independent, that is the variance of $G = A - L$ is maximal over BD, the constant $k = \frac{\sigma_A + \sigma_L}{\mu_A - \mu_L}$ may be viewed as a *bivariate coefficient of variation* for differences of random variables with positive mean difference.

By arbitrary ranges, the maximum in (5.1) is better replaced by a combined maximum as in (V.6.3). The obtained upper bound for the excess-of-loss reserve will not in general be attained over BD. However, there exist tight distribution-free upper bounds, which are attained by Hoeffding-Fréchet extremal distributions constructed from the stop-loss ordered maximal distributions by given ranges, means and variances, and which directly generalize the bivariate inequality of Bowers. By a linear transformation of variable, it suffices to consider the distribution-free upper bounds for expected positive differences determined in Theorem V.6.1.

The upper bound in Table V.6.1 is an increasing function of the marginal variances. If one replaces them by their upper bounds $\sigma_X^2 = (\mu_X - A_X)(B_X - \mu_X)$, $\sigma_Y^2 = (\mu_Y - A_Y)(B_Y - \mu_Y)$, one obtains a very simple upper bound, which depends only on the given ranges and the marginal means.

Theorem 5.1. Let $(X,Y) \in BD([A_X, B_X]x[A_Y, B_Y]; \mu_X, \mu_Y)$ be a bivariate pair of random variables with the given marginal ranges and means. Then the following inequality holds :

$$E[(X-Y)_+] \le \begin{cases} \mu_X - \underline{A}, & a_X a_Y \le 1, \\ \overline{B} - \mu_Y, & a_X a_Y \ge 1. \end{cases} \tag{5.3}$$

Proof. By maximal marginal variances, one has the relationships

$$b_X = \bar{a}_X = \sqrt{\frac{B_X - \mu_X}{\mu_X - A_X}}, \quad b_Y = \bar{a}_Y = \sqrt{\frac{B_Y - \mu_Y}{\mu_Y - A_Y}}.$$

This implies that $a_X a_Y \le 1$ if and only if $b_X b_Y \ge 1$. Therefore only the cases (I) and (III) in Table V.6.1 are possible. ◊

Based on Table V.6.1 and Theorem 5.1, it is possible to derive bivariate versions of Theorem 4.3 and Corollary 4.1. Let us begin with the mathematically more tractable situation.

Corollary 5.1. Let $G = A - L$ be a financial gain with $(A,L) \in BD([A_m, A_M]x[L_m, L_M]; \mu_A, \mu_L)$. Denote by R^* the distribution-free upper bound for the excess-of-loss reserve obtained from Theorem 5.1. Then two situations are possible.
Case 1 : If $\Pr(A \geq \mu_L) = 1$, that is the expected liabilities should always (with probability one) be covered by the assets, and $\mu_L - L_m \leq \left(\frac{L_M - L_m}{A_M - A_m}\right) \cdot (A_M - \mu_A)$, then one has $R^* = \mu_L - L_m$.
Case 2 : If $\Pr(L \leq \mu_A) = 1$, that is the liabilities should never (with probability one) exceed the expected assets, and $A_M - \mu_A \leq \left(\frac{A_M - A_m}{L_M - L_m}\right) \cdot (\mu_L - L_m)$, then one has $R^* = A_M - \mu_A$.

Proof. The result follows from Theorem 5.1 by setting $X=A$, $Y = L + R^*$. One notes that

$$a_X^2 = \frac{\mu_A - A_m}{A_M - \mu_A}, \quad a_Y^2 = \frac{\mu_L - L_m}{L_M - \mu_L}.$$

In case 1, that is $a_X a_Y \leq 1$, one must solve the equation

$$\mu_A - \min\{A_m, L_m + R^*\} = \mu_A - \mu_L .$$

Under the condition $L_m + R^* \leq A_m$ one gets $R^* = \mu_L - L_m$. Then the required conditon is equivalent with $A_m \geq \mu_L$, that is $\Pr(A \geq \mu_L) = 1$. Case 2 is shown similarly. ◊

Based on Table V.6.1, let us now derive a bivariate version of Theorem 4.3. For technical reasons, we restrict ourselve to a special case, which is, however, strong enough to generate the desired result in case A, L are defined on the one-sided infinite ranges $[0,\infty)$. A more general result can be obtained by the same technic, but seems a bit tedious.

Theorem 5.2. Let $G = A - L$ be a financial gain, with positive mean, such that $(A,L) \in BD([A_m, A_M]x[L_m, L_M]; \mu_A, \mu_L, \sigma_A, \sigma_L)$, and set $k = \frac{\sigma_A + \sigma_L}{\mu_A - \mu_L}$, $\mu = \mu_A - \mu_L$, $a_A = \frac{A_m - \mu_A}{\sigma_A}$, $b_A = \frac{A_M - \mu_A}{\sigma_A}$, $a_L = \frac{L_m - \mu_L}{\sigma_L}$, $b_L = \frac{L_M - \mu_L}{\sigma_L}$. Denote by R^* the distribution free upper bound for the excess-of-loss reserve obtained from Table V.6.1, and suppose that $R^* \in [A_m - L_m, A_M - L_M]$, as well as $a_A a_L \geq 1, b_A b_L \geq 1$. Then R^* is determined by Table 5.1.

Specializing Theorem 5.1 to $A_m = L_m = 0$, $A_M = L_M = \infty$, one obtains the following result.

Corollary 5.2. Let $G = A - L$ be a financial gain, with positive mean, such that $(A,L) \in BD([0,\infty)x[0,\infty);\mu_A,\mu_L,\sigma_A,\sigma_L)$, and set $k = \frac{\sigma_A + \sigma_L}{\mu_A - \mu_L}$, $\mu = \mu_A - \mu_L$, $k_A = \frac{\sigma_A}{\mu_A}$, $k_L = \frac{\sigma_L}{\mu_L}$. If $k_A k_L \le 1$, then a distribution-free upper bound for the excess-of-loss reserve is given by

$$R^* = \begin{cases} k_A \cdot (k - k_A) \cdot \mu, & \text{if } \tfrac{1}{2}k \le k_A, \\ \tfrac{1}{4}k^2 \cdot \mu, & \text{if } k_A \le \tfrac{1}{2}k \le (-\bar{k}_L), \\ (-\bar{k}_L) \cdot (k - (-\bar{k}_L)) \cdot \mu, & \text{if } (-\bar{k}_L) \le \tfrac{1}{2}k < (-\bar{k}_L) \cdot (1 + \frac{\sigma_L}{\sigma_A}). \end{cases} \tag{5.4}$$

Table 5.1 : Distribution-free upper bound for the excess-of-loss reserve associated to a financial gain by given ranges, means and variances of the marginal assets and liabilities

case	conditions	excess-of-loss reserve upper bound
(1)	$a_A + \bar{a}_A + b_L + \bar{b}_L \ge 0$ $\tfrac{1}{2}k \le \bar{a}_A$	$\bar{a}_A \cdot (k - \bar{a}_A) \cdot \mu$
(2)	$a_L + \bar{a}_L + b_A + \bar{b}_A \ge 0$ $-a_L \le \tfrac{1}{2}k < (-a_L) \cdot (1 + \frac{\sigma_L}{\sigma_A})$	$(-a_L) \cdot (k - (-a_L)) \cdot \mu$
(3)	$\max\{\bar{a}_A, -\bar{b}_L\} \le \tfrac{1}{2}k \le \min\{-a_L, b_A\}$	$\tfrac{1}{2}k^2 \cdot \mu$
(4)	$a_A + \bar{a}_A + b_L + \bar{b}_L \le 0$ $(-\bar{b}_L) \cdot (1 - \frac{\sigma_L}{\sigma_A + \sigma_L}) < \tfrac{1}{2}k \le -\bar{b}_L$	$(-\bar{b}_L) \cdot (k - (-\bar{b}_L)) \cdot \mu$
(5)	$a_L + \bar{a}_L + b_A + \bar{b}_A \le 0$ $\tfrac{1}{2}k \ge b_A$	$b_A \cdot (k - b_A) \cdot \mu$

Proof of Theorem 5.2. The result follows from Theorem V.6.1 by setting X=A, $Y = L + R^*$, hence $\mu_X = \mu_A, \sigma_X = \sigma_A, \mu_Y = \mu_L + R^*, \sigma_Y = \sigma_L, a_X = a_A, b_X = b_A, a_Y = a_L, b_Y = b_L$. Under the given assumptions, one has always $\bar{A} = R^* + L_m$, $\underline{B} = R^* + L_M$, and Table 5.1 follows from a detailed analysis of the subcases (1) to (5) of the case (IIc) in Table V6.1.

<u>Case (1)</u> :

The expected value equation for R^* equals

$$\frac{\sigma_A + \sigma_L + (-a_A)(\mu_A - \mu_L - R^*)}{\bar{a}_A - a_A} = \mu_A - \mu_L$$

and has the solution

$$R^* = \bar{a}_A(\sigma_A + \sigma_L) - \bar{a}_A^2(\mu_A - \mu_L) = \bar{a}_A \cdot (k - \bar{a}_A) \cdot \mu .$$

The second condition, which must be fulfilled, reads

$$\frac{R^* - (\mu_A - \mu_L)}{\sigma_A + \sigma_L} \le \frac{1}{2}(a_A + \bar{a}_A).$$

Inserting R^* this is seen equivalent to $\frac{1}{2}k \le \bar{a}_A$. The fact that $\alpha_0^1 = R^* + \mu_L - \frac{1}{2}\sigma_L \cdot (a_A + \bar{a}_A) \in (R^* + L_m, R^* + L_M)$ is seen equivalent to the two conditions $a_A + \bar{a}_A + 2a_L < 0$, $a_A + \bar{a}_A + 2b_L > 0$. The first one is fulfilled because

$$a_A + \bar{a}_A + 2a_L < a_A + \bar{a}_A + a_L + \bar{a}_L \le 0,$$

and the second one because

$$a_A + \bar{a}_A + 2b_L > a_A + \bar{a}_A + b_L + \bar{b}_L \ge 0.$$

<u>Case (2)</u> :

From the expected value equation

$$\frac{\sigma_A + \sigma_L + \bar{a}_L(\mu_A - \mu_L - R^*)}{\bar{a}_L - a_L} = \mu_A - \mu_L,$$

one gets

$$R^* = (-a_L)(\sigma_A + \sigma_L) - a_L^2(\mu_A - \mu_L) = (-a_L) \cdot (k - (-a_L)) \cdot \mu .$$

One verifies that the second condition

$$\frac{\mu_A - \mu_L - R^*}{\sigma_A + \sigma_L} \le \frac{1}{2}(a_L + \bar{a}_L).$$

is equivalent to $\frac{1}{2}k \ge (-a_L)$. One shows that $\alpha_0^2 = \mu_A - \frac{1}{2}\sigma_A \cdot (a_L + \bar{a}_L) \in (R^* + L_m, R^* + L_M)$ if and only if

$$\frac{1}{2}\bar{a}_L \cdot \sigma_A < \mu_A - \mu_L < \frac{1}{2}\bar{a}_L \cdot \sigma_A + \left(\frac{b_L - a_L}{1 + a_L^2}\right) \cdot \sigma_L .$$

By the condition $\frac{1}{2}k \ge (-a_L)$, that is $\mu_A - \mu_L < \frac{1}{2}\bar{a}_L \cdot (\sigma_A + \sigma_L)$, the right hand side inequality is fulfilled provided

$$\mu_A - \mu_L \le \frac{1}{2}\bar{a}_L \cdot (\sigma_A + \sigma_L) < \frac{1}{2}\bar{a}_L \cdot \sigma_A + \left(\frac{b_L - a_L}{1 + a_L^2}\right) \cdot \sigma_L .$$

This holds provided $a_L + \bar{a}_L < 2b_L$. Since $a_L < \bar{b}_L < \bar{a}_L < b_L$ this is clearly fulfilled. A restatement of the left hand side inequality implies the condition $\tfrac{1}{2}k < (-a_L)\cdot(1+\frac{\sigma_L}{\sigma_A})$.

<u>Case (3)</u> :

Solving the expected value equation

$$\frac{1}{2}\left\{\sqrt{(\sigma_A+\sigma_L)^2+(\mu_A-\mu_L-R^*)^2}+(\mu_A-\mu_L-R^*)\right\}=\mu_A-\mu_L\,,$$

one finds

$$R^* = \frac{1}{4}\cdot\frac{(\sigma_A+\sigma_L)^2}{\mu_A-\mu_L} = \frac{1}{4}\cdot k^2\cdot\mu\,.$$

The two conditions about $\lambda_{X,Y} = \dfrac{R^* - (\mu_A - \mu_L)}{\sigma_A + \sigma_L}$ are shown to be equivalent with

(C1) $$\tfrac{1}{2}(a_A+\bar{a}_A) \le \tfrac{1}{2}\left\{\tfrac{1}{2}k + \overline{(\tfrac{1}{2}k)}\right\} \le \tfrac{1}{2}(b_A+\bar{b}_A)\,,$$

(C2) $$\tfrac{1}{2}(b_L+\bar{b}_L) \le \tfrac{1}{2}\left\{\tfrac{1}{2}k + \overline{(\tfrac{1}{2}k)}\right\} \le -\tfrac{1}{2}(a_L+\bar{a}_L)\,.$$

But, if $y > 0$ then $x + \bar{x} \le y + \bar{y}$ holds exactly when either $\bar{x} \le y$ if $x < 0$ or $x \le y$ if $x > 0$. Therefore these conditions are equivalent with $\max\{\bar{a}_A, -\bar{b}_L\} \le \tfrac{1}{2}k \le \min\{-a_L, b_A\}$. The condition $\alpha_0^3 = \dfrac{\mu_A\sigma_L + (\mu_L + R^*)\cdot\sigma_A}{\sigma_A+\sigma_L} \in \left(R^* + L_m, R^* + L_M\right)$ is equivalent with

$$\frac{R^*}{\sigma_A+\sigma_L} + a_L < \frac{\mu_A-\mu_L}{\sigma_A+\sigma_L} < \frac{R^*}{\sigma_A+\sigma_L} + b_L,$$

or, by inserting the value of R^*,

$$\frac{1}{4}\cdot\frac{\sigma_A+\sigma_L}{\mu_A-\mu_L} + a_L < \frac{\mu_A-\mu_L}{\sigma_A+\sigma_L} < \frac{1}{4}\cdot\frac{\sigma_A+\sigma_L}{\mu_A-\mu_L} + b_L.$$

Since $a_L < \tfrac{1}{2}(a_L + \bar{a}_L)$ and $\tfrac{1}{2}(b_L + \bar{b}_L) < b_L$, this is always fulfilled by (C2).

<u>Case (4)</u> :

The expected value equation

$$\frac{\sigma_A + \sigma_L + b_L(\mu_A - \mu_L - R^*)}{b_L - \bar{b}_L} = \mu_A - \mu_L,$$

implies that

$$R^* = (-\bar{b}_L)(\sigma_A+\sigma_L) - \bar{b}_L^2(\mu_A-\mu_L) = (-\bar{b}_L)\cdot(k - (-\bar{b}_L))\cdot\mu\,.$$

One shows that the second condition

$$\frac{\mu_A - \mu_L - R^*}{\sigma_A + \sigma_L} \le \frac{1}{2}(b_L + \overline{b}_L).$$

is equivalent to $\frac{1}{2}k \le (-\overline{b}_L)$. Now, one shows that $\alpha_0^4 = \mu_A - \frac{1}{2}\sigma_A \cdot \left(b_L + \overline{b}_L\right) \in \left(R^* + L_m, R^* + L_M\right)$ if and only if

$$b_L \cdot (\frac{1}{2}\sigma_A + \sigma_L) - \left(\frac{b_L - a_L}{1 + \overline{b}_L^2}\right) \cdot \sigma_L < \mu_A - \mu_L < \frac{1}{2} b_L \cdot (\sigma_A + \sigma_L) + \frac{1}{2} b_L .$$

By the condition $\frac{1}{2}k \le (-\overline{b}_L)$, that is $\frac{1}{2} b_L \cdot (\sigma_A + \sigma_L) \le \mu_A - \mu_L$, the left hand side inequality is fulfilled provided

$$\frac{1}{2} b_L < \frac{b_L - a_L}{1 + \overline{b}_L^2},$$

or equivalently $b_L + \overline{b}_L > 2a_L$, which always holds because $a_L < \overline{b}_L < \overline{a}_L < b_L$. A restatement of the right hand side inequality implies the condition $\frac{1}{2}k > (-\overline{b}_L) \cdot (\frac{\sigma_A + \sigma_L}{\sigma_A + 2\sigma_L})$.

Case (5) :

The expected value equation for R^* equals

$$\frac{\sigma_A + \sigma_L + (-\overline{b}_A)(\mu_A - \mu_L - R^*)}{b_A - \overline{b}_A} = \mu_A - \mu_L$$

and has the solution

$$R^* = b_A(\sigma_A + \sigma_L) - b_A^2(\mu_A - \mu_L) = b_A \cdot (k - b_A) \cdot \mu .$$

The second condition, which must be fulfilled, reads

$$\frac{R^* - (\mu_A - \mu_L)}{\sigma_A + \sigma_L} \ge \frac{1}{2}(b_A + \overline{b}_A).$$

Inserting R^* this is seen equivalent to $\frac{1}{2}k \ge b_A$. The fact that $\alpha_0^5 = R^* + \mu_L - \frac{1}{2}\sigma_L \cdot \left(b_A + \overline{b}_A\right) \in \left(R^* + L_m, R^* + L_M\right)$ is seen equivalent to the two conditions $b_A + \overline{b}_A + 2a_L < 0$, $b_A + \overline{b}_A + 2b_L > 0$. The first one is fulfilled because

$$b_A + \overline{b}_A + 2a_L < b_A + \overline{b}_A + a_L + \overline{a}_L \le 0,$$

and the second one because

$$b_A + \overline{b}_A + 2b_L > b_A + \overline{b}_A + b_L + \overline{b}_L \ge 0.$$

The proof of Theorem 5.1 is complete. ◊

Example 5.1 : maximal financial risk premium for a guaranteed random rate of return

It is possible to improve on Example 4.1. It is in general more realistic to assume that the accumulated rate of return, which should be guaranteed, also variies randomly through time, and can be represented by a random variable R_g with mean $r_g = E[R_g]$. Setting $A=R$, $r = E[R]$, $r_{min} \le R \le r_{max}$, $L = R_g$, $r_{g,min} \le R_g \le r_{g,max}$, $G = A - L$, the distribution-free bivariate modelling results of the present Section can be applied to this dual random environment.

6. Distribution-free safe layer-additive distortion pricing.

In global (re)insurance and financial markets, where often only incomplete information about risks is available, it is useful and desirable to use distribution-free pricing principles with the property that the prices of layers are safe. Since splitting in an arbitrary number of (re)insurance layers is world-wide widespread, it is important to construct distribution-free pricing principles, which are simultaneously layer-additive and safe for each layer.

The Hardy-Littlewood majorant $(X^*)^H$ of the stop-loss ordered maximal random variable X^* to an arbitrary risk $X \in D_2 = D_2([0,\infty);\mu,\sigma)$ with given range $[0,\infty)$ and known finite mean and variance has been defined in Section IV.2. Consider the modified simpler stochastic majorant $X^{**} \ge_{st} (X^*)^H$, as defined in the proof of Theorem IV.2.4. Then, apply a distribution-free implicit price loading method by setting prices at $H^{**}[X] = E[X^{**}]$, which is the expected value of the two-stage transform X^{**} of X. Here and in the following, notations of the type $H[\cdot]$ define and denote pricing principles, which are real functionals defined on some space of risks. As a main result, our Theorem 6.2 shows that the obtained *Hardy-Littlewood pricing* principle is both layer-additive and safe for each layer. As a consequence, the same property is shared by the class of distribution-free *distortion pricing* principles obtained setting prices at $H_g^{**}[X] = H_g[X^{**}] = \int_0^\infty g(\overline{F}^{**}(x))dx$, where $g(x)$ is an arbitrary increasing concave *distortion function* $g(x)$ with $g(0)=0$, $g(1)=1$, and $\overline{F}^{**}(x)$ is the survival function of X^{**}.

The proposed methodology is related to modern Choquet pricing theory (e.g. Chateauneuf et al.(1996)) and risk-neutral (distribution-free) valuation. In the more specific (re)insurance context, it is related to (**P**roportional **H**azard) *PH-transform pricing*, which has been justified on an axiomatic basis in Wang et al.(1997). Moreover, it fulfills the following traditional requirements. It uses only the first two moments of the risk. While it differs from classical economics utility theory, it does preserve the partial ordering of risks shared by all risk-averse decision makers. Furthermore, for Hardy-Littlewood pricing, no decision parameter must be evaluated, as is the case with traditional premium calculation principles (e.g. Goovaerts et al.(1984)). In general, however, a distortion function must be chosen, which will involve some decision rule.

Subsection 6.1 recalls elementary facts about layer-additive distortion pricing derived from Choquet pricing, and introduces the notions of layer safeness and distribution-free safe layer-additive pricing. Subsection 6.2 presents three applications of main interest. Theorem 6.1 describes a quite general distribution-free safe stop-loss distortion pricing principle. The layer safeness property of Hardy-Littlewood pricing, which implies distribution-free safe layer additive distortion pricing as explained above, is proved in Theorem 6.2. Finally, the Karlsruhe pricing principle, introduced by Heilmann(1987), turns out to be a valid linear approximation to the more sophisticated Hardy-Littlewood pricing principle provided the coefficient of variation of risks is sufficiently high.

6.1. Safe layer-additive distortion pricing.

Let (Ω, P, A) be a probability space such that Ω is the space of outcomes, A is the σ-algebra of events, and P is a probability measure on (Ω, A). For non-negative risks X, which are random variables defined on Ω taking values in $[0,\infty)$, one knows that the (special) *Choquet pricing* principle

$$H_g[X] = \int X d\mu = \int_0^\infty (g \circ P)\{X > x\} dx = \int_0^\infty g(\overline{F}(x))dx, \tag{6.1}$$

where the monotone set function $\mu = g \circ P$ is a so-called *distorted probability measure* of P by an increasing concave *distortion function* g with $g(0) = 0, g(1) = 1$, is *layer-additive* (e.g. Wang(1996), Section 4.1). In precise mathematical language, this property is described as follows. A *layer* at $(D, D+L]$ of a risk X is defined as the loss from an (excess-of-loss) insurance cover

$$I_{(D,D+L]}(X) = (X - D)_+ - (X - D - L)_+, \tag{6.2}$$

where D is called the *deductible*, and the width L is the maximal payment of this insurance cover and is called the *limit*. The expected value of this limited stop-loss reinsurance is denoted by $\pi_X(D, L)$ and, as difference of two stop-loss transforms, equals

$$\pi_X(D,L) = E\left[I_{(D,D+L]}(X)\right] = \pi_X(D) - \pi_X(D+L) = \int_D^{D+L} \overline{F}(x)dx. \tag{6.3}$$

In case $D + L \ge \sup\{X\}$ one recovers the stop-loss cover. A general pricing principle $H[\cdot]$ is called *layer-additive* if the property

$$H[X] = \sum_{k=0}^m H\left[I_{(D_k, D_{k+1}]}(X)\right] \tag{6.4}$$

holds for any splitting of X into layers at $(D_k, D_{k+1}]$, k=0,...,m, such that $0 = D_0 < D_1 < ... < D_m < D_{m+1} = \infty$. In particular, the m-th layer corresponds to a stop-loss cover. It is in this sense that the *distortion pricing* principle (6.1) is layer-additive. In this situation, note that the price of a layer at $(D, D+L]$ equals

$$H_g\left[I_{(D,D+L]}(X)\right] = \int_D^{D+L} g(\overline{F}(x))dx. \tag{6.5}$$

Under a *safe layer-additive pricing* principle, we mean a general pricing principle $H[\cdot]$ such that (6.4) as well as the following *layer safeness* criterion holds :

$$H\left[I_{(D,D+L]}(X)\right] \ge \pi^*(D,L) = \max_{Z \in D_n}\{\pi_Z(D,L)\}, \tag{6.6}$$

where $D_n = D_n([0,\infty);\mu_1,...,\mu_n)$ is the set of all non-negative random variables with given first n moments, and $X \in D_n$. In general, the distortion pricing principle (6.1) does not satisfy the layer safeness property, as will be shown through counterexample in the next Subsection. However, a pricing principle of type (6.1) and satisfying (6.6) is explicitely constructed for the simplest case n=2.

6.2. Distribution-free safe layer-additive pricing.

For use in (re)insurance and financial markets, where splitting in an arbitrary number of layers is widespread, we derive distribution-free pricing principles, which are both *layer-additive* and *safe* for each layer. We omit the lower index in $H_g[\cdot]$.

First of all, one observes that it is easy to construct distribution-free layer-additive distortion pricing principles. Since a distortion function g(x) is increasing, the sharp ordering relations (IV.2.10) induce a series of similarly ordered distribution-free pricing principles, which are defined and ordered as follows. For all $X \in D_n$, n=2,3,4, setting $H_u^{(n)}[X] := H[X_u^{(n)}]$, $H_*^{(n)}[X] := H[X_*^{(n)}]$, $H^{*(n)}[X] := H[X^{*(n)}]$, $H^{*H(n)}[X] := H[(X^{*H(n)}]$, $H_l^{(n)}[X] := H[X_l^{(n)}]$, one obtains from these relations the distribution-free pricing inequalities:

$$H_u^{(n)}[X] \le H_*^{(n)}[X] \le H_l^{(n)}[X],\ H_u^{(n)}[X] \le H^{*(n)}[X] \le H^{*H(n)}[X] \le H_l^{(n)}[X], \quad \text{for all } X \in D_n,\ n=2,3,4. \tag{6.7}$$

The crucial point is clearly the verification of the layer safeness property (6.6). In the special case of a splitting into two layers $(0,D]$, (D,∞), such that $Y = X-(X-D)_+$, $Z=(X-D)_+$ are the splitting components of X, one obtains immediately the following result.

Theorem 6.1. (*Distribution-free safe stop-loss distortion pricing*) Let $X \in D_n$, n arbitrary, and set $Y = X-(X-D)_+$, $Z=(X-D)_+$, $D \ge 0$. Moreover, let g(x) be an increasing concave distortion function such that g(0)=0, g(1)=1. Then the stop-loss ordered maximal random variable $X^{*(n)}$ for the set D_n with survival function $\overline{F}^{*(n)}(x)$ defines a distribution-free distortion pricing principle through the formulas

$$\begin{aligned} H^{*(n)}[X] &:= H[X^{*(n)}] = \int_0^\infty g(\overline{F}^{*(n)}(x))dx, X \in D_n, \\ H^{*(n)}[Z] &:= H[(X^{*(n)}-D)_+] = \int_D^\infty g(\overline{F}^{*(n)}(x))dx, \\ H^{*(n)}[Y] &= H^{*(n)}[X] - H^{*(n)}[Z]. \end{aligned} \tag{6.8}$$

Furthermore, this pricing principle is stop-loss safe such that

$$H^{*(n)}[Z] := \int_D^\infty g(\overline{F}^{*(n)}(x))dx \ge \int_D^\infty \overline{F}^{*(n)}(x)dx = \pi_{X^{*(n)}}(D) = \pi^{*(n)}(D). \tag{6.9}$$

Proof. Since $H^{*(n)}[\cdot]$ is a distortion pricing principle, the layer-additive property $H^{*(n)}[X] = H^{*(n)}[Y] + H^{*(n)}[Z]$ is clearly satisfied, which shows the third relation. By assumption on the distortion function, one has $g(x) \ge x$ for all $x \in (0,1)$, which implies immediately (6.9). ◊

In general, the distribution-free distortion pricing principle induced by $X^{*(n)}$ will not satisfy the layer safeness property (6.6). Over the space D_2, a simple counterexample is PH-transform pricing with $g(x) = x^{\frac{1}{\rho}}$, $\rho \ge 1$. For a layer at $(D, D+L]$ with $D+L<\mu$, one has $H[I_{(D,D+L]}(X^*)] = (1+k^2)^{-\frac{1}{\rho}} \cdot L < L = \pi^*(D,L)$ (see the table in the proof of Theorem 6.2). However, as we will show, distortion pricing induced by the modified Hardy-Littlewood majorant X^{**} of X^* over the set D_2 does fulfill it. Note that the Chebyshev-Markov majorant $X_l^{(2)}$ can be ruled out. Indeed, since $H_l^{(2)}[X] \ge H^{**}[X]$ uniformly for all $X \in D_2$, the Chebyshev-Markov price is not enough competitive.

Theorem 6.2. (*Distribution-free safe layer-additive distortion pricing*) Let $D_2 = D_2([0,\infty); \mu, \sigma)$ be the set of all random variables X with finite mean μ and standard deviation σ, and let X^{**} be the modified Hardy-Littlewood majorant of X^* over D_2 with survival function

$$\overline{F}^{**}(x) = \begin{cases} 1, & x < (1+k^2)\mu, \\ \dfrac{\sigma^2}{\sigma^2 + (x-\mu)^2}, & x \ge (1+k^2)\mu, \end{cases} \tag{6.10}$$

where $k = \frac{\sigma}{\mu}$ is the coefficient of variation. Then the layer safeness property holds, that is

$$\begin{aligned} & H^{**}[I_{(D,D+L]}(X)] := H[I_{(D,D+L]}(X^{**})] = \int_D^{D+L} g(\overline{F}^{**}(x))dx \\ & \ge \int_D^{D+L} \overline{F}^{**}(x)dx \ge \pi^*(D,L) = \max_{X \in D_2}\{\pi_X(D,L)\} \end{aligned} \tag{6.11}$$

for all $X \in D_2$, all $L, D \ge 0$, all increasing concave distortion functions $g(x)$ such that $g(0)=0$, $g(1)=1$. The uniformly lowest pricing formula obtained for the "risk-neutral" distortion function $g(x)=x$ induces a so-called *Hardy-Littlewood pricing* principle.

Proof. Since $g(x) \ge x$ for all $x \in [0,1]$, only the last inequality in (6.11) must be verified. From Table II.5.3 one obtains the maximum expected layer in the following tabular form :

case	condition	maximum $\pi^*(D,L)$
(1)	$D+L<\mu$	L
(2)	$\mu \le D+L \le (1+k^2)\cdot\mu$	$\left(\dfrac{L}{D+L}\right)\cdot\mu$
(3)	$D+L \ge (1+k^2)\cdot\mu$	
(3a)	$D \le \frac{1}{2}(1+k^2)\cdot\mu$	$\mu - \dfrac{D}{1+k^2}$
(3b)	$\frac{1}{2}(1+k^2)\cdot\mu \le D \le \mu + \frac{1}{2}(D+L-\mu-\frac{\sigma^2}{D+L-\mu})$	$\frac{1}{2}\left\{\sqrt{\sigma^2+(D-\mu)^2} - (D-\mu)\right\}$
(3c)	$D \ge \mu + \frac{1}{2}(D+L-\mu-\frac{\sigma^2}{D+L-\mu})$	$\dfrac{\sigma^2}{\sigma^2+(D+L-\mu)^2}\cdot L$

In the following set $\pi^{**}(D,L) := \int_D^{D+L} \overline{F}^{**}(x)dx$ for the expected layer valued with the Hardy-Littlewood majorant (6.10). The verification of the inequality $\pi^{**}(D,L) \ge \pi^{*}(D,L)$ is done case by case.

Case (1) : $D+L<\mu$

One has $\pi^{**}(D,L) = \int_D^{D+L} 1 \cdot dx = L = \pi^{*}(D,L)$.

Case (2) : $\mu \le D+L \le (1+k^2)\cdot\mu$

One has $\pi^{**}(D,L) = L \ge \left(\dfrac{\mu}{D+L}\right)\cdot L = \pi^{*}(D,L)$.

Case (3) : $D+L \ge (1+k^2)\cdot\mu$

Case (3a) : $D \le \frac{1}{2}(1+k^2)\cdot\mu$

A calculation shows that

$$\pi^{**}(D,L) - \pi^{*}(D,L) \ge \int_D^{D+L} 1 \cdot dx - \pi^{*}(D,L) = k^2 \cdot \left(\mu - \frac{D}{1+k^2}\right) \ge \frac{1}{2}k^2\mu > 0 .$$

Case (3b) : $\frac{1}{2}(1+k^2)\cdot\mu \le D \le \mu + \frac{1}{2}(D+L-\mu-\dfrac{\sigma^2}{D+L-\mu})$

The right hand side condition is equivalent with the inequality $L \ge \sqrt{\sigma^2+(D-\mu)^2}$. Set $\alpha = \left(\dfrac{D-\mu}{\sigma}\right)$ and distinguish between two subcases.

Subcase (i) : $\alpha \ge 0$

Since $D+L-\mu \ge \sigma\cdot(\alpha+\sqrt{1+\alpha^2})$, one obtains without difficulty that $\pi^{**}(D,L) \ge \sigma \cdot \int_\alpha^{\alpha+\sqrt{1+\alpha^2}} \dfrac{dz}{1+z^2} = \sigma \cdot \left\{\arctan(\alpha+\sqrt{1+\alpha^2}) - \arctan(\alpha)\right\}$.

Using the trigonometric difference formula $\arctan(x) - \arctan(y) = \arctan\left(\dfrac{x-y}{1+xy}\right)$, one obtains further $\pi^{**}(D,L) \ge \sigma\cdot\arctan(\sqrt{1+\alpha^2}-\alpha)$. Since $\arctan(x) \ge \frac{1}{2}x$ for $x^2 \le 1$, this implies for $\alpha \ge 0$ that $\pi^{**}(D,L) \ge \frac{1}{2}\sigma\cdot(\sqrt{1+\alpha^2}-\alpha) = \pi^{*}(D,L)$.

Subcase (ii) : $\alpha < 0$

Since $D < \mu$ one obtains

$$\pi^{**}(D,L) = \int_D^{(1+k^2)\mu} dx + \int_{(1+k^2)\mu}^{D+L} \frac{\sigma^2 dx}{\sigma^2 + (x-\mu)^2} \geq (1+k^2)\mu - D + \sigma \cdot \int_k^{\alpha+\sqrt{1+\alpha^2}} \frac{dz}{1+z^2}$$
$$= \sigma \cdot \left\{ [k - \arctan(k)] + \arctan(\alpha + \sqrt{1+\alpha^2}) - \alpha \right\}.$$

Since $\arctan(k) \leq k$ and $\arctan(\alpha + \sqrt{1+\alpha^2}) \geq \frac{1}{2}(\alpha + \sqrt{1+\alpha^2})$ for $\alpha < 0$, one concludes as in subcase (i).

Case (3c) : $D \geq \mu + \frac{1}{2}(D + L - \mu - \frac{\sigma^2}{D+L-\mu})$

One has immediately

$$\pi^{**}(D,L) \geq \int_D^{D+L} \frac{\sigma^2 dx}{\sigma^2 + (x-\mu)^2} \geq \int_D^{D+L} \frac{\sigma^2 dx}{\sigma^2 + (D+L-\mu)^2} = \frac{\sigma^2}{\sigma^2 + (D+L-\mu)^2} \cdot L = \pi^*(D,L).$$

This completes the proof of Theorem 6.2. ◊

A close look at the Hardy-Littlewood price appears to be instructive. For all $X \in D_2([0,\infty);\mu,\sigma)$ one has

(6.12)
$$H^{**}[X] := E[X^{**}] = (1+k^2)\mu + \sigma \cdot \int_k^{\infty} \frac{dz}{(1+z^2)}$$
$$= (1+k^2)\mu + \left(\frac{\pi}{2} - \arctan(k)\right) k \cdot \mu,$$

which always exceeds the *Karlsruhe price* $(1+k^2)\mu$, introduced by Heilmann(1987).

The order of magnitude of (6.12) is obtained using the pricing expansions

(6.13)
$$H^{**}[X] = \begin{cases} (1 + \frac{\pi}{2}k + \frac{1}{3}k^4 - \frac{1}{5}k^6 + \frac{1}{7}k^8 \mp \ldots) \cdot \mu, & \text{if } k \leq 1, \\ (1 + k^2 - \frac{1}{3}\lambda^2 + \frac{1}{5}\lambda^4 - \frac{1}{7}\lambda^6 \pm \ldots) \cdot \mu, & \text{if } \lambda = \frac{1}{k} \leq 1, \end{cases}$$

which are obtained from series expansions of the function $\arctan(x)$.

Clearly only empirical work about distribution-free pricing can decide upon which formula should be most adequate in real-life situations. Let us illustrate with an empirical study by Lemaire and Zi(1994), p. 292, which have obtained a coefficient of variation of the average order $k=6.4$ for the aggregate claims distribution of a non-life business, which has been fitted by means of a compound Poisson distribution with a lognormal claim size density. In this situation the Hardy-Littlewood price (6.12) differs from $(1+k^2)\mu = 41.96$ by the relatively small amount $(0.99)\mu$, that is approximately 2% relative error. Truncating the expansion (6.13) at the quadratic term in λ, the difference is a negligible $(0.008)\mu$.

It is remarkable that quite different theoretical explanations can lead to very similar answers. To sum up, we have obtained a rational justification of the fact that Karlsruhe pricing is a valid *linear approximation* to distribution-free safe layer-additive pricing via

Hardy-Littlewood pricing in the situation of "large" risks, defined here as risks with high volatility or coefficient of variation, as suggeted by Heilmann(1987).

To conclude, let us close the circle of our short excursion on applications. If the Karlsruhe price is viewed as stable price in the sense of Section 1, than the corresponding distribution-free probability of loss, by given mean and variance, will be less than ε provided $k \geq \sqrt{(1-\varepsilon)\cdot\varepsilon^{-1}}$, in accordance with the notion of large risk in Actuarial Science. For comparison, if X is lognormally distributed with parameters ν, τ, hence $1+k^2 = \exp(\tau^2)$, than the probability of loss of the Karlsruhe price equals $\overline{F}_X(P = (1+k^2)\cdot\mu) = \overline{N}(\frac{3}{2}\tau)$, with $N(x)$ the standard normal distribution. This is less than ε provided $\tau \geq \frac{2}{3}\cdot N^{-1}(1-\varepsilon)$, in accordance with the notion of high volatility in Finance.

7. Notes.

In a distribution dependent context, the stability criterion defines the so-called ε-percentile pricing principle (see e.g. Goovaerts et al.(1984), Heilmann(1987b)). The stable pricing principle, interpreted as distribution-free percentile principle, has also been considered in Hürlimann(1993b), where attention is paid to the link between reinsurance and solvability (see also Hürlimann(1995a)). The actuarial interest in the Chebyshev-Markov extremal distributions has been pointed out in Kaas and Goovaerts(1985).

The optimal mean self-financing portfolio insurance strategy has been discussed first in Hürlimann(1994b) (see also Hürlimann(1996b/98b)). Further informations about the Dutch pricing principle are found in Heerwaarden and Kaas(1992), and Hürlimann(1994c/95a/d/e). There are several reasons to call (2.3) special Dutch pricing principle. The name is an allusion to a paper by Benktander(1977). The actuarial relevance of this choice is quite significant. Besides the given interpretation as minimal price of a mean self-financing strategy, this choice satisfies several other remarkable properties and characterizations found in the mentioned papers above. Moreover, according to Borch(1967), the loading functional $E\left[(X-\mu)_+\right]$ is a quite old measure of risk associated to an insurance contract, which has been considered by Tetens(1786), who defined risk as expected loss to an insurance company given the insurance contract leads to a loss. The fundamental identity of portfolio insurance can be generalized to include more complex so-called "perfectly hedged" reinsurance and option strategies (see Hürlimann(1994a/c/d,1995b)).

As a mathematical discipline, Risk Theory is a quite recent subject. After the pioneering work by Cramér(1930/55) and surveys by Segerdhal(1959) and Borch(1967), the first books on this subject are Seal(1969), Beard et al.(1969) and Bühlmann(1970). From the second title, there has been three new editions by Beard et al.(1977/84) and Daykin et al.(1994). At present there exist an increasing number of books and monographs dealing with whole or parts of this today widely enlarged topic. Among others, let us mention in chronological order Seal(1978), Gerber(1979), Goovaerts et al.(1984/90), Hogg and Klugman(1984), Sundt(1984/91/93), Kremer(1985), Bowers et al.(1986), Heilmann(1987a), Straub(1988), Drude(1988), Hipp and Michel(1990), Grandell(1991), Panjer and Willmot(1992), Kaas et al.(1994), De Vylder(1996), Embrechts et al.(1997), Mack(1997).

Section 3 is based on Hürlimann(1996a). By given range, mean and variance, the obtained lower and upper bounds for stop-loss premiums and ruin probabilities in Section 3 are tighter than those in Steenackers and Goovaerts(1991), which are based on extremal random variables with respect to the dangerousness order relation constructed in Kaas and

Goovaerts(1986). In the recent actuarial literature the concolution formula (3.8) is often attributed to Beekman(1985) (e.g. Hipp and Michel(1990), p.169). However, this classical formula about the ladder heights of random walk is much older and known as Khintchin-Pollaczek formula in Probability Theory. The linear representation (3.11) for a compound Poisson random variable is found in many places, for example in Gerber(1979), chap. 1.7, Bowers et al.(1986), Theorem 11.2, Hürlimann(1988), Hipp and Michel(1990), p. 27-30. An early generalization is Jànossy et al.(1950) (see also Aczél and Dhombres(1989), Chapter 12), and a more recent development has been made by the author(1990a), which led to a novel application in Hürlimann(1993d). The formulas (3.12), (3.13) are in Kaas(1991) and Kaas et al.(1994), Chapter XI. According to Bühlmann(1996), the handy formula (3.16) solves the most famous classical actuarial optimization problem. The corresponding maximizing diatomic claim size random variable can be interpreted as the safest risk with fixed mean and finite range (see Bühlmann et al.(1977) and Kaas et al.(1994), Example III.1.2). Schmitter's original problem has been discussed in Brockett et al.(1991) and Kaas(1991). The modified problem has been considered first in Hürlimann(1996a). The most recent contributions are by De Vylder et al.(1996a/b/c). Section 3.5 is related to findings of Benktander(1977), as explained in Hürlimann(1996a). The method of Section 3 is more widely applicable. A possible use in life insurance is exposed in Hürlimann(1997l).

Further information and references to the actuarial literature about the topic of "excess-of-loss reserves" is found in Hürlimann(1998b), which contains in particular the results presented in Sections 4 and 5. Some statistical knowledge about the coefficient of variation has been collected in Hürlimann(1997c). In the special case of an infinite range $(-\infty, \infty)$ for the financial gain, the maximal excess-of-loss reserve has been derived earlier by the author(1990b) (see also author(1991a), (4.12), author(1992a), (3.11) and author(1992b), (3.1)).

Section 6 follows closely Hürlimann(1997d). Theorem 6.1 is in the spirit of the insurance market based distribution-free stop-loss pricing principle presented first in Hürlimann(1993a), and later refined in Hürlimann(1994a), Theorem 5.1. The Karlsruhe pricing principle can been shown plausible on the basis of several other arguments, as exposed in Hürlimann(1997f). The given interpretation is compatible with the fact that Karlsruhe pricing can be derived from the *insurance CAPM* (read Capital Asset Pricing Model), which can be viewed as a *linear approximation* to an arbitrage-free insurance pricing model (see Hürlimann(1997g)).

Distribution-free methods and results in Actuarial Science and Finance are numerous. A very important subject, not touched upon here, is Credibility Theory, going back to Whitney(1918), whose modern era has been justified by Bühlmann(1967) (see De Vylder(1996), Part III, and references). Interesting results by working actuaries include Schmitter(1987) and Mack(1993). Another illustration for the use of bounds and optimization results in Risk Theory is Waldmann(1988).

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"Mathematics is never lost, it is always used. And it will always be used, the same mathematics; once it's discovered and understood, it will be used forever. It's a tremendous resource in that respect, and it's not one that we should neglect to develop."

Andrew Wiles

AUTHOR INDEX

LIST OF TABLES

LIST OF SYMBOLS AND NOTATIONS

Chapter I

$\mu_1,\ldots,\mu_n$	moments of a random variable (r.v.)
$\mu, \sigma^2, \gamma, \gamma_2$	mean, variance, skewness and kurtosis of a r.v.
$\mu_3, \delta=\mu_4$	third and fourth order central moment of a standard r.v. (s.r.v.)
$\gamma=\mu_3, \gamma_2=\delta-3$	skewness and kurtosis of a s.r.v.
$\Delta=\delta-(\gamma^2+1)$	excess of kurtosis to skewness of a s.r.v.
$\{x_1,\ldots,x_n\}$	support .
$\{p_1^{(n)},\ldots,p_n^{(n)}\}$	probabilities of a finite n-atomic r.v.
D(a,b)	set of all s.r.v. with range $[a,b]$
$D_k^{(n)}(a,b)$ or $D^{(n)}(a,b;\mu_3,\ldots,\mu_k)$	set of all n-atomic s.r.v. with range $[a,b]$ and known moments up to the order k
$D_{k,m}^{(n)}(a,b)$	a s.r.v. in $D_k^{(n)}(a,b)$ with m atoms fixed
$\{\ldots\}$	set with the properties ...
$\cong$	isomorphism between sets
$\overline{x}=-x^{-1}$	involution mapping a non-zero x to its negative inverse
$c=\frac{1}{2}(\gamma-\sqrt{4+\gamma^2})$, $\overline{c}=\frac{1}{2}(\gamma+\sqrt{4+\gamma^2})$	zeros of the standard quadratic orthogonal polynomial associated to a s.r.v.
$S_2(a,b)=\{[a,\overline{b}];^-\}$	algebraic set isomorphic $D_2^{(2)}(a,b)$
$\varphi(x,y)=\dfrac{\gamma-(x+y)}{1+xy}$	third atom of a standard triatomic r.v. with support $\{x,y,\varphi(x,y)\}$
$S_3(a,b)=\{[a,c]x[\overline{c},b];\varphi\}$	algebraic set isomorphic $D_3^{(3)}(a,b)$
x^*	involution mapping an atom of a standard triatomic r.v. to another one by known skewness and kurtosis
$S_4(a,b)=\{[a,b^*];\varphi,^*\}$	algebraic set isomorphic $D_4^{(3)}(a,b)$
$D_S(a)$	set of all symmetric s.r.v. with range $[a,b]$
$D_{S,2k}^{(n)}(a)$ or $D_S^{(n)}(a;\mu_4,\ldots,\mu_{2k})$	set of all n-atomic symmetric s.r.v. with range $[a,b]$ and known moments of even order up to the order 2k
$D_{S,2k,2m}^{(n)}(a)$	a symmetric s.r.v. in $D_{S,2k}^{(n)}(a)$ with 2m atoms fixed

Chapter II

$X=\{u,v\},\{u,v,w\}$	s.r.v. X with support $\{u,v\}$, $\{u,v,w\}$
$\ell_i(x)$	linear function
$\nabla_{ij}\ell(x)=\ell_j(x)-\ell_i(x)$	backward functional operator
$d_{ij}=d_{ji}$	abscissa of intersection point of two non-parallel $\ell_j(x)$, $\ell_i(x)$
$\overline{F}(x)$, $\pi(x)$	survival function and stop-loss transform of a r.v.
$I_E(x)$	indicator function of an event E

Chapter III

$F_l(x), F_u(x)$	Chebyshev-Markov extremal standard distributions (refined notation below)
$\pi_*(x), \pi^*(x)$	minimal and maximal stop-loss transform of a s.r.v.
$d_i(x)$	deductible functions

Chapter IV

$\le_{st}$	(usual) stochastic order or stochastic dominance of first order
$\le_{sl}$ or $\le_{icx}$	stop-loss order or increasing convex order
$\le_{sl,=}$ or $\le_{cx}$	stop-loss order by equal means or convex order
$\le_D$	dangerousness order or once-crossing condition
$\le_{D,=}$	dangerousness order by equal means
$D_n = D_n([a,b];\mu_1,...,\mu_n)$	set of all r.v. with range $[a,b]$ and known moments up to the order n
$F_l^{(n)}(x), F_u^{(n)}(x)$	Chebyshev-Markov extremal standard distributions over D_n
$X_l^{(n)}, X_u^{(n)}$	r.v. with distributions $F_l^{(n)}(x), F_u^{(n)}(x)$
$\pi_*^{(n)}(x), \pi^{*(n)}(x)$	extremal stop-loss transforms over D_n
$F_*^{(n)}(x), F^{*(n)}(x)$	extremal stop-loss ordered distributions over D_n
$X_*^{(n)}, X^{*(n)}$	r.v. with distributions $F_*^{(n)}(x), F^{*(n)}(x)$
X^{**}	Hardy-Littlewood majorant of $X^* := X^{*(n)}$
F^{**}	distribution of X^{**}

Chapter V

$BD_i, i = 1,2,3$	sets of bivariate distributions
$BD_3^{(2)} \subseteq BD_3$	subset of diatomic couples
$H_u(x,y)$	bivariate Chebyshev-Markov maximal distribution over BD_3
$H_*(x,y), H^*(x,y)$	Hoeffding-Fréchet bivariate extremal distributions
BD(F,G)	set of bivariate r.v. with fixed marginals

Chapter VI

$H[\cdot]$	pricing principle
$\theta_\varepsilon^{(n)}$	ε-percentile of $F_l^{(n)}(x)$
$\pi_\varepsilon^{(n)}(d)$	stable stop-loss price to the deductible d
$H_*[\cdot], H^*[\cdot]$	extremal pricing principles
$\psi(u)$	probability of ruin to the initial reserve u
$A = \{A(t)\}$	stochastic process of assets
$L = \{L(t)\}$	stochastic process of liabilities
$R = \{R(t)\}$	stochastic process of the excess-of-loss reserve
$G = A - L$	stochastic process of financial gain
$V = L - A$	stochastic process of financial loss
$G_+ = V_-, V_+ = G_-$	stochastic process of positive gain (=negative loss) and positive loss (=negative gain)

$U = (R - G)_+$	stochastic process of excess-of-loss
$D = (G - R)_+$	stochastic process of excess-of-gain
$NO = G - D$	stochastic process of net outcome
$R^* = \min\{B, G_+\}$	stochastic process of stable excess-of-loss reserve
$H^{**}[\cdot]$	Hardy-Littlewood pricing principle
$H_g[\cdot]$	distortion pricing principle
$H_g^{**}[\cdot]$	Hardy-Littlewood distortion pricing principle
$\pi(D, L)$	limited stop-loss transform
$\pi^*(D, L)$	maximum limited stop-loss transform

SUBJECT INDEX I : MATHEMATICS AND STATISTICS

SUBJECT INDEX II : ACTUARIAL SCIENCE AND FINANCE