A FAST IMFES FORMULATION FOR SOLVING 1D THREE-PHASE BLACK-OIL EQUATIONS

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Abstract. A new approach for solving the flux equations governing a three-phase black-oil model in porous media is proposed. The approach consists of solving the total flux implicitly and the saturations explicitly. This formulation avoids solving a costly second order differential equation in pressure. In the proposed approach, the total flux is expressed as an asymptotic expansion of ascending powers of the total fluid compressibility. Contributions to the total flux are obtained from solving first order differential equations (gradient and divergence operators). Discretizing these operators by finite differences, the resulting linear system coefficients are fixed during the whole simulation.

Resumen. Se propone nuevo enfoque para la solución de las ecuaciones de flujo que gobiernan el modelo de tres fases de petróleo en medios porosos. El enfoque consiste en resolver el total de flujo de forma implícita y la saturación de forma explícita. De esta forma se evita la solución de una costosa ecuación diferencial de segundo orden en la presión. En el enfoque propuesto, el flujo total se expresa como una expansión asintótica de potencias crecientes de la compresibilidad total del fluido. Las contribuciones al flujo total se obtienen de la resolución de ecuaciones diferenciales de primer orden (los operadores gradiente y divergencia). Discretizando los operadores por diferencias finitas, resulta en un sistema lineal de coeficientes fijos durante toda la simulación.

1 Introduction

Nowadays, there is an increasing demand for performing fast reservoir simulation studies. This has been mainly driven by the need to model larger reservoirs (consisting of millions of gridblocks) and to perform as many realizations as possible in order to quantify the uncertainty associated to the exploitation plans (see [1,3,4,8,9]). Parallel computing strategies have been an important vehicle to achieve this goal. On the other hand, new numerical paradigms may provide an alternative solution to this problem. An approach for solving the flux in porous media equations governing a three-phase black-oil model is proposed. The approach consists of solving the total flux implicitly and the saturations explicitly. Pressures are then computed directly from the total flux without incurring in a significant additional cost. In contrast to more conventional approaches (see [1,2,3,4]), this formulation avoids solving a costly second order differential equation in pressure. Compared to streamline codes, this formulation includes solubility of gas, gravity and capillary pressures. In fact, solutions are not restricted to lower dimensional domains (streamtubes). In the proposed approach, the total flux is expressed as an asymptotic expansion of ascending powers of the total fluid compressibility. Contributions to the total flux are obtained from solving first order differential equations (gradient and divergence operators). Discretizing these operators by finite differences, the resulting linear system coefficients are fixed during the whole simulation. The method is amenable for carrying out only one matrix factorization at the beginning of the simulation, allowing to reduce the number of floating point operations for solving the associated linear system in each time step.

2 The fluid flow equations

Darcy's law states that the volumetric flow rate Q of a homogeneous fluid through a porous medium is proportional to the pressure (or hydraulic gradient) and to the cross-sectional area A normal to the direction of flow and inversely proportional to the viscosity μ of the fluid (see [1,2,3,4]). The law defines a concept of permeability K of the rock, which quantifies the ability of the rock to transmit fluid. We can write the superficial fluid velocity of the phase l (Darcy velocity u_l) with $l = o, w, g$ (oil, water and gas respetively) by

$$u_l \;=\; K_x \,\frac{K_{rl}}{\mu_l}\,\frac{\partial P_l}{\partial x}\ , \tag{1}$$

where P_l is the fluid pressure for phase l in psi, μ_l is the viscosity of phase l in cp, K_{rl} is the relative permeability of phase l between 0 and 1, and K_x is an absolute permeability in units of milidarcies (length squared).

The mathematical models for displacement process (1 dimensional 3 phases) considered in this work, equations (2) below, are partial differential equations of convection-diffusion type, under the asumption of no gas solubility in water (i.e. $R_{sw} = 0$).

That is,

$$\frac{\partial}{\partial x}[K_x(\frac{\lambda_o}{B_o})\frac{\partial P_o}{\partial x}] - \frac{q_o}{\rho_{osc}} = \frac{\partial}{\partial t}(\phi\frac{S_o}{B_o}) \quad (2A)$$

$$\frac{\partial}{\partial x}[K_x(\frac{\lambda_w}{B_w})\frac{\partial P_w}{\partial x}] - \frac{q_w}{\rho_{wsc}} = \frac{\partial}{\partial t}(\phi\frac{S_w}{B_w}) \quad (2B)$$

$$\frac{\partial}{\partial x}[K_x(\frac{\lambda_g}{B_g})\frac{\partial P_g}{\partial x} + R_{so}K_x(\frac{\lambda_o}{B_o})\frac{\partial P_o}{\partial x}] - \frac{q_g}{\rho_{gsc}} = \frac{\partial}{\partial t}[\phi(\frac{S_g}{B_g} + R_{so}\frac{S_o}{B_o})], \quad (2C)$$

with the mobility of the phase i $\lambda_i = K_{ri}/\mu_i$, for $i = o, w, g$.

The system (2) is taken to hold over a bounded domain Ω, representing a reservoir or part of a reservoir, through a time interval $[0, T]$. The equations (2A-2C) are coupled via the constraints

$$S_o + S_w + S_g = 1, \quad (2D)$$

$$P_{cow} = P_o - P_w \quad \text{and} \quad P_{cgo} = P_g - P_o , \quad (2E)$$

where S_i, $i = o, w, g$ are saturations for each phase, P_{cow} and P_{cgo} the oil-water and gas-oil capillary pressures, respectively.

Phase densities are related to formation volume factors and gas solubilities by

$$\rho_o = \frac{1}{B_o}[\rho_{osc} + R_{so}\rho_{gsc}] , \quad (2F)$$

$$\rho_w = \frac{\rho_{wsc}}{B_w} , \quad (2G)$$

$$\rho_g = \frac{\rho_{gsc}}{B_g} . \quad (2H)$$

The system (2) requires boundary and initial conditions. In reservoir modeling, the usual boundary conditions are no-flow, representing an axis of simmetry or an impermeable boundary, or constant pressure, applicable when the reservoir is repressurized at the boundary during depletion, say by an aquifer.

The oil, water, gas and rock compressibilities are defined as

$$c_o = -\frac{1}{B_o}\frac{\partial B_o}{\partial P_o} + \frac{B_g}{B_o}\frac{\partial R_{so}}{\partial P_o}, \quad (2I)$$

$$c_w = -\frac{1}{B_w}\frac{\partial B_w}{\partial P_o}, \quad (2J)$$

$$c_g = -\frac{1}{B_g}\frac{\partial B_g}{\partial P_o}, \quad (2K)$$

$$c_r \;=\; -\frac{1}{\phi}\frac{\partial \phi}{\partial P_o}, \qquad (2L)$$

and the total compressibility

$$c_t \;=\; c_r + c_o S_o + c_w S_w + c_g S_g \;. \qquad (2M)$$

Definition 1: From the diffusion terms from equations (2A) to (2C), let define L_o, L_w and L_g for the oil, water and gas fases as follows

$$L_o \;=\; \frac{\partial}{\partial x}[K_x(\frac{\lambda_o}{B_o})\frac{\partial P_o}{\partial x}], \quad L_w \;=\; \frac{\partial}{\partial x}[K_x(\frac{\lambda_w}{B_w})\frac{\partial P_w}{\partial x}],$$

$$L_g \;=\; \frac{\partial}{\partial x}[K_x(\frac{\lambda_g}{B_g})\frac{\partial P_g}{\partial x}]. \qquad (3)$$

Observation: The oil, water and gas fluid velocities, according to equations (1), (2E) and the definitions of λ_o, λ_w and λ_g, can be rewritten as follows

$$u_o \;=\; K_x \,\frac{K_{ro}}{\mu_o}\,\frac{\partial P_o}{\partial x} = K_x\,\lambda_o\,\frac{\partial P_o}{\partial x}\;, \qquad (4A)$$

$$u_w \;=\; K_x \,\frac{K_{rw}}{\mu_w}\,\frac{\partial P_w}{\partial x} = K_x\,\lambda_w\,\frac{\partial P_w}{\partial x} = K_x\,\lambda_w\,[\frac{\partial P_o}{\partial x} - \frac{\partial P_{cow}}{\partial x}]\;, \qquad (4B)$$

$$u_g \;=\; K_x \,\frac{K_{rg}}{\mu_g}\,\frac{\partial P_g}{\partial x} = K_x\,\lambda_g\,\frac{\partial P_g}{\partial x} = K_x\,\lambda_g\,[\frac{\partial P_o}{\partial x} + \frac{\partial P_{cgo}}{\partial x}]\;. \qquad (4C)$$

Lemma 1: L_o, L_w and L_g can be expressed in function of u_o, u_w and u_g the oil, water and gas fluid velocities respectively, by

$$L_o \;=\; \frac{\partial}{\partial x}[\frac{1}{B_o}u_o], \quad L_w \;=\; \frac{\partial}{\partial x}[\frac{1}{B_w}u_w], \quad L_g \;=\; \frac{\partial}{\partial x}[\frac{1}{B_g}u_g + \frac{R_{so}}{B_o}u_o]\;. \qquad (5)$$

Proof: it follows easily from (3), (1) and the definition of phase mobility λ_i. $\diamond$

Definition 2: Let define the total flow velocity u_t by

$$u_t = u_o + u_w + u_g \qquad (6)$$

where u_o, u_w and u_g are the oil, water and gas fluid velocities respectively.

Definition 3: Let define the total mobility λ_t by

$$\lambda_t = \lambda_o + \lambda_w + \lambda_g \qquad (7)$$

where λ_o, λ_w and λ_g are the oil, water and gas fluid mobilities repectively.

Lemma 2: The total flow velocity u_t can be expressed as

$$u_t \; = \; K_x \lambda_t \frac{\partial P_o}{\partial x} + K_x (\lambda_g \frac{\partial P_{cgo}}{\partial x} - \lambda_w \frac{\partial P_{cow}}{\partial x}) \; . \tag{8}$$

Proof: it follows easily from the definition of the total velocity u_t, the total mobility λ_t and equations (4). $\diamond$

Lemma 3: The oil pressure gradient $\frac{\partial P_o}{\partial x}$ can be expressed as

$$\frac{\partial P_o}{\partial x} \; = \; \frac{1}{\lambda_t} {K_x}^{-1} [u_t - K_x (\lambda_g \frac{\partial P_{cgo}}{\partial x} - \lambda_w \frac{\partial P_{cow}}{\partial x})] \; . \tag{9}$$

Proof: Equation (9) for the oil pressure gradiente follows directly from (8). $\diamond$

Lemma 4: The u_o, u_w and u_g oil, water and gas fluid velocities respectively can be expressed as

$$u_o \; = \; \frac{\lambda_o}{\lambda_t} u_t + C_{og} + C_{ow} \; , \tag{10}$$

$$u_w \; = \; \frac{\lambda_w}{\lambda_t} u_t + C_{wg} - C_{ow} \; , \tag{11}$$

$$u_g \; = \; \frac{\lambda_g}{\lambda_t} u_t - C_{wg} - C_{og} \; , \tag{12}$$

where C_{og}, C_{ow} and C_{wg} group the capilarity terms.
Proof: Replacing (9) in (4A), (4B) and (4C), and after some calculations follows equations (10), (11) and (12) respectively by setting

$$C_{og} \; = \; -K_x \frac{\lambda_o \lambda_g}{\lambda_t} \frac{\partial P_{cgo}}{\partial x} \; , \qquad C_{ow} \; = \; K_x \frac{\lambda_o \lambda_w}{\lambda_t} \frac{\partial P_{cow}}{\partial x} \; ,$$

$$C_{wg} \; = \; K_x \frac{\lambda_w \lambda_g}{\lambda_t} (- \frac{\partial P_{cgo}}{\partial x} - \frac{\partial P_{cow}}{\partial x}) \; . \diamond$$

Lemma 5: The oil pressure gradient can be expressed as a function of the total flow velocity u_t and the total mobility λ_t by

$$\frac{\partial P_o}{\partial x} \; = \; \frac{1}{\lambda_t} K_x^{-1} u_t + T, \tag{13A}$$

where T depends on C_{og}, C_{ow}, C_{wg}, λ_o, λ_w and λ_g.

Proof: From equation (9), the oil pressure gradient yields (13A) with

$$T = \frac{1}{2\lambda_o K_x}\,(\frac{1}{\lambda_w}[\lambda_w C_{og} + \lambda_g C_{ow} + \lambda_o C_{wg}] + \frac{1}{\lambda_g}[\lambda_w C_{og} + \lambda_g C_{ow} - \lambda_o C_{wg}]), \tag{13B}$$

using the definitions for C_{og}, C_{ow} and C_{wg} from Lemma 4. $\diamond$

Definition 4: Let define c_f the total fluid compressibility as follows

$$c_f = c_o + c_w + c_g. \tag{14}$$

Lemma 6: Using the definitions of L_o, L_w y L_g, the linear combination $(B_o - R_{so}B_g)L_o + B_w L_w + B_g L_g$ becomes

$$(B_o - R_{so}B_g)L_o + B_w L_w + B_g L_g \;=$$

$$\frac{\partial}{\partial x}u_t + \frac{1}{\lambda_t^2}c_f\beta_\lambda K_x^{-1}u_t^2 + [\frac{K_x^{-1}}{\lambda_t}c_f\beta_C + \frac{T}{\lambda_t}c_f\beta_\lambda]u_t + Tc_f\beta_C \tag{15A}$$

with

$$\alpha_o = \frac{c_o}{c_f}, \quad \alpha_w = \frac{c_w}{c_f}, \quad \alpha_g = \frac{c_g}{c_f},$$

$$\beta_\lambda \;=\; \lambda_o\alpha_o + \lambda_w\alpha_w + \lambda_g\alpha_g \;,$$

$$\beta_C \;=\; \alpha_o(C_{og} + C_{ow}) + \alpha_w(C_{wg} - C_{ow}) - \alpha_g(C_{wg} + C_{og}) \;.$$

Proof: Using the definitions for L_o, L_w y L_g, given in (5), and the definition of the total velocity u_t, then

$$(B_o - R_{so}B_g)L_o + B_w L_w + B_g L_g \;=$$

$$\frac{\partial u_t}{\partial x} + B_o\frac{\partial}{\partial x}[\frac{1}{B_o}]u_o + B_w\frac{\partial}{\partial x}[\frac{1}{B_w}]u_w + B_g\frac{\partial}{\partial x}[\frac{1}{B_g}]u_g + \frac{B_g}{B_o}u_o\frac{\partial R_{so}}{\partial x} \;.$$

From the fact B_i, $i = o, w, g$, and R_{so} are functions of P_o,

$$\frac{\partial}{\partial x}[\frac{1}{B_i}] = -\frac{1}{B_i^2}\,\frac{\partial B_i}{\partial x} = -\frac{1}{B_i^2}\,\frac{\partial B_i}{\partial P_o}\,\frac{\partial P_o}{\partial x} \;,$$

and introducing the definitions of the fluid compressibilities equations (2I) to (2K), the following expresion arises

$$(B_o - R_{so}B_g)L_o + B_w L_w + B_g L_g = \frac{\partial u_t}{\partial x} + c_o\,\frac{\partial P_o}{\partial x}u_o + c_w\,\frac{\partial P_o}{\partial x}u_w + c_g\,\frac{\partial P_o}{\partial x}u_g \;.$$

Using equation (13A) for the oil pressure gradient and the equations (10), (11) and (12) for u_o, u_w and u_g respectively

$$(B_o - R_{so}B_g)L_o + B_wL_w + B_gL_g = \frac{\partial u_t}{\partial x} + \frac{1}{\lambda_t^2}(\lambda_o c_o + \lambda_w c_w + \lambda_g c_g)K_x^{-1}u_t^2$$

$$+[\frac{K_x^{-1}}{\lambda_t}(c_o(C_{og} + C_{ow}) + c_w(C_{wg} - C_{ow}) + c_g(-C_{wg} - C_{og})$$

$$+\frac{T}{\lambda_t}(\lambda_o c_o + \lambda_w c_w + \lambda_g c_g)]u_t + T(c_o(C_{og} + C_{ow}) + c_w(C_{wg} - C_{ow}) + c_g(-C_{wg} - C_{og})).$$

Introducing the definitions for α_o, α_w, α_g, β_λ and β_C, in addition to definition 4 for c_f, follows

$$(B_o - R_{so}B_g)L_o + B_wL_w + B_gL_g =$$

$$\frac{\partial u_t}{\partial x} + \frac{1}{\lambda_t^2}c_f\beta_\lambda K_x^{-1}u_t^2 + [\frac{K_x^{-1}}{\lambda_t}c_f\beta_C + \frac{T}{\lambda_t}c_f\beta_\lambda]u_t + Tc_f\beta_C \ . \ \diamond$$

Now we take a look at the accumulation terms, i.e. equations (2A) to (2C).

Definition 5: Let define R_o, R_w and R_g for the oil, water and gas fases as follows

$$R_o = \frac{\partial}{\partial t}(\phi\frac{S_o}{B_o}), \quad R_w = \frac{\partial}{\partial t}(\phi\frac{S_w}{B_w}), \quad R_g = \frac{\partial}{\partial t}[\phi(\frac{S_g}{B_g} + R_{so}\frac{S_o}{B_o})] \ .$$

Lemma 7: Using the definitions of R_o, R_w y R_g, follows

$$(B_o - R_{so}B_g)R_o + B_wR_w + B_gR_g = \phi c_t\frac{\partial P_o}{\partial t} \ , \qquad (15B)$$

with $c_t = c_f\ \beta_S$, $\beta_S = \alpha_r + \alpha_o S_o + \alpha_w S_w + \alpha_g S_g$, $\alpha_r = c_r/c_f$.
Proof: Recognizing that the formation volume factor, gas solubilities and porosity are functions of pressure, we use the chain rule to expand the accumulation terms of equations (2A) through (2C), representing in this case by R_o, R_w y R_g (see definition 5),

$$R_o = \frac{\phi}{B_o}\frac{\partial S_o}{\partial t} + [\frac{S_o}{B_o}\frac{\partial \phi}{\partial P_o} - \frac{S_o\phi}{{B_o}^2}\frac{\partial B_o}{\partial P_o}]\frac{\partial P_o}{\partial t},$$

$$R_w = \frac{\phi}{B_w}\frac{\partial S_w}{\partial t} + [\frac{S_w}{B_w}\frac{\partial \phi}{\partial P_o} - \frac{S_w\phi}{{B_w}^2}\frac{\partial B_w}{\partial P_o}]\frac{\partial P_o}{\partial t},$$

$$R_g = \frac{\phi}{B_g}\frac{\partial S_g}{\partial t} + [\frac{S_g}{B_g}\frac{\partial \phi}{\partial P_o} - \frac{S_g\phi}{{B_g}^2}\frac{\partial B_g}{\partial P_o}]\frac{\partial P_o}{\partial t}$$

$$+\frac{\phi R_{so}}{B_o}\frac{\partial S_o}{\partial t} + [\frac{S_o R_{so}}{B_o}\frac{\partial \phi}{\partial P_o} + \frac{\partial \phi S_o}{B_o}\frac{\partial R_{so}}{\partial P_o} - \frac{\phi S_o R_{so}}{{B_o}^2}\frac{\partial B_o}{\partial P_o}]\frac{\partial P_o}{\partial t}.$$

The equality $S_o + S_w + S_g = 1$ is now used to remove $\frac{\partial S_g}{\partial t}$ from the equations for R_g, i.e. $\frac{\partial S_g}{\partial t} = -\frac{\partial S_o}{\partial t} - \frac{\partial S_w}{\partial t}$.
Substituting the last equation into equation for R_g and simplifying yields

$$R_g = (\frac{\phi R_{so}}{B_o} - \frac{\phi}{B_g})\frac{\partial S_o}{\partial t} - \frac{\phi}{B_g}\frac{\partial S_w}{\partial t}$$

$$+[\frac{S_g}{B_g}\frac{\partial \phi}{\partial P_o} - \frac{S_g\phi}{{B_g}^2}\frac{\partial B_g}{\partial P_o} + \frac{S_o R_{so}}{B_o}\frac{\partial \phi}{\partial P_o} + \frac{\partial \phi S_o}{B_o}\frac{\partial R_{so}}{\partial P_o} - \frac{\phi S_o R_{so}}{{B_o}^2}\frac{\partial B_o}{\partial P_o}]\frac{\partial P_o}{\partial t} .$$

Multiplying R_o by $(B_o - R_{so}B_g)$, R_w by B_w, and R_g by B_g, and adding the results gives

$$(B_o - R_{so}B_g)R_o + B_w R_w + B_g R_g =$$

$$= [(S_g+S_w+S_o)\frac{\partial \phi}{\partial P_o} - \frac{\phi S_g}{B_g}\frac{\partial B_g}{\partial P_o} + \phi S_o(\frac{B_g}{B_o}\frac{\partial R_{so}}{\partial P_o} - \frac{1}{B_o}\frac{\partial B_o}{\partial P_o}) - \phi S_w\frac{1}{B_w}\frac{\partial B_w}{\partial P_o}]\frac{\partial P_o}{\partial t} .$$

Employing the definitions for the oil, water, gas, rock and total compressibilities from equations (2I) through (2M) gives

$$(B_o - R_{so}B_g)L_o + B_w L_w + B_g L_g = \phi c_t \frac{\partial P_o}{\partial t} . \ \diamond$$

Definition 6: Let define the total rate q_t by

$$q_t = \frac{B_o q_o}{\rho_{osc}} + \frac{B_w q_w}{\rho_{wsc}} + B_g(\frac{q_g}{\rho_{gsc}} - \frac{R_{so}q_o}{\rho_{osc}}) . \tag{16}$$

Theorem 1: The systems of equations (2) is equivalent to the system

$$\frac{\partial}{\partial x}u_t + c_f A K_x^{-1}u_t^2 + c_f B u_t + c_f D - q_t = c_f E\frac{\partial P_o}{\partial t} \tag{17A}$$

$$\frac{\partial}{\partial x}P_o = \frac{1}{\lambda_t}K_x^{-1}u_t + T \tag{17B}$$

$$\frac{\partial}{\partial x}[\frac{1}{B_o}u_o] - \frac{q_o}{\rho_{osc}} = \frac{\partial}{\partial t}(\phi\frac{S_o}{B_o}) \tag{17C}$$

$$\frac{\partial}{\partial x}[\frac{1}{B_w}u_w] - \frac{q_w}{\rho_{wsc}} = \frac{\partial}{\partial t}(\phi\frac{S_w}{B_w}) \qquad (17D)$$

$$S_o + S_w + S_g = 1, \qquad (17E)$$

with q_t given in definition 6, T in (13B),

$$A = \frac{1}{\lambda_t^2}\beta_\lambda \ , \quad B = \frac{K_x^{-1}}{\lambda_t}\beta_C + T\frac{\beta_\lambda}{\lambda_t} \ , \quad D = T\beta_C \ , \quad E = \phi\beta_S \ .$$

Proof: Multiplying equation (2A) by $(B_o - R_{so}B_g)$, equation (2B) by B_w, and equations (2C) by B_g, adding the results and doing some algebra gives

$$(B_o - R_{so}B_g)L_o + B_wL_w + B_gL_g - (B_o - R_{so}B_g)\frac{q_o}{\rho_{osc}} - B_w\frac{q_w}{\rho_{wsc}} - B_g\frac{q_g}{\rho_{gsc}} =$$

$$(B_o - R_{so}B_g)R_o + B_wR_w + B_gR_g$$

employing definitions 1 and 5.
Using equations (15A-B) and (16) the last equation becomes

$$\frac{\partial}{\partial x}u_t + \frac{1}{\lambda_t^2}c_f\beta_\lambda K_x^{-1}u_t^2 + [\frac{K_x^{-1}}{\lambda_t}c_f\beta_C + \frac{T}{\lambda_t}c_f\beta_\lambda]u_t + Tc_f\beta_C - q_t = \phi c_f\beta_S\frac{\partial P_o}{\partial t} \ .$$

Defining

$$A = \frac{1}{\lambda_t^2}\beta_\lambda \ , \quad B = \frac{K_x^{-1}}{\lambda_t}\beta_C + T\frac{\beta_\lambda}{\lambda_t} \ , \quad D = T\beta_C \ , \quad E = \phi\beta_S$$

finally the last equation becomes (17A), that is to say

$$\frac{\partial}{\partial x}u_t + c_fAK_x^{-1}u_t^2 + c_fBu_t + c_fD - q_t = c_fE\frac{\partial P_o}{\partial t} \ ,$$

which is complemented with equation (13A), in other words equation (17B)

$$\frac{\partial}{\partial x}P_o = \frac{1}{\lambda_t}K_x^{-1}u_t + T \ .$$

Equations (17C-D) follow from sustitution of equations (3A-B) and (5A-B) into (2A-B), and equation (17E) corresponds to (2D). $\diamond$

Observation: The system (17) is complemented with equations (10) to (12) in lemma 4, which relate u_t with the phase velocities u_o, u_w and u_g.
System (17) represents a way to decouple the original system (2) in such a way that equations (17A-B) could be used to calculate P_o and u_t alone, and get S_o and S_w from equations (17C-D) and fianlly S_g from (17E). This idea is based on the well-known IMPES formulation for the black oil equations (see [1-3]).

The solution precedure to system (17) will be discussed in subsequent sections.

3 Solving the fluid flow equations

The proposed approach to solve the system (17A-B) is to express the total flux as an asymptotic expansion of ascending powers of the total fluid compressibility called here ϵ. Assuming $\frac{\partial \epsilon}{\partial x} = 0$, u_t can be written as

$$u_t \ = \ u_0 + \epsilon u_1 + \epsilon^2 u_2, \tag{18A}$$

with

$$u_0 \ = \ \lambda_t K_x (\frac{\partial P_0}{\partial x} - T), \quad u_1 \ = \ \lambda_t K_x \frac{\partial P_1}{\partial x}, \quad u_2 \ = \ \lambda_t K_x \frac{\partial P_2}{\partial x}, \tag{18B}$$

where P_0, P_1 and P_2 are pressures associated to the velocities u_0, u_1 and u_2 respectively. It can be done because the total flow velocity u_t depends on the total fluid compressibility c_f, being c_f linear on x for weakly compressible flow in porous media.
It follows from equation (17B), (18A) and (18B) that

$$\lambda_t K_x (\frac{\partial P_o}{\partial x} - T) = \lambda_t K_x (\frac{\partial P_0}{\partial x} - T) + \epsilon \lambda_t K_x \frac{\partial P_1}{\partial x} + \epsilon^2 \lambda_t K_x \frac{\partial P_2}{\partial x},$$

that is,

$$\frac{\partial P_o}{\partial x} \ = \ \frac{\partial P_0}{\partial x} + \epsilon \frac{\partial P_1}{\partial x} + \epsilon^2 \frac{\partial P_2}{\partial x} \ . \tag{19A}$$

Finally, it can be assumed without loss of generality that

$$P_o \ = \ P_0 + \epsilon P_1 + \epsilon^2 P_2 \ . \tag{19B}$$

Replacing equations (17B), (19A) and (19B) in (17A)

$$\frac{\partial}{\partial x}[\lambda_t K_x (\frac{\partial P_0}{\partial x} + \epsilon \frac{\partial P_1}{\partial x} + \epsilon^2 \frac{\partial P_2}{\partial x} - T)] + \epsilon A K_x^{-1} [\lambda_t K_x (\frac{\partial P_0}{\partial x} + \epsilon \frac{\partial P_1}{\partial x} + \epsilon^2 \frac{\partial P_2}{\partial x} - T)]^2$$

$$+ \epsilon B \lambda_t K_x (\frac{\partial P_0}{\partial x} + \epsilon \frac{\partial P_1}{\partial x} + \epsilon^2 \frac{\partial P_2}{\partial x} - T) + \epsilon D - q_t \ = \ \epsilon E \frac{\partial (P_0 + \epsilon P_1 + \epsilon^2 P_2)}{\partial t} \ .$$

Gathering terms according to powers of epsilon, i.e. ϵ^0, ϵ^1 and ϵ^2, and Using the definitions of u_0, u_1 and u_2 given by (18B), follows the set of equations

$$u_0 - \lambda_t K_x \frac{\partial P_0}{\partial x} \ = \ -\lambda_t K_x T \ , \tag{20A}$$

$$\frac{\partial u_0}{\partial x} \ = \ q_t \ , \tag{20B}$$

$$u_1 - \lambda_t K_x \frac{\partial P_1}{\partial x} \;=\; 0 \;, \tag{20C}$$

$$\frac{\partial u_1}{\partial x} \;=\; -AK_x^{-1}u_0^2 - Bu_0 - D + E\frac{\partial P_0}{\partial t} \;, \tag{20D}$$

$$u_2 - \lambda_t K_x \frac{\partial P_2}{\partial x} \;=\; 0 \;, \tag{20E}$$

$$\frac{\partial u_2}{\partial x} \;=\; -2AK_x^{-1}u_1 u_0 - Bu_1 + E\frac{\partial P_1}{\partial t} \;, \tag{20F}$$

where P_o and u_t are given by

$$P_o \;=\; P_0 + \epsilon P_1 + \epsilon^2 P_2 \;,$$

$$u_t \;=\; u_0 + \epsilon u_1 + \epsilon^2 u_2 \;,$$

with the boundary conditions of zero flux across the boundary, i.e. $u_i = u_t = 0$ for $i = 0, 1, 2$, and zero pressure gradient in the boundary, that is,

$$\frac{\partial P_i}{\partial x} = \frac{\partial P_o}{\partial x} = 0 \quad \text{for} \quad i = 0, 1, 2 \;.$$

From u_t the phase velocities u_o and u_w can be computed using equations (10), (11) and (12). Now the phase saturations are estimated from equations (17C) to (17E).

4 Discretizing the fluid flow equations

The basic idea of any approximation method is to replace the original problem by another problem that is easier to solve and whose solution is, in some sense, close to the solution of the original problem.
The usual approach for discretizing the fluid flow equations entail block centered finite differences with upstream weighting. The upstream weighting is used in the industry because it suppresses nonphysical oscillations in the finite difference solution. The time stepping may be either explicit or implicit.
Each set of two equation in (20) can be seen as

$$\frac{1}{\lambda_t} K_x^{-1} u - \frac{\partial p}{\partial x} \;=\; f \;, \tag{21A}$$

$$\frac{\partial u}{\partial x} \;=\; g \;. \tag{21B}$$

where p stands P_0, P_1 or P_2, and u stands u_0, u_1 or u_2, and g includes the discretization in time of $\partial p/\partial t$ corresponding to equations (20D) and (20F).

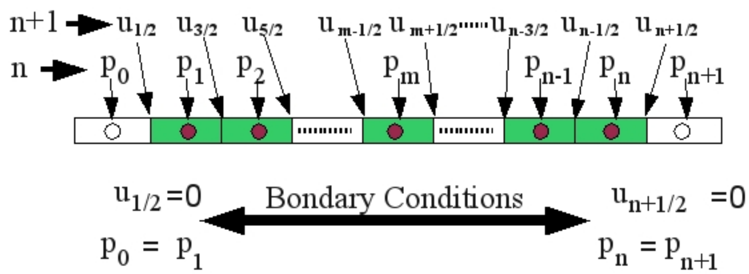

Figure 1: Discretizing the spatial domain (the reservoir) por u and p.

For the proposed equations (21A) and (21B) with boundary conditions of zero flux across the boundary, the methodology to develop will be cell-centered finite difference for the first order spatial operators and implicit forward difference for discretisation in time.
Given N, let $\{x_i\}_{1\leq i\leq N}$ be an uniform subdivision of the spatial domain (the reservoir), with mesh length Δx along the direction x. Let denote $x_{i-1/2} = (x_i - x_{i-1})/2$, $p_i = p(x_i)$ for $1 \leq i \leq N$, $u_{i-1/2} = u(x_{i-1/2})$ for $1 \leq i \leq N-1$ (see Fig.1).
The following are the finite difference used for the first order operators

$$(\frac{\partial p}{\partial x})_{i+1/2} = \frac{p_{i+1} - p_i}{\Delta x} \quad \text{for} \quad 1 \leq i \leq N-1\ , \tag{22A}$$

$$(\frac{\partial u}{\partial x})_{i} = \frac{u_{i+1/2} - u_{i-1/2}}{\Delta x} \quad \text{for} \quad 1 \leq i \leq N\ . \tag{22B}$$

The boundary conditions impose

$$u_{1/2} = 0, \quad u_{N+1/2} = 0, \quad p_1 = p_0, \quad p_{n+1} = p_N\ . \tag{23}$$

The discretized system associated to (21) is therefore as follows

$$\frac{1}{\lambda_t K_x}\ u_{i+1/2} - \frac{p_{i+1} - p_i}{\Delta x} = f(x_{i+1/2})\ , \tag{24A}$$

$$\frac{u_{i+1/2} - u_{i-1/2}}{\Delta x} = g(x_i)\ , \tag{24B}$$

for $1 \leq i \leq N$, with the above boundary conditions.
Finally, from (24), the following system of linear equations arises

$$\begin{pmatrix} D & B \\ B^t & 0 \end{pmatrix} \begin{pmatrix} U \\ P \end{pmatrix} = \begin{pmatrix} F \\ G \end{pmatrix}. \tag{25}$$

This is a saddle point formulation. Here D is a diagonal matrix of dimension $N-1$ by $N-1$, B is a matrix of dimension $N-1$ by N and given by

$$B = \begin{pmatrix} 1 & -1 & & \\ & \ddots & \ddots & \\ & & 1 & -1 \end{pmatrix} .$$

Here B^t is the transpose of B; U is a vector of dimension $(N-1)$; P is a vector of dimension N; F is a vector of dimension $(N-1)$; and G is a vector of dimension N. By construction B^t is full rank.
Rewriting equation (25) we get the system

$$Du + BP = F \qquad (26A)$$

$$B^t U = G \qquad (26B)$$

The next step is to solve the system (25). Let suppose that exist $\hat{P}$ such that $P = B^t \hat{P}$, and from (26A) the following expression is obtained

$$\hat{P} = (BB^t)^{-1}(F - DU).$$

Multilpying both sides by B^t and defining $B^+ = B^t(BB^t)^{-1}$, the pseudoinverse of B, it follows that

$$P = B^+(F - DU).$$

Let suppose that

$$P = B^+(F - DU) + v \ , \qquad (27)$$

for an arbitrary vector v of dimension N.
Substituting equation (27) in (26A) and using the fact that $BB^+ = I$, it follows $Bv = 0$, i.e. v belongs to the null space of B, namely $null(B)$.
An important property of pseudoinverse is that it gives simple expressions for the orthogonal projections onto the four fundamental subspaces of B

$$\begin{aligned} P_{rank(B)} &= BB^+ = I \ , \\ P_{rank(B^t)} &= B^+B \ , \\ P_{null(B^t)} &= I - BB^+ = 0 \ , \\ P_{null(B)} &= I - B^+B \ . \end{aligned}$$

Given $\tilde{P}$, let

$$v = P_{null(B)}\tilde{P} \ . \qquad (28)$$

From equation (26B), i.e. $B^tU = G$, multipling by B both sides and using the fact that $(B^t)^+ = (BB^t)^{-1}B$, it follows that

$$U = (B^t)^+ G \ . \qquad (29)$$

Replacing (29) in (27) then

$$P \;=\; B^{+}(F - D(B^t)^{+}G) + v \ , \quad v \in null(B). \tag{30}$$

Finally equations (28), (29) and (30) represent the solution to the discretized equation (26) of the problem (21).

4.1 Calculating saturations

The other equations to be discretized are the phase saturations. After using the same approximation for the first order operators given at the begining of section 4, it follows the vectorial equations for the oil and water saturations

$$S_o \;=\; (V_b\frac{\Delta t}{h}B^t\frac{u_o}{B_o} - \Delta t\ q_o + V_b\phi\frac{S_o}{B_o})\frac{B_o}{V_b\phi} \tag{31A}$$

$$S_w \;=\; (V_b\frac{\Delta t}{h}B^t\frac{u_w}{B_w} - \Delta t\ q_w + V_b\phi\frac{S_w}{B_w})\frac{B_w}{V_b\phi} \tag{31B}$$

with V_b a vector of dimension N with entries the volume of each cell, S_o a vector of dimension N with entries $(S_o)_i = S_o(x_i)$, u_o and u_w vectors of dimension $N-1$ with entries of type $u_{i-1/2}$, B_o and B_w vectors of dimension N with entries of type $(B_o)_i = B_o(x_i)$, q_o and q_w vectors of dimension N with entries of type $(q_o)_i = q_o(x_i)$, ϕ a vector of dimension N with entries $\phi_i = \phi(x_i)$, and B a matrix of dimension $(N-1)\times N$ defined before.
The gas saturation follows from equation (2D)

$$S_g \;=\; 1 - S_o - S_w \ , \tag{31C}$$

5 Algorithm for solving the fluid flow equations

For each time step:

1. Solve the three systems of differential equations given by (21) through the methodology for linear systems of type (25).

2. Compute the total velocity $u_t = u_0 + \epsilon u_1 + \epsilon^2 u_2$ and pressure $P_o = P_0 + \epsilon P_1 + \epsilon^2 P_2$.

3. Compute the phase velocities u_o, u_w and u_g according to (10), (11) and (12).

4. Compute the new oil, water and gas saturations, i.e. S_o, S_w and S_g, according to (31).

It is important to note that the matrix B in (25) is fixed along the whole simulation. It provides to the methodology a great advantage over the traditional methods for the solution of the discretized system associated to the fluid flow equations (2). It can be shown that the number of floating point operations associated to find the solution of the linear system can be reduced significantly for each time step. It represents a significant reduction in the number of floating point operations performed to solve the linear system of equations (25) (it will be part of a coming article in the same subject).

6 Conclusions

1. A new methodology to build the solution of the black oil formulation of the fluid flow equations is presented. It includes solubility of gas, gravity and capillary pressures.

2. The use of first order operators (gradient and divergence) leads to simpler approximation schemes.

3. The factorization of the matrix BB^t is carried out only once along the simulation.

4. The number of floating point operations in the solution of the linear system can be reduced significantly, compared with the tradicional IMPES method for the same black oil fluid flow equations.

References

[1] Aziz, K., and Settari, A.: *Petroleum Reservoir Simulation*, Elsevier Science, London, Applied Science Publishers Ltd., 1979.

[2] Franchi, J.R., Harpole, K.J., Bujnowski, S.W.: *BOAST: A three dimensional, three phase black oil applied simulation tool.* Vol. 1 y 2, Department of Energy, Washington, U.S., 1982.

[3] Mattax, C.: *Reservoir Simulation*, Society of Petroleum Engineers, Richardson, 1990.

[4] Helming, R.: *Multiphase Flow and Transport Processes in the Subsurphase*, Springer-Verlag Berlin Heidelberg, 1997.

[5] Holmes, M.H.: *Introduction to Perturbation Methodos*, Sringer-Verlag NY, 1995.

[6] Bramley, R.: "An Orthogonal Projection Algorithm for linear Stokes Problems". Tech. report 1190 CSRD, University of Illinois Urbana-Champaign, IL, 1992.

[7] Golub, G.H., Van Loan, C.F.:*Matrix Computation*, The Johns Hopkins University Press, 1991.

[8] Buitrago, S.: *Técnicas Secuenciales y Paralelas para la Solución de los Sistemas Lineales en la Simulación de Yacimientos. Proc* First International Congress On Numerical Methods in Engineering, (ISBN 980-00-0667-2), UCV, Caracas, Venezuela, March 1993.

[9] Kuhn J. C.: *A New Method for the Fast Estimation Reservoir Performance*, SPE 39743, *Proc* Asia Pacific Conference on Integreted Modeling for Asset Management, Kuala Lumpur,Malaysia, March 23-24, 1998.

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