Metric properties of a tensor norm defined by $\ell_p\{\ell_q\}$ spaces and some characteristics of its associated operator ideals.

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Abstract. The present article examines the study stated in [4] regarding a tensor norm defined in a Bochner Banach sequence space $\ell_p\{\ell_q\}$ and its associated operator ideals in the sense of Defant and Floret [2]. In particular, we analyze the coincidence between components of the minimal and maximal operator ideals, and we prove some metric properties of the tensor norm and its dual.

Resumen. Este artículo examina el estudio indicado en [4] en cuanto a una norma de tensor definida en un espacio de secuencias de Bochner Banach $\ell_p\{\ell_q\}$ y sus ideales de operador asociados en el sentido de Defant y Floret [2]. En particular, analizamos la coincidencia entre los componentes de los ideales de operador mínimos y máximos, y demostramos algunas propiedades métricas de la norma de tensor y su duales.

1 Introduction

The equality between components of the maximal and the minimal operator ideals is a classical problem of the operator ideals theory and its solution has always been related to the Radon-Nikodým property, and the use of some concepts of the local theory. We use results found in [4] about the tensor norm defined by $\ell_p\{\ell_q\}$ spaces and the associated operator ideals.

The notation is standard. All the spaces are Banach spaces over the real field which allow us to use known results in Banach lattices. When we want to emphasize the space E where a norm is defined, we shall write $\|.\|_E$. The canonical inclusion of a Banach space E into its bidual E'' will be denoted by

J_E. Using FIN, we denote the class of all finite dimensional normed spaces, and for a normed space E we define

$$FIN(E) := \{M \subset E : M \in FIN\} \text{ and } COFIN(E) := \{L \subset E : E/L \in FIN\}$$

the sets of finite dimensional subspaces and finite codimensional, closed subspaces with induced norm.

We suppose the reader is familiar with the theory of operator ideals and tensor norms. The fundamental references about these matters are the books [14] and [2] of Pietsch and Defant and Floret respectively. We set the notation and some definitions to be used.

Given a pair of Banach spaces E and F and a tensor norm α, $E \otimes_\alpha F$ represents the space $E \otimes F$ endowed with the α-normed topology. The completion of $E \otimes_\alpha F$ is denoted by $E \hat{\otimes}_\alpha F$, and the norm of z in $E \otimes_\alpha F$ by $\alpha(z; E, F)$ (or $\alpha(z)$ if there is no risk of mistake). These are three tensor norms relative to α: transposed, α^t, finite hull, $\overrightarrow{\alpha}$, and cofinite hull, $\overleftarrow{\alpha}$. They are defined by

$$\alpha^t(z; E, F) := \alpha(z^t; F, E)$$

$$\overrightarrow{\alpha}(z) := \inf\{\alpha(z; M, N) : M \in FIN(E), N \in FIN(F), z \in M \otimes N\}$$

$$\overleftarrow{\alpha}(z) := \sup\left\{\alpha\left(Q_K^E \otimes Q_L^F(z); E/K, F/L\right) : K \in COFIN(E), L \in COFIN(F)\right\}$$

where $Q_K^E : E \twoheadrightarrow E/K$ is the canonical mapping.

A tensor norm α is called right-accessible if $\overleftarrow{\alpha}(\cdot; M, F) := \overrightarrow{\alpha}(\cdot; M, F)$ for all $(M, F) \in FIN \times NORM$, left-accessible if α^t is right-accessible, and accessible if it is right- and left-accessible. α is called totally accessible if $\overleftarrow{\alpha} = \overrightarrow{\alpha}$.

For $1 \leq p \leq \infty$ the Saphar's tensor norm g_p is defined by

$$g_p(z; E \otimes F) := \inf\left\{\pi_p((x_i))\varepsilon_{p'}((y_i)) : z = \sum_{i=1}^n x_i \otimes y_i \in E \otimes F\right\}$$

where

$$\pi_p((x_i)) := \left(\sum_{i=1}^\infty \|x_i\|^p\right)^{1/p} \quad \varepsilon_p((x_i)) := \sup_{x' \in B_{E'}} \left(\sum_{i=1}^\infty |\langle x', x_i\rangle|^p\right)^{1/p}$$

for $1 \leq p < \infty$, and $\pi_\infty((x_i)) = \varepsilon_\infty((x_i)) = \sup_i \|x_i\|$

Concerning Banach lattices, we refer the reader to [1]. We recall the more relevant definitions and properties for our purposes: A Banach lattice E is order complete or Dedekind complete if every order bounded set in E has a least upper bound in E; it is order continuous if every order convergent filter is norm convergent. Every dual Banach sequence lattice E' is order complete, and the reflexive spaces are even order continuous. A linear map T between Banach lattices E and F is said to be positive if $T(x) \geq 0$ in F for every $x \in E, x \geq 0$. T is called order bounded if $T(A)$ is order bounded in F for every order bounded set A in E.

For $1 \leq p, q \leq \infty$ define the Bochner Banach sequence space

$$\ell_p\{\ell_q\} := \{a = (a_{ij})_{i,j=1}^{\infty} : \ \ \|a\|_{p\{q\}} := (\sum_{i=1}^{\infty}(\sum_{j=1}^{\infty}|a_{ij}|^q)^{\frac{p}{q}})^{\frac{1}{p}} < \infty\}$$

with the usual modifications if p or q are infinite. If $1 \leq p, q < \infty$, $\ell_p\{\ell_q\}$ is an order continuous Banach sequence lattice. By $\ell_p^n\{\ell_q^m\}, n, m \in \mathbb{N}$, we denote the sectional subspace of $\ell_p\{\ell_q\}$ of those sequences (α_{ij}) such that $\alpha_{ij} = 0$ for every $i \geq n,\ j \geq m$. $\ell_p^n\{\ell_q^m\}$ is 1-complemented in $\ell_p\{\ell_q\}$. By $\ell_{p'}\{\ell_{q'}\}$, we denote the dual space of $\ell_p\{\ell_q\}$, as usual.

According to J. Hoffman-Jörgensen's definition, a Banach space X is of type p for $1 < p \leq 2$, and cotype q for $q \geq 2$, respectively, if there is a constant $0 \leq M < \infty$ such that for all finite vector set $\{x_j\}_{j=1}^n$ in X,

$$\int_0^1 \left\| \sum_{j=1}^n r_j(t)x_j \right\| dt \leq M \left(\sum_{j=1}^n \|x_j\|^p \right)^{1/p}$$

respectively,

$$\int_0^1 \left\| \sum_{j=1}^n r_j(t)x_j \right\| dt \geq M^{-1} \left(\sum_{j=1}^n \|x_j\|^q \right)^{1/q}$$

where $r_n(t) := sing(\sin 2^n \pi t)$ is a Rademacher functions sequence for all $t \in [0, 1]$ and $n = 0, 1, 2, 3....$ In particular, for $1 \leq p < \infty$ the space $L_p(\mu)$ is of type $\min(2, p)$ and cotype $\max(2, p)$, see [2].

Let (Ω, Σ, μ) be a measure, we denote $L_0(\mu)$ the space of equivalence classes, modulo equality μ-almost everywhere, of μ-measurable real-valued functions, endowed with the topology of local convergence in measure. The space of all equivalence classes of μ-measurable X-valued functions is denoted $L_0(\mu, X)$. By a Köthe function space $\mathcal{K}(\mu)$ on (Ω, Σ, μ), we shall mean an order dense ideal of $L_0(\mu)$, which is equipped with a norm $\|.\|_{\mathcal{K}(\mu)}$ that makes it a Banach lattice (if $f \in L_0(\mu)$ and $g \in \mathcal{K}(\mu)$ $|f| \leq |g|$, then $f \in \mathcal{K}(\mu)$ with $\|f\|_{\mathcal{K}(\mu)} \leq \|g\|_{\mathcal{K}(\mu)}$). Likewise, $\mathcal{K}(\mu, X) = \{f \in L_0(\mu, X) : \|f(.)\|_X \in \mathcal{K}(\mu)\}$, endowed with the norm $\|f\|_{\mathcal{K}(\mu,X)} = \|\|f(.)\|_X\|_{\mathcal{K}(\mu)}$.

2 Local theory and $\mathcal{L}^{p,q}$ spaces.

A standard reference on ultraproducts of Banach spaces, is [6].

Let D be a non empty index set and $\mathcal{U}$ a non-trivial ultrafilter in D. Given a family $\{X_d,\ d \in D\}$ of Banach spaces, $(X_d)_{\mathcal{U}}$ denotes the corresponding ultraproduct Banach space. If every X_d, $d \in D$, coincides with a fixed Banach space X, the corresponding ultraproduct is called an ultrapower of X, and it is denoted by $(X)_{\mathcal{U}}$. Notice that if every $X_d,\ d \in D$ is a Banach lattice, $(X_d)_{\mathcal{U}}$ has a canonical order which makes it a Banach lattice. If we have another family of Banach spaces $\{Y_d,\ d \in D\}$ and a family of operators $\{T_d \in \mathcal{L}(X_d, Y_d),\ d \in D\}$ such that $\sup_{d\in D} \|T_d\| < \infty$, then $(T_d)_{\mathcal{U}} \in \mathcal{L}((X_d)_{\mathcal{U}}, (Y_d)_{\mathcal{U}})$ denotes the canonical ultraproduct operator.

Two strong notions of finite representability are included in the following extension of the $\mathcal{L}^p$ spaces due to Lindenstrauss and Pelczyński [7]:

Definition 2.1 *Let $1 \leq p, q \leq \infty$. We say that a Banach space X is an $\mathcal{L}^{p,q}$ space if for every $F \in FIN(X)$ and every $\varepsilon > 0$ there is $G \in FIN(X)$, $\dim(G) = n$, containing F such that $d(G, \ell_p^n\{\ell_q^n\}) \leq 1 + \varepsilon$.*

Being an $\mathcal{L}^{p,q}$ space turns out to be very strong a condition with bad stability properties under ultraproducts; therefore, we need a weaker condition:

Definition 2.2 *Let $1 \leq p, q \leq \infty$. We say that a Banach space X is a quasi $\mathcal{L}^{p,q}$ space if there are $a > 0$ and $b > 0$ such that for every $M \in FIN(X)$ there are $M_1 \in FIN(X)$ containing M and a b-complemented subspace $H \subset \ell_p\{\ell_q\}$, such that $d(M_1, H) \leq a$. Moreover, if X is a Banach lattice, we say that it is a quasi $\mathcal{L}^{p,q}$ lattice.*

Of course, $\ell_p\{\ell_q\}$ is a *quasi* $\mathcal{L}^{p,q}$ space. Furthermore, from the the following definition of the uniform projection property introduced by Pelczyński and Rosenthal [13] and theorem 9.4 [6], $(\ell_p\{\ell_q\})_{\mathcal{U}}$ is a *quasi* $\mathcal{L}^{p,q}$ space too.

Definition 2.3 *A Banach space X has the uniform projection property if there is a number $b > 0$ such that for every $k \in \mathbb{N}$ there is $m(k) \in \mathbb{N}$ with the following property: for every $F \in FIN(X)$ with dimension k there is a b-complement subspace $G \in FIN(X)$ containing F with dimension $dim(G) \leq m(k)$.*

3 Tensor norm $g_{p\{q\}}$ and its associated operator ideals.

Given a Banach space E, a sequence of sequences $(x_{ij})_{i,j=1}^{\infty} \subset E$ is strongly $p\{q\}$-summing if $\pi_{p\{q\}}((x_{ij})) := \|(\|x_{ij}\|)\|_{p\{q\}} < \infty$ and it is weakly $p\{q\}$-summing if $\varepsilon_{p\{q\}}((x_{ij})) := \sup_{\|x'\|\leq 1} \|(|\langle x_{ij}, x'\rangle|)\|_{p\{q\}} < \infty$.

Let E and F be Banach spaces and $1 \leq p, q < \infty$. For every $z \in E \otimes F$, we define

$$g_{p\{q\}}(z) := \inf\{\pi_{p\{q\}}((x_{ij}))\ \varepsilon_{p'\{q'\}}((y_{ij})) \quad / \quad z = \sum_{i=1}^{n}\sum_{j=1}^{m} x_{ij} \otimes y_{ij}\}.$$

The functional $g_{p\{q\}}$ is a tensor norm for $E \otimes F$. A suitable representation of the elements of a completed tensor product is a basic tool in the study of the operator ideals involved. It can be shown that $z \in E\hat{\otimes}_{g_{p\{q\}}}F$ may be represented as $z = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} x_{ij} \otimes y_{ij}$ where $\{(x_{ij})_{j=1}^{\infty}, \quad / \quad i \in \mathbb{N}\} \subset E^{\mathbb{N}}, \{(y_{ij})_{j=1}^{\infty}, \quad / \quad i \in \mathbb{N}\} \subset F^{\mathbb{N}}$ and $\pi_{p\{q\}}((x_{ij}))\ \varepsilon_{p'\{q'\}}((y_{ij})) < \infty$. Moreover, $g_{p\{q\}}(z) = \pi_{p\{q\}}((x_{ij}))\ \varepsilon_{p'\{q'\}}((y_{ij}))$ where the infimum is taken over all representations of z.

There are three important associated operator ideals to $g_{p\{q\}}$, which we define and characterize now.

Definition 3.1 *Let $T \in \mathcal{L}(E, F)$. We say that T is $p\{q\}$-absolutely summing if it exists a real number $C > 0$, such that for all sequence of sequences (x_{ij}) in E, with $\varepsilon_{p\{q\}}((x_{ij})) < \infty$, it satisfies that*

$$\pi_{p\{q\}}((T(x_{ij}))) \leq C\varepsilon_{p\{q\}}((x_{ij})). \tag{1}$$

By $\mathcal{P}_{p\{q\}}(E, F)$, we denote the Banach ideal of the $p\{q\}$-absolutely summing operators $T : E \to F$ endowed with the topology of the norm $\Pi_{p\{q\}}(T) := \inf\{C \geq 0 : C \text{ satisfies (1)}\}$

Theorem 3.2 *Let E, F be Banach spaces, then $\left(E \otimes_{g_{p\{q\}}} F\right)' = \mathcal{P}_{p'\{q'\}}(F, E')$ isometrically.*

Proof. For all $T \in \mathcal{P}_{p'\{q'\}}(F, E')$, we define $\varphi_T : E \otimes_{g_{p\{q\}}} F \to \mathbb{R}$ by

$$\langle \varphi_T, z\rangle = \sum_{i=1}^{n}\sum_{j=1}^{l_i}\langle x_{ij}, T(y_{ij})\rangle \ \text{ for every } \ z = \sum_{i=1}^{n}\sum_{j=1}^{l_i} x_{ij} \otimes y_{ij} \in E \otimes_{g_{p\{q\}}} F,$$

The definition does not depend on the representation of z used and it can be proved that $\varphi_T \in (E \otimes_{g_{p\{q\}}} F)'$ with $|\langle \varphi_T, z\rangle| \leq \Pi_{p'\{q'\}}(T)\, g_{p\{q\}}(z; E, F)$, and therefore, $\|\varphi_T\| \leq \Pi_{p'\{q'\}}(T)$.

On the other hand, for $\varphi \in \left(E \otimes_{g_{p\{q\}}} F\right)'$, we define $T_\varphi : F \longrightarrow E'$ by

$$\langle T_\varphi(y), x\rangle = \langle \varphi, x \otimes y\rangle\ \forall y \in F,\ x \in E.$$

Then, if $(y_{ij}) \in F^{\mathbb{N}}$ such as $\varepsilon_{p'\{q'\}}((y_{ij})) < \infty$, as B_E is weakly dense in $B_{E''}$, given $\epsilon > 0$ and $(\delta_{ij}) \in \ell_{p'}\{\ell_{q'}\}$ with $\|(\delta_{ij})\|_{p'\{q'\}} \leq 1$, for all $i, j \in \mathbb{N}$ there is

$x_{ij} \in E$ such as $\|x_{ij}\| \leq 1$ and $\|T_\varphi(y_{ij})\| \leq |\langle \varphi, x_{ij} \otimes y_{ij}\rangle| + \epsilon\delta_{ij}$ hence

$$\|(\|T_\varphi\,(y_{ij})\,\|)\|_{p'\{q'\}} \leq \sup_{\|(\eta_{ij})\|_{p\{q\}}\leq 1} |\sum_{i=1}^{\infty} \eta_{ij}\langle \varphi, x_{ij} \otimes y_{ij}\rangle| + \epsilon$$

but $\pi_{p\{q\}}((\eta_{ij}x_{ij})) = \|(\|\eta_{ij}x_{ij}\|)\|_{p\{q\}} \leq \|(\eta_{ij})\|_{p\{q\}} \leq 1$ and $\varepsilon_{p'\{q'\}}((y_{ij})) < \infty$ then $\sum_{i=1}^{\infty} \eta_{ij}x_{ij} \otimes y_{ij} \in E\widehat{\otimes}_{g_{p\{q\}}}F$, hence

$$\|(\|T_\varphi\,(y_{ij})\,\|)\|_{p'\{q'\}} \leq \sup_{\|(\eta_{ij})\|_{p\{q\}}\leq 1} \|\varphi\| g_{p\{q\}}(\sum_{i=1}^{\infty} \eta_{ij}x_{ij}\otimes y_{ij}) + \epsilon \leq \|\varphi\|\varepsilon_{p'\{q'\}}((y_{ij})) + \epsilon$$

and, as ϵ is arbitrary, it follows that $\|(\|T_\varphi(y_{ij}\|)\|_{p'\{q'\}} \leq \|\varphi\|\varepsilon_{p'\{q'\}}((y_{ij}))$ and $\Pi_{p'\{q'\}}(T_\varphi) \leq \|\varphi\|$. ■

Now, every representation of $z \in E'\hat{\otimes}_{g_{p\{q\}}}F$ of the form

$$z = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} x'_{ij} \;\otimes\; y_{ij}$$

defines a map $T_z \in \mathcal{L}(E, F)$ such that for every $x \in E$,

$$T_z(x) := \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \langle x'_{ij}, x\rangle \; y_{ij}.$$

We remark that all representations of z define the same map T_z. Let Φ_{EF} : $E'\hat{\otimes}_{g_{p\{q\}}}F \to \mathcal{L}(E, F)$ be defined by $\Phi_{EF}(z) := T_z$.

Definition 3.3 *Let E, F be Banach spaces. An operator $T : E \to F$ is said to be $p\{q\}$- nuclear if $T = \Phi_{EF}(z)$, for some $z \in E'\hat{\otimes}_{g_{p\{q\}}}F$.*

$\mathcal{N}_{p\{q\}}(E, F)$ denotes the space of the $p\{q\}$-nuclear operators $T : E \to F$ endowed with the topology of the norm $\mathbf{N}_{p\{q\}}(T) := \inf\{g_{p\{q\}}(z) \;/\; \Phi_{EF}(z) = T\}$.

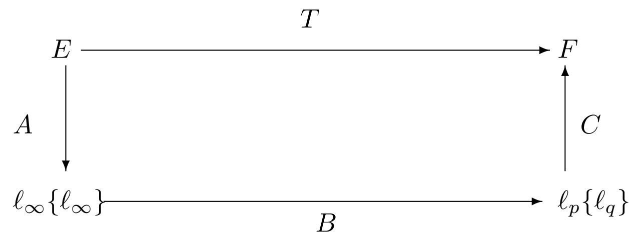
For every pair of Banach spaces E and F, $(\mathcal{N}_{p\{q\}}(E, F), \mathbf{N}_{p\{q\}})$ is a component of the minimal Banach operator ideal $(\mathcal{N}_{p\{q\}}, \mathbf{N}_{p\{q\}})$ associated to the tensor norm $g_{p\{q\}}$.

We have the following characterization of $p\{q\}$-nuclear operators:

Theorem 3.4 *[4] Let E and F be Banach spaces and let T be an operator in $\mathcal{L}(E, F)$. Then the following statements are equivalent:*

1) T is $p\{q\}$-nuclear.

2) T factors continuously in the following way:



where B is a diagonal multiplication operator defined by a positive sequence $((b_{ij})) \in \ell_p\{\ell_q\}$.

Furthermore $\mathbf{N}_{p\{q\}}(T) = \inf\{\|C\|\|B\|\|A\|\}$, *taking it over all such factors.*

The normed ideal of $p\{q\}$-integral operators $(\mathcal{I}_{p\{q\}}, \mathbf{I}_{p\{q\}})$ is the maximal operator ideal associated to the tensor norm $g_{p\{q\}}$ in the sense of Defant and Floret [2], which coincides with the maximal normed associated operator ideal to the normed ideal of $p\{q\}$-nuclear operators in the sense of Pietsch [14]. From [2], for every pair of Banach spaces E and F, an operator $T : E \to F$ is $p\{q\}$-integral if and only if $J_F T \in (E \otimes_{g'_{p\{q\}}} F')'$.

Given Banach spaces E, F, we define the finitely generated tensor norm $g'_{p\{q\}}$ such that if $M \in FIN(E)$ and $N \in FIN(F)$, for every $z \in M \otimes N$,

$$g'_{p\{q\}}(z; M \otimes N) := \sup\left\{ |\langle z, w\rangle| \ / \ g_{p\{q\}}(w; M' \otimes N') \le 1 \right\}.$$

We remark that $E' \otimes_{g_{p\{q\}}} F'$ is an isometric subspace of $(E \otimes_{g'_{p\{q\}}} F)'$ because $g_{p\{q\}}$ is finitely generated, see [2], 15.3.

In this case, we define $\mathbf{I}_{p\{q\}}(T)$ to be the norm of $J_F\ T$ considered as an element of the topological dual of the Banach space $E \otimes_{g'_{p\{q\}}} F'$. Notice that $\mathbf{I}_{p\{q\}}(T) = \mathbf{I}_{p\{q\}}(J_F\ T)$ as a consequence of F' be canonically complemented in F'''.

An essential example of $p\{q\}$-integral operators is given in the following theorem:

Theorem 3.5 *Let $1 < p, q < \infty$ and let G be an abstract M-space. Then every order bounded operator $T : G \to \ell_p\{\ell_q\}$ is $p\{q\}$-integral with $\boldsymbol{I}_{p\{q\}}(T) = \|T\|$.*

Proof. As G is an abstract M-space, its dual G' is lattice isomorphic to $L_1(\mu)$ for some measure space (Ω, Σ, μ) and hence there is an isometric order isomorphism $B : G'' \to L_\infty(\mu)$ from the bidual G'' onto $L_\infty(\mu)$. Noting that $T = T''\ B^{-1}\ B\ J_G$ with $T''\ B^{-1} : L_\infty(\mu) \to \ell_p\{\ell_q\}$, it is enough to see that every bounded operator $S : L_\infty(\mu) \longrightarrow \ell_p\{\ell_q\}$ is $p\{q\}$-integral. Let $\mathcal{T}$ be the linear span of the set $\{\mathbf{e}_{ij},\ i, j \in \mathbb{N}\}$ which is dense in $\ell_p\{\ell_q\}$. Then, by the representation theorem of maximal operator ideals (see 17.5 in [2]) and the density lemma (theorem 13.4 in [2]), we only have to show that $S \in (L_\infty(\mu) \otimes_{g'_{p\{q\}}} \mathcal{T})'$.

Given $z \in L_\infty(\mu) \otimes_{g'_{p\{q\}}} \mathcal{T}$ and $\varepsilon > 0$, let X and Y be finite dimensional subspaces of $L_\infty(\mu)$ and $\mathcal{T}$ respectively such that $z \in X \otimes Y$ and

$$g'_{p\{q\}}(z; X \otimes Y) \leq g'_{p\{q\}}(z; L_\infty(\mu) \otimes \mathcal{T}) + \varepsilon. \tag{2}$$

Let $\{\mathbf{g}_s\}_{s=1}^m$ be a basis for Y and let $k_1, k_2 \in \mathbb{N}$ be such that

$$\mathbf{g}_s = \sum_{i=1}^{k_1} \sum_{j=1}^{k_2} c_{sij} \mathbf{e}_{ij} \text{ for every } 1 \leq s \leq m.$$

Then for every $f \in X$ and $1 \leq s \leq m$

$$\begin{aligned} \langle S, f \otimes \mathbf{g}_s \rangle &= \langle f, S'(\mathbf{g}_s) \rangle = \left\langle f, \left(\sum_{i=1}^{k_1} \sum_{j=1}^{k_2} c_{sij} \right) S'(\mathbf{e}_{ij}) \right\rangle \\ &= \left\langle f \otimes \sum_{t=1}^{k_1} \sum_{l=1}^{k_2} c_{stl}\, \mathbf{e}_{tl}, \sum_{i=1}^{k_1} \sum_{j=1}^{k_2} S'(\mathbf{e}_{ij}) \otimes \mathbf{e}_{ij} \right\rangle. \end{aligned}$$

Then if U denotes the tensor

$$U := \sum_{i=1}^{k_1} \sum_{j=1}^{k_2} S'(\mathbf{e}_{ij}) \otimes \mathbf{e}_{ij} \in L_\infty(\mu)' \otimes \ell_p\{\ell_q\},$$

by bylinearity we get for all $z \in X \otimes Y$ $\langle z, S \rangle = \langle U, z \rangle$.

Given $\nu > 0$, for every $1 \leq i \leq k_1$ and $1 \leq j \leq k_2$ there is $f_{ij} \in L_\infty(\mu)$ such that $\|f_{ij}\| \leq 1$ and $\|S'(\mathbf{e}_{ij})\| \leq |\langle S'(\mathbf{e}_{ij}), f_{ij} \rangle| + \nu$. Then $f := \sup_{1 \leq i \leq k_1} \sup_{1 \leq j \leq k_2} f_{ij}$ lies in the closed unit ball of $L_\infty(\mu)$. On the other hand, we know that $\ell_p\{\ell_q\}$ is order complete. By the Riesz-Kantorovich theorem (see theorem 1.13 in [1] for instance), the modulus $|S|$ of the operator S exists in

$\mathcal{L}(L_\infty(\mu), \ell_p\{\ell_q\})$. By the lattice properties of $\ell_p\{\ell_q\}$, we have

$$\begin{aligned}
\pi_{p\{q\}}((S'(e_{ij})) &= \left\| \sum_{i=1}^{k_1}\sum_{j=1}^{k_2} \|S'(\mathbf{e}_{ij})\| \, \mathbf{e}_{ij} \right\|_{p\{q\}} \\
&\leq \left\| \sum_{i=1}^{k_1}\sum_{j=1}^{k_2} |\langle S'(\mathbf{e}_{ij}), f_{ij}\rangle| \, \mathbf{e}_{ij} \right\|_{p\{q\}} + \nu \left\| \sum_{i=1}^{k_1}\sum_{j=1}^{k_2} \mathbf{e}_{ij} \right\|_{p\{q\}} \\
&\leq \left\| \sum_{i=1}^{k_1}\sum_{j=1}^{k_2} |\langle S(f_{ij}), \mathbf{e}_{ij}\rangle| \, \mathbf{e}_{ij} \right\|_{p\{q\}} + \nu \left\| \sum_{i=1}^{k_1}\sum_{j=1}^{k_2} \mathbf{e}_{ij} \right\|_{p\{q\}} \\
&\leq \left\| \sum_{i=1}^{k_1}\sum_{j=1}^{k_2} \langle |S(f_{ij})|, \mathbf{e}_{ij}\rangle \right\|_{p\{q\}} + \nu \left\| \sum_{i=1}^{k_1}\sum_{j=1}^{k_2} \mathbf{e}_{ij} \right\|_{p\{q\}} \\
&\leq \left\| \sum_{i=1}^{k_1}\sum_{j=1}^{k_2} \langle |S|(|f_{ij}|), \mathbf{e}_{ij}\rangle \, \mathbf{e}_{ij} \right\|_{p\{q\}} + \nu \left\| \sum_{i=1}^{k_1}\sum_{j=1}^{k_2} \mathbf{e}_{ij} \right\|_{p\{q\}} \\
&\leq \left\| \sum_{i=1}^{k_1}\sum_{j=1}^{k_2} \langle |S|(|f|), \mathbf{e}_{ij}\rangle \, \mathbf{e}_{ij} \right\|_{p\{q\}} + \nu \left\| \sum_{i=1}^{k_1}\sum_{j=1}^{k_2} \mathbf{e}_{ij} \right\|_{p\{q\}} \\
&= \| \, |S|(|f|)\|_{p\{q\}} + \nu \left\| \sum_{i=1}^{k_1}\sum_{j=1}^{k_2} \mathbf{e}_{ij} \right\|_{p\{q\}} \\
&\leq \| \, |S| \, \| + \nu \left\| \sum_{i=1}^{k_1}\sum_{j=1}^{k_2} \mathbf{e}_{ij} \right\|_{p\{q\}} .
\end{aligned}$$

Moreover

$$\begin{aligned}
\varepsilon_{p'\{q'\}}(((\mathbf{e}_{ij})_{i=1}^{k_1})_{j=1}^{k_2}) &= \sup_{\|(\beta_{tl})\|_{p'\{q'\}}\leq 1} \left\| \sum_{i=1}^{k_1}\sum_{j=1}^{k_2} \langle \mathbf{e}_{ij}, (\beta_{tl})\rangle \, \mathbf{e}_{ij} \right\|_{p'\{q'\}} \\
&= \left\| \sum_{i=1}^{k_1}\sum_{j=1}^{k_2} \beta_{ij} \, \mathbf{e}_{ij} \right\|_{p'\{q'\}} \leq 1.
\end{aligned}$$

Hence, denoting by I_X and I_Y the corresponding inclusion maps into $L_\infty(\mu)$ and $\ell_p\{\ell_q\}$ respectively, we have

$$\begin{aligned}
|\langle S,z\rangle| &= |\langle U,z\rangle| = |\langle U,((I_X)'\otimes(I_Y)')(z)\rangle| \\
&\leq g_{p\{q\}}(U;X\otimes Y)\; g'_{p\{q\}}(((I_X)'\otimes(I_Y)')(z);X'\otimes Y') \\
&\leq g_{p\{q\}}(U;X\otimes Y)\; g'_{p\{q\}}(((I_X)'\otimes(I_Y)')(z);X'\otimes Y') \\
&\leq (g_{p\{q\}}(U;L_\infty\otimes\ell_p\{\ell_q\})+\varepsilon) g'_{p\{q\}}(z;L_\infty(\mu)\otimes\ell_{p'}\{\ell_{q'}\}) \\
&\leq g'_{p\{q\}}(z;L_\infty(\mu)\otimes\ell_{p'}\{\ell_{q'}\}\})\ \left(\pi_{p\{q\}}((S'(\mathbf{e}_{ij}))\ \varepsilon_{p'\{q'\}}((\mathbf{e}_{ij}))+\varepsilon\right) \\
&\leq g'_{p\{q\}}(z;L_\infty(\mu)\otimes\ell_{p'}\{\ell_{q'}\}\})\left(\|\,|S|\,\|+\nu\left\|\sum_{i=1}^{k_1}\sum_{j=1}^{k_2}\mathbf{e}_{ij}\right\|_{p\{q\}}+\varepsilon\right)
\end{aligned}$$

and ν being arbitrary $|\langle S,z\rangle| \leq g'_{p\{q\}}(z;L_\infty(\mu)\otimes\ell_{p'}\{\ell_{q'}\})(\|\,|S|\,\|+\varepsilon)$. Finally, by the arbitrariness of ε, we get

$$|\langle S,z\rangle| \leq g'_{p\{q\}}(z;L_\infty(\mu)\otimes\ell_{p'}\{\ell_{q'}\})\ \|\,|S|\,\|.$$

But from [1] theorem 1.10, $|S|(\chi_\Omega) = \sup\{|S(f)|, |f| \leq \chi_\Omega\}$ and as $\ell_p\{\ell_q\}$ is order continuous

$$\|\,|S|\,\| = \|\,|S|(\chi_\Omega)\| = \sup\{\|\,|S(f)|\,\|, \|f\| \leq 1\} = \|S\|.$$

Then S is $p\{q\}-$integral with $\mathbf{I}_{p\{q\}}(S) \leq \|S\|$. As $(\mathcal{I}_{p\{q\}}, \mathbf{I}_{p\{q\}})$ is a Banach operators ideal, $\|S\| \leq \mathbf{I}_{p\{q\}}(S)$, hence $\mathbf{I}_{p\{q\}}(S) = \|S\|$. ∎

Corollary 3.6 *Let G be an abstract M-space with unit. Then every operator $T: G \to \ell_p^n\{\ell_q^m\}$ is $p\{q\}$-integral with $\mathbf{I}_{p\{q\}}(T) = \|T\|$.*

Proof. The results follows easily from theorem 3.5 because every operator $T: G \to \ell_p^n\{\ell_q^m\}$ is order bounded and $\ell_p^n\{\ell_q^m\}$ is order continuous. ∎

For our next theorem, we need a very deep technical result due to Lindenstrauss and Tzafriri [9] which gives us a kind of "uniform approximation" of finite dimensional subspaces by finite dimensional sublattices in Banach lattices.

Lemma 3.7 *Let $\varepsilon > 0$ and $n \in \mathbb{N}$ be fixed. There is a natural number $h(n,\varepsilon)$ such that for every Banach lattice X and every subspace $F \subset X$ of dimension* $\dim(F) = n$ *there are $h(n,\varepsilon)$ disjoints elements $\{z_i,\ 1 \leq i \leq h(n,\varepsilon)\}$ and an operator A from F into the linear span G of $\{z_i,\ 1 \leq i \leq h(n,\varepsilon)\}$ such that for every $x \in F$ $\|A(x) - x\| \leq \varepsilon\ \|x\|$*

Theorem 3.8 *For $1 \leq p,q < \infty$, G an abstract M-space, and X a quasi $\mathcal{L}^{p,q}$ space or a complemented subspace of a quasi $\mathcal{L}^{p,q}$ space. Then every operator $T: G \longrightarrow X$ is $p\{q\}$-integral and there is a constant $K > 0$ such that $\mathbf{I}_{p\{q\}}(T) \leq K\ \|T\|$.*

Proof. By the representation theorem of maximal operator ideals (see 17.5 in [2]), if J_X is the inclusion map of X into X'', we only need to show that $J_X T \in (G \otimes_{g'_{p\{q\}}} X')'$.

Given $z \in G \otimes X'$ and $\varepsilon > 0$, let $M \subset G$ and $N \subset X'$ be finite dimensional subspaces, and let $z = \sum_{i=1}^n f_i \otimes x'_i$ be a fixed representation of z with $f_i \in M$ and $x'_i \in N$, $i = 1, 2, .., n$ such that

$$g'_{p\{q\}}(z; G \otimes X') \leq g'_{p\{q\}}(z; M \otimes N) \leq g'_{p\{q\}}(z; G \otimes X') + \varepsilon.$$

By lemma 3.7, we obtain a finite dimensional sublattice M_1 of G and an operator $A : M \to M_1$ so that for every $f \in M$, $\|A(f) - f\| \leq \varepsilon \|f\|$. Then, if id_G denotes the identity map on G, we have

$$|\langle T, z\rangle| = \left|\sum_{i=1}^n \langle T(f_i), x'_i\rangle\right| \leq \left|\sum_{i=1}^n \langle\ (id_G - A)(f_i), x'_i\rangle\right| + \left|\sum_{i=1}^n \langle T\ A(f_i), x'_i\rangle\right|$$

$$\leq \varepsilon \|T\| \sum_{i=1}^n \|f_i\|\ \|x'_i\| + \left|\sum_{i=1}^n \langle T\ A(f_i), x'_i\rangle\right|.$$

Put $X_1 := T(M_1)$. By hypothesis, X is a *quasi*-$\mathcal{L}^{p,q}$-space. Hence, there are a finite dimensional subspace X_2 of X, some complemented finite dimensional subspace H of $\ell_p\{\ell_q\}$ with projection $P_H : \ell_p\{\ell_q\} \to H$ such that $\|P_H\| \leq b$ for some $b > 0$, and an isomorphism $V : X_2 \longrightarrow H$ such that $X_1 \subset X_2$ and $\|V\|\ \|V^{-1}\| \leq a$ for some positive real constant a. Let $I_{X_1} : X_1 \longrightarrow X_2$ be the inclusion map. To simplify notation, we denote $R : M_1 \longrightarrow H$ such that $R := V\ I_{X_1}\ T$. Let $K_2 : X''' \longrightarrow X'_2 = X'''/X_2^\circ$ be the canonical quotient map. Then

$$\begin{aligned}
\sum_{i=1}^n \langle T\ (A(f_i)), x'_i\rangle &= \sum_{i=1}^n \langle I_{X_1}\ T\ (A(f_i)), K_2(x'_i)\rangle \\
&= \sum_{i=1}^n \langle V^{-1}\ V\ I_{X_1}\ T\ (A(f_i)), K_2(x'_i)\rangle \\
&= \sum_{i=1}^n \langle R\ A(f_i), (V^{-1})' K_2(x'_i)\rangle \\
&= \left\langle R, \sum_{i=1}^n A(f_i) \otimes (V^{-1})'\ K_2(x'_i)\right\rangle
\end{aligned}$$

with $\sum_{i=1}^n A(f_i) \otimes (V^{-1})'\ K_2(x'_i) \in M_1 \otimes H'$. As $\ell_p\{\ell_q\}$ is an $\mathcal{L}^{p,q}$ space, H is contained in some r-dimensional subspace W of $\ell_p\{\ell_q\}$ such that $d(W, \ell_p^r\{\ell_q^r\}) < 1 + \varepsilon$.

We denote I_H the inclusion of H in W and by $C : W \to \ell_p^r\{\ell_q^r\}$ such that $\|C\|\|C^{-1}\| \leq 1+\varepsilon$.

Since M_1 is an abstract M-space with unit, the map $C\ I_H\ R\ : M_1 \to \ell_p^r\{\ell_q^r\}$ is order bounded by corollary 3.6, hence $\mathbf{I}_{p\{q\}}(CI_H R) \leq \|C\|\|R\| \leq \|C\|\|V\|\|T\|$. Then, as

$$R = (P_H)_{|W}\ C^{-1}\ C\ I_H\ R,$$

R is again $p\{q\}$-integral with $\mathbf{I}_{p\{q\}}(R) \leq \|P_H\|\ \|C^{-1}\|\ \|C\|\ \|V\|\ \|T\| \leq (1+\varepsilon)\ b\ \|V\|\ \|T\|$. Therefore,

$$\left|\sum_{i=1}^{n}\langle T\ (A(f_i)), x_i'\rangle\right| = \left|\left\langle R, \sum_{i=1}^{n} A(f_i) \otimes (V^{-1})'\ K_2(x_i')\right\rangle\right| \leq$$

$$\leq \mathbf{I}_{p\{q\}}(R)\ g_{p\{q\}}(\sum_{i=1}^{n} A(f_i) \otimes (V^{-1})'\ K_2(x_i'); M_1 \otimes H') \leq$$

$$\leq (1+\varepsilon)\ b\ \|V\|\ \|T\|\ g_{p\{q\}}\left((A \otimes (V^{-1})'\ K_2)(z); M_1 \otimes H'\right) \leq$$

$$\leq (1+\varepsilon)\ b\ \|V\|\ \|T\|\ \|A\|\ \|(V^{-1})'\|\ \|K_2\|\ g_{p\{q\}}(z; M \otimes N) \leq$$

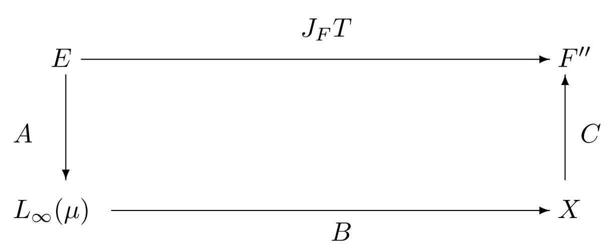
$$\leq (1+\varepsilon)^2\ a\ b\ \|T\|\ g_{p\{q\}}(z; M \otimes N) \leq (1+\varepsilon)^2\ a\ b\ \|T\|\ (g_{p\{q\}}(z; G \otimes X) + \varepsilon)$$

and by the arbitrariness of $\varepsilon > 0$ we obtain

$$|\langle T, z\rangle| \leq a\, b\, \|T\|\ \ g'_{p\{q\}}\,(z; G \otimes X')\,.\blacksquare$$

Concerning necessary conditions for an operator to be $p\{q\}$-integral we have:

Theorem 3.9 *[4] Let $1 \leq p, q < \infty$. For every pair of Banach spaces E, F if $T \in \mathcal{I}_{p\{q\}}(E,F)$ then $J_F T$ factors as:*

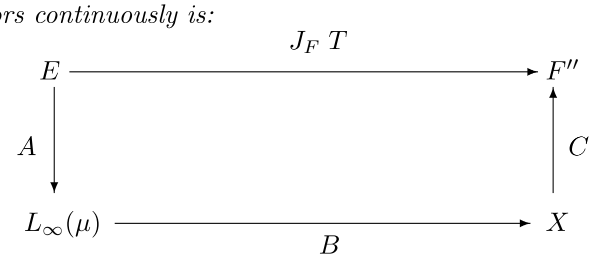


where X is some ultrapower $(\ell_p\{\ell_q\})_{\mathcal{U}_1}$ of $\ell_p\{\ell_q\}$ and B is a lattice homomorphism. Moreover $\mathbf{I}_{p\{q\}}(T) \geq \inf \|C\|\ \|B\|\ \|A\|$ taking the infimum over all such factorizations.

Theorem 3.10 *Let $1 \leq p, q < \infty$ and let E and F be Banach spaces. The following statements are equivalent:*

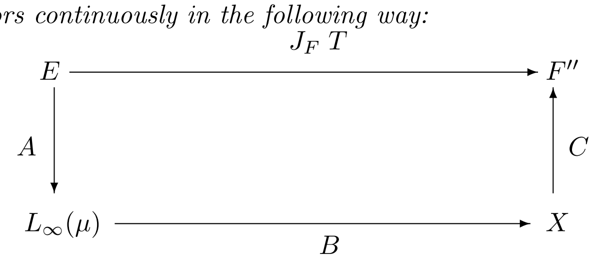
1) $T \in \mathcal{I}_{p\{q\}}(E, F)$.

2) $J_F T$ factors continuously is:



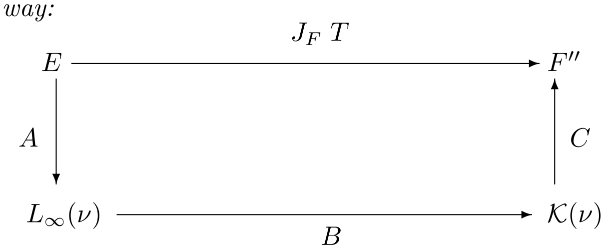
where X is a quasi $\mathcal{L}^{p,q}$-space. Furthermore, the norm $\mathbf{I}_{p\{q\}}(T)$ is equal to $\inf\{\|C\|\|B\|\|A\|\}$*, taking the infimum over all such factorizations.*

3) $J_F T$ factors continuously in the following way:



where X is a quasi $\mathcal{L}^{p,q}$-lattice and B is a lattice homomorphism. Furthermore $\mathbf{I}_{p\{q\}}(T)$ is equal to $\inf\{\|C\|\|B\|\|A\|\}$*, taking the infimum over all such factorizations.*

4) It exists a σ-finite measure space $(\mathcal{O}, \mathcal{S}, \nu)$ and a Köthe function space $\mathcal{K}(\nu)$ which is complemented in a quasi $\mathcal{L}^{p,q}$, such that $J_F T$ factors continuously in the following way:



where B is a multiplication operator for a positive function of $\mathcal{K}(\nu)$. Furthermore, $\mathbf{I}_{p\{q\}}(T)$ is equal to $\inf\{\|C\|\|B\|\|A\|\}$*, taking the infimum over all such factorizations.*

Proof. For 1) ⇔ 2) and 1) ⇔ 3), we remember that $(\ell_p\{\ell_q\})_{\mathcal{U}}$ is a *quasi* $\mathcal{L}^{p,q}$ space, and we use theorems 3.9 and 3.8. For 1) ⇔ 4), we have to keep in mind that $\ell_p\{\ell_q\}$ has finite cotype; hence, every ultrapower of $\ell_p\{\ell_q\}$ is order continuous (Henson and Moore, [5], 4.6), and by [8] theorem 1.a.9, it may be decomposed into an unconditional direct sum of a family of mutually disjoint ideals $\{X_h, h \in H\}$ having a positive weak unit. Then, from 1.b.14 in [8], as every X_h is order isometric to a Köthe function space defined on a probability space $(\mathcal{O}_h, \mathcal{S}_h, \nu_h)$, then $(\ell_p\{\ell_q\})_{\mathcal{U}}$ is order isometric to a Köthe function space $\mathcal{K}(\nu^1)$ on a measure space $(\mathcal{O}^1, \mathcal{S}^1, \nu^1)$, hence we may replace $(\ell_p\{\ell_q\})_{\mathcal{U}}$ by $\mathcal{K}(\nu^1)$ in 2). If we denote $z := B(\chi_\Omega)$ as $z = \sum_{i=1}^{\infty} y_{h_i}$ with $y_{h_i} \in X_{h_i}$ for every $i \in \mathbb{N}$, then $B(L_\infty(\mu))$ is contained in the unconditional direct sum of $\{X_{h_i}, i \in \mathbb{N}\}$ which is order isometric to a Köthe function space $\mathcal{K}(\nu)$ on a σ-finite measure space $(\mathcal{O}, \mathcal{S}, \nu)$, which is 1-complemented in $\mathcal{K}(\nu^1)$.

Finally, as $\mathcal{K}(\nu)$ is order complete, it exists $g := \sup_{\|f\|_{L_\infty(\mu)}} B(f)$ in $\mathcal{K}(\nu)$. Then, the operators $B_1 : L_\infty(\mu) \to L_\infty(\nu)$ and $B_2 : L_\infty(\nu) \to \mathcal{K}(\nu)$, defined as $B_1(f)(\omega) := B(f)(\omega)/g(\omega)$, for all $f \in L_\infty(\mu)$, $\omega \in \mathcal{O}$ with $g(\omega) \neq 0$ and $B_1(f)(\omega) = 0$ otherwise, and $B_2(h)(\omega) := g(\omega)h(\omega)$ for all $h \in L_\infty(\nu)$, $\omega \in \mathcal{O}$, satisfy $B = B_2 B_1$ and B_2 is a multiplication operator by a positive element $g \in \mathcal{K}(\nu)$. ■

4 Coincidence between $p\{q\}$-nuclear and $p\{q\}$-integral operators.

For $1 \leq p, q < \infty$, we introduce a new operator ideal, which is contained in the ideal of the $p\{q\}$-integral operators .

Definition 4.1 *Let $1 \leq p, q < \infty$. We say that $T \in \mathcal{L}(E, F)$ is **strictly** $p\{q\}$-**integral** if a σ-finite measure space $(\mathcal{O}, \mathcal{S}, \nu)$ and a Köthe function space $\mathcal{K}(\nu)$ exist which is complemented in some quasi $\mathcal{L}^{p,q}$ space, such that T factors continuously in the following way:*

$$\begin{array}{ccc} E & \xrightarrow{\quad T \quad} & F \\ {\scriptstyle A}\downarrow & & \uparrow{\scriptstyle C} \\ L_\infty(\nu) & \xrightarrow[\quad B \quad]{} & \mathcal{K}(\nu) \end{array}$$

where B is a multiplication operator for a positive function of $\mathcal{K}(\nu)$.

We denote by $\mathcal{SI}_{p\{q\}}(E,F)$ the set of the strictly $p\{q\}$-integral operators between E and F which is closed subspace of $\mathcal{I}_{p\{q\}}(E,F)$ and $\mathbf{SI}_{p\{q\}}(T) = \mathbf{I}_{p\{q\}}(T)$ for every $T \in \mathcal{SI}_{p\{q\}}(E,F)$. It is clear that if F is a dual space, or it is complemented in its bidual space, then $\mathcal{SI}_{p\{q\}}(E,F) = \mathcal{I}_{p\{q\}}(E,F)$.

Theorem 4.2 *Let $1 \leq p,q < \infty$, and let E and F be Banach spaces, such that E' satisfies the Radon-Nikodým property, then $\mathcal{N}_{p\{q\}}(E,F) = \mathcal{SI}_{p\{q\}}(E,F)$.*

Proof. We suppose that E' has the Radon-Nikodým property and let $T \in \mathcal{SI}_{p\{q\}}(E,F)$.

a) We suppose that B is a multiplication operator by a function $g \in \mathcal{K}(\nu)$ with support on D, a set of finite measure. We denote ν_D the restriction of ν to D.

As $(\chi_D A) : E \to L_\infty(\nu_D)$, then $(\chi_D A)' : (L_\infty(\nu_D))' \to E'$ and the restriction of $(\chi_D A)' \restriction_{L_1(\nu_D)}: L_1(\nu_D) \to E'$, thus, for every $x \in E$ and $f \in L_1(\nu_D)$

$$\langle x, (\chi_D A)'(f)\rangle = \langle \chi_D A(x), f\rangle = \int_D \chi_D A(x)\, f d(\nu_D).$$

As E' has the Radon-Nikodým property, applying theorem III(5) of [3], we have that $(\chi_D A)'$ has a Riesz representation; therefore, it exists a function $\phi \in L_\infty(\nu_D, E')$ such that for every $f \in L_1(\nu_D)$

$$(\chi_D A)'(f) = \int_D f\phi d(\nu_D).$$

Then, for every $x \in E$, we have that $\chi_D A(x)(t) = \langle \phi(t), x\rangle$, ν_D-almost everywhere in D, and then $B(\chi_D A)(x) =< g\phi(.), x >$, ν_D-almost everywhere in D. We denote by $g\phi$ this last operator, and we can consider it as an element of $\mathcal{K}(\nu_D, E')$.

Now, as the simple functions are dense in $\mathcal{K}(\nu_D, E')$, $g\phi$ can be approximated by a sequence of simple functions $(S_k)_{k=1}^{\infty}$.

We suppose $S_k = \sum_{j=1}^{m_k} x'_{kj}\chi_{A_{kj}}$, where $\{A_{ki} : i = 1,..,m\}$ is a family of pairwise disjoint ν-measurable sets of Ω . For each $k \in \mathbb{N}$, we can interpret S_k as a map $S_k : E \to \mathcal{K}(\nu)$ such that $S_k(x) = \sum_{j=1}^{m_k} < x'_{kj}, x > \chi_{A_{kj}}$ with norm less than or equal to the norm of S_k in $\mathcal{K}(\nu, E')$.

Obviously, for all $k \in \mathbb{N}$, S_k is $p\{q\}$-nuclear because it has finite rank , but we need to evaluate its $p\{q\}$-nuclear norm $\mathbf{N}_{p\{q\}}(S_k)$ which coincides with its $p\{q\}$-integral norm $\mathbf{I}_{p\{q\}}(S_k)$.

Let $S_k^1 : E \to L_\infty(\nu)$ be defined as $S_k^1(x) := \sum_{j=1}^{m_k} \frac{\langle x'_{kj}, x\rangle}{\|x'_{kj}\|}\chi_{A_{kj}}$ and $S_k^2 : L_\infty(\nu) \to \mathcal{K}(\nu)$ as $S_k^2(f) := \sum_{j=1}^{m_k} \|x'_{kj}\| f\chi_{A_{kj}}$. It is easy to see that $\|S_k^1\| \leq 1$, $\|S_k^2\| \leq \|S_k\|_{\mathcal{K}(\nu,E')}$ and $S_k = S_k^2\, S_k^1$.

As $\mathcal{K}(\nu)$ is a complemented subspace of a quasi $\mathcal{L}^{p,q}$, from 3.8, there is $K > 0$ such that $\mathbf{N}_{p\{q\}}(S_k^2) = \mathbf{I}_{p\{q\}}(S_k^2) \leq K\ \|S_k^2\| \leq K\ \|S_k\|_{\mathcal{K}(\nu,E')}$, hence

$\mathbf{N}_{p\{q\}}(S_k) \leq K \; \|S_k\|_{\mathcal{K}(\nu,E')}$. Then, as $(S_k)_{k=1}^{\infty}$ converges in $\mathcal{K}(\nu_D, E')$, it is a Cauchy sequence in the complete space $\mathcal{N}_{p\{q\}}(E, \mathcal{K}(\nu_D))$, so $(S_k)_{k=1}^{\infty}$ converges to $g\phi$, i.e., $g\phi \in \mathcal{N}_{p\{q\}}(E, \mathcal{K}(\nu_D))$. Therefore, $g\phi = B\chi_D A$ is $p\{q\}$-nuclear and so is T.

b) If g is any element of $\mathcal{K}(\nu)$, it can be approximated in norm by means of a sequence $(t_n)_{n=1}^{\infty}$ of simple functions with finite measure support therefore, by a), the sequence $T_n = CB_{t_n}A$ is a Cauchy sequence in $\mathcal{N}_{p\{q\}}(E, F)$ converging to T in $\mathcal{L}(E, F)$, and then $T \in \mathcal{N}_{p\{q\}}(E, F)$. ■

As a consequence of the former result and the factorization theorems 3.10 and 3.4, we obtain the following metric properties of $g_{p\{q\}}$ and $(g_{p\{_q\}})'$.

Theorem 4.3 *$(g_{p\{q\}})'$ is a totally accessible tensor norm.*

Proof. As $(g_{p\{q\}})'$ is finitely generated, it is sufficient to prove that the map $F \otimes_{(g_{p\{q\}})'} E \hookrightarrow \mathcal{P}_{p'\{q'\}}(E', F'')$ is an isometry.

In fact, let $z = \sum_{i=1}^{n} \sum_{j=1}^{l_i} y_{ij} \otimes x_{ij} \in F \otimes_{(g_{p\{q\}})'} E$, and let $H_z \in \mathcal{P}_{p'\{q'\}}(E', F'')$ be the canonical map associated to z,

$$H_z\left(x'\right) = \sum_{i=1}^{n} \sum_{j=1}^{l_i} \langle x_{ij}, x' \rangle \, y_{ij}$$

for all $x' \in E'$, where $H_z \in \mathcal{L}\left(E', F\right) \subset \mathcal{L}\left(E', F''\right)$.

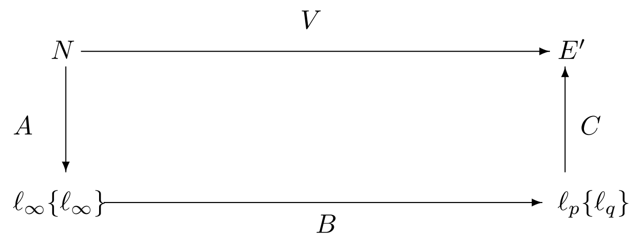
Applying theorem 15.5 of [2] with $\alpha = (g_{p\{q\}})'$, theorem 3.2, and the equality $(g_{p\{q\}})'' = g_{p\{q\}}$, since $g_{p\{q\}}$ is finitely generated, the inclusion

$$F \otimes_{\overleftarrow{(g_{p\{q\}})'}} E \hookrightarrow \left(F' \otimes_{g_{p\{q\}}} E'\right)' \to \mathcal{P}_{p'\{q'\}}\left(E', F''\right)$$

is an isometry, therefore by proposition 12.4 in [2], we obtain

$$\Pi_{p'\{q'\}}\left(H_z\right) = \overleftarrow{(g_{p\{q\}})'}\left(z; F \otimes E\right) \leq (g_{p\{q\}})'\left(z; F \otimes E\right).$$

On the other hand, given N, a finite dimensional subspace of F such that $z \in N \otimes_{(g_{p\{q\}})'} E$, there exists $V \in (N \otimes_{(g_{p\{q\}})'} E)' = \mathcal{I}_{p\{q\}}(N, E')$ such that $\mathbf{I}_{p\{q\}}(V) \leq 1$ and $(g_{p\{q\}})'(z; N \otimes E) = \langle z, V \rangle$. Clearly enough $V \in \mathcal{SI}_{p\{q\}}(N, E') = \mathcal{I}_{p\{q\}}(N, E')$ because E' is a dual space, and N', being finite dimensional, has the Radon-Nikodým property. Therefore by theorem 4.2, $V \in \mathcal{N}_{p\{q\}}\left(N, E'\right)$ and by theorem 3.4, given $\varepsilon > 0$, there is a factorization of V of the form



such that $\|C\|\|B\|\|A\| \leq \mathbf{N}_{p\{q\}}(V) + \varepsilon = \mathbf{I}_{p\{q\}}(V) + \varepsilon \leq 1 + \varepsilon$.

As $\ell_\infty\{\ell_\infty\}$ has the extension metric property, (see proposition 1, C.3.2. in [14]), A may be extended to a continuous map $\overline{A} \in \mathcal{L}\left(F, \ell_\infty\{\ell_\infty\}\right)$ such that $\left\|\overline{A}\right\| = \|A\|$. By theorem 3.4 again, $W := CB\overline{A}$ is in $\mathcal{N}_{p\{q\}}\left(F, E'\right)$, so there is a representation $w =: \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} y'_{ij} \otimes x'_{ij} \in F'\widehat{\otimes}_{g_{p\{q\}}} E'$ of W satisfying

$$\sum_{i=1}^{\infty} \pi_{p\{q\}}\left(\left(y'_{ij}\right)\right) \varepsilon_{p'\{q'\}}\left(\left(x'_{ij}\right)\right) \leq \mathbf{N}_{p\{q\}}\left(W\right) + \varepsilon \leq \|C\|\,\|B\|\,\left\|\overline{A}\right\| + \varepsilon \leq 1 + 2\varepsilon.$$

Then, $(g_{p\{q\}})'(z; F \otimes E) \leq (g_{p\{q\}})'(z; N \otimes E) = \langle z, V\rangle = \langle z, W\rangle$ and it follows that

$$(g_{p\{q\}})'(z; F \otimes E) \leq g_{p\{q\}}(w)\Pi_{p'\{q'\}}(H_z) \leq (1 + 2\varepsilon)\Pi_{p'\{q'\}}(H_z)$$

whence $(g_{p\{q\}})'(z; F \otimes E) \leq \Pi_{p'\{q'\}}(H_z)$, and the equality becomes obvious. ■

Corollary 4.4 $g_{p\{q\}}$ *is an accesible tensor norm.*

Proof. It is a direct consequence of the former theorem and proposition 15.6 of [2].■

In the following theorem, related to the approximation property, we have to keep in mind that every $p\{q\}$-absolutely summing operator T is a p-summing operator with the Saphar's tensor norm g_p , and therefore, it is absolutely continuous according to Niculescu's definition.

Theorem 4.5 *Let E, F be Banach spaces such that E' has the approximation property and E does not contain a copy of ℓ_1. Then $\mathcal{F}(E, F)$ is dense in $\mathcal{P}_{p'\{q'\}}(E, F)$.*

Proof. Let $T \in \mathcal{P}_{p'\{q'\}}(E, F)$. Then, T is absolutely continuous, and by theorem 2.2. in Niculescu [12], as E contains no isomorphic copy of ℓ_1, T is compact. Finally, as E' has the approximation property, by proposition 5.3 (2) in [2], $T \in E'\widehat{\otimes}_\varepsilon F = \overline{\mathcal{F}}(E, F)$. ■

Theorem 4.6 *If F has the approximation property, then $E\widehat{\otimes}_{g_{p\{q\}}}F$, $E\otimes_{g'_{p\{q\}}}F$, $\mathcal{N}_{p\{q\}}(E,F)$, $\mathcal{P}_{p'\{q'\}}(E,F)$ and $\mathcal{I}_{p\{q\}}(E,F)$ are reflexive if and only if E y F are reflexive.*

Proof. The necessity part is evident. For the sufficiency, suppose that E and F are reflexive. As F is reflexive and it has the approximation property, it does not contain a copy of ℓ_1 and by corollary 9 (p. 244) of [3] F' has the approximation property. Then, applying theorems 4.5 and 3.2

$$\left(E\widehat{\otimes}_{g_{p\{q\}}}F\right)' = \mathcal{P}_{p'\{q'\}}(F,E') = E'\widehat{\otimes}_{g'_{p\{q\}}}F'.$$

By corollary 4 (pg. 82) of [3], it follows that E'' has the Radon-Nikodym property and then, by theorems, 4.2 and 4.3

$$\left(E'\widehat{\otimes}_{g'_{p\{q\}}}F'\right)' = \mathcal{I}_{p\{q\}}(E',F) = \mathcal{SI}_{p\{q\}}(E',F) = \mathcal{N}_{p\{q\}}(E',F).$$

As F has the approximation property, by corollary 1, 22.2 of [2], the equality

$$\mathcal{N}_{p\{q\}}(E',F) = E\widehat{\otimes}_{g_{p\{q\}}}F$$

follows. Then $E\widehat{\otimes}_{g_{p\{q\}}}F$, and therefore, all the spaces, as stated in the theorem, are reflexive. ■

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