DIVULGACIÓN MATEMÁTICA

Another perspective on a famous problem, IMO 1988: The equation $\frac{x^2+y^2}{xy+1} = n^2$

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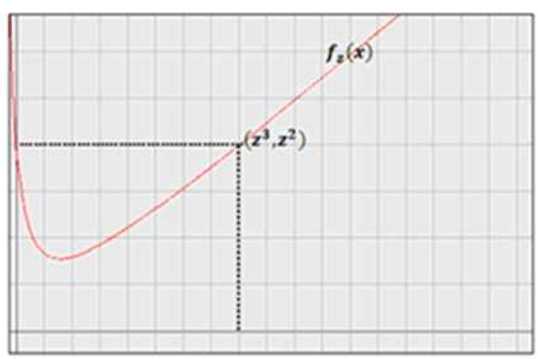
Abstract. In this work we apply a simple property of the function F below to study an interesting IMO problem proposed in 1988 of which we give a solution. We analyze with some detail the diophantine equation $F(x, y) = n^2$ in connection with this problem.

Resumen. En este trabajo se aplica una simple propiedad de la función F, ver abajo, para estudiar un interesante problema propuesto en la OMI de 1988, del cual damos una solución. Se analiza con cierto detalle la ecuación diofántica $F(x, y) = n^2$ en relación con este problema.

The symmetrical function $F(x, y) = \frac{x^2+y^2}{xy+1}$ of $\mathbb{R}_+ \times \mathbb{R}_+$ in $\mathbb{R}_+$ has the remarkable property, trivial to verify: $F(x, x^3) = F(x, 0) = x^2$ for all x.



$f_z(x) = \frac{z^2+x^2}{zx+1}$, $x \geq 0, z \geq 0$. *Always* $f_z(0) = f_z(z^3) = z^2$

Here we use basically this property to determine an infinity of integer solutions of the equation $F(x, y) = n^2$ for all $n \geq 2$. We give first a solution, apparently new, to a famous problem [see (9) below] proposed by Stephan Beck, Federal Germany, in the 29° International Olympic Games of Mathematics held at Canberra, Australia, in 1988. The statement of this problem implies that if n is not a perfect square, the equation $F(x,y)=n$ does not have integer solutions.

Let us define the function f_n from $\mathbb{R}_+$ to $\mathbb{R}_+$ by $f_n(x)=F(n,x)$, i.e.,

$$f_n(x)=\tfrac{n^2+x^2}{nx+1};\ x\geq 0$$

For $n\in\mathbb{N}$ we have the following properties which are elementary results:

1. $f_n(m)=f_m(n)$ and $f_n(0)=f_n(n^3)=n^2$.
2. f_n is $1-1$ over $x>n^3$.
3. f_n has a unique minimum at $n_0=\frac{-1+\sqrt{n^4+1}}{n}<n$.
4. f_n decreases over $[0,n_0]$ and increases over $x>n_0$

$$f_n(n_0)=\frac{2(\sqrt{1+n^4}-1)}{n^2}=m_0<2 \text{ for all } n;\ 1<m_0<2;\ n\neq 1$$

5. For all $x\neq n_0$ in $[0,n^3]$ there exists a unique

$$y\neq x \text{ such that } f_n(x)=f_n(y);\ \text{in fact } y=\frac{n^3-x}{nx+1}\in[0,n^3]$$

Let h_n be the function defined by $h_n(x)=\frac{n^3-x}{nx+1};\ 0\leq x\leq n^3$.

Thus $h_n(x)=y$. Note the function h_n is involutive, i.e., $h_n(h_n(x))=x$.

6. If $x, f_n(x)$ are nonnegative integers, with $0\leq x<n^3$ then $h_n(x)$ is a nonnegative integer.

$$\text{Moreover, } n_0<x<n^3 \Longleftrightarrow 0<h_n(x)<n_0$$

Proof:

$$\frac{n^2+x^2}{nx+1}=\frac{n^2+[h_n(x)]^2}{nh_n(x)+1}=k\Rightarrow\frac{x+h_n(x)}{n}=k \text{ therefore}$$

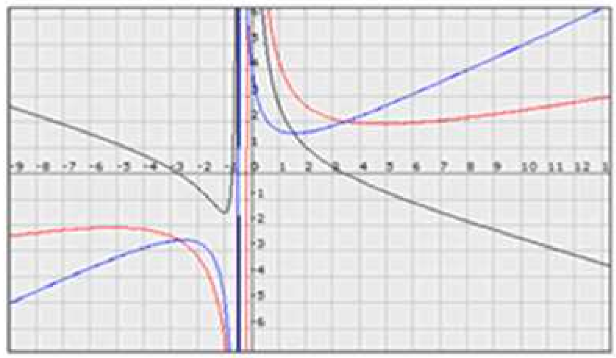
$h_n(x)=kn-x$ is an integer; it must be positive by definition of $h_n(x)$.

7. If $0\leq a<b$ then $f_a(x)>f_b(x)$ for all $x>\alpha$ where α is the unique positive root of $x^3-abx-(a+b)=0$.

Proof: Consider the difference function

$$g(x)=f_b(x)-f_a(x)=\frac{-(b-a)[x^3-abx-(a+b)]}{(ax+1)(bx+1)};\ x\geq 0.$$

It is easily seen, using the derivative, that $g(x)$ is decreasing over $x\geq 0$ going from $g(0)=b^2-a^2$ to $-\infty$ so the equation $g(x)=0$ has a unique positive root α; consequently $f_a(x)>f_b(x)$ if $x>\alpha$.



$f_2(x) = \frac{4+x^2}{2x+1}$, $f_5(x) = \frac{25+x^2}{5x+1}$, $g(x) = f_5(x) - f_2(x) = \frac{-3(x^2-10x-7)}{(2x+1)(5x+1)}$

We are dealing with $x \geq 0$ but the corresponding part to $x < 0$ is showed in order to see what happen with the other two roots. The function g has the unique positive root $\alpha = 3.46686$ and the negative roots $\beta = -0.740625$, $\gamma = -2.766235$.

8. If $0 \leq a < b$ then $f_b(x) = f_a(x) = \beta$ at a unique point $x = \alpha$ where α is the positive root of $x^3 - abx - (a+b) = 0$.

   Furthermore $a + b = \alpha\beta$.

   Proof:
   $$\frac{b^2+x^2}{bx+1} = \frac{a^2+x^2}{ax+1} \Rightarrow x^3 - abx - (a+b) = 0$$
   On the other side $$\frac{b^2+\alpha^2}{b\alpha+1} = \frac{a^2+\alpha^2}{a\alpha+1} = \beta \Rightarrow a + b = \alpha\beta$$

9. **PROBLEM 6 (IMO 1988).-** Let a and b positive integers such that $ab+1$ divides a^2+b^2. Show that $\frac{a^2+b^2}{ab+1}$ is the square of an integer.

   SOLUTION: With $a < b$ ($a = b$ would give $1 < k < 2$ where $\frac{a^2+b^2}{ab+1} = k$) consider the functions f_a and f_b so, $k = f_b(a) = f_a(b)$ as in (1).

   When $k = a^2$ there is nothing to prove. Suppose $f_a(b) = k > a^2$. There exists always a real $c \neq b > a$ such that $k = \frac{a^2+b^2}{ab+1} = \frac{a^2+c^2}{ac+1}$ from which, as in the proof of (6), we have $b + c = ak$ hence c is an integer. On the other hand, when $k > a^2$, it is easily seen that $-\frac{1}{a} < c < 0$. This is a contradiction and therefore we consider only $k < a^2$. *

   We know, by (3) and (4), that $f_a(x)$ is increasing at $x = b$ because $b > a > a_0$ where a_0 is the unique point in which f_a takes its minimum. Applying

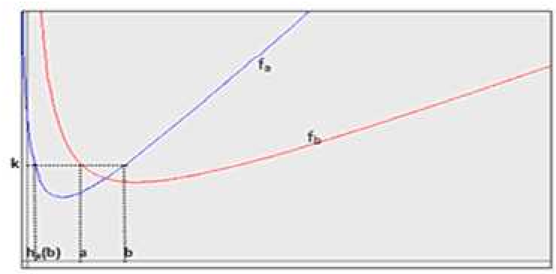
*This is indeed the proposition (13) given below but stated otherwise.

(5) and (6) we obtain the integers $k = f_a(b) = f_a(a_1) = f_{a_1}(a)$ where $0 < a_1 = h_a(b) < a_0 < a < b$ and obviously $a^2 > a_1^2$. Now $f_{a_1}(x)$ is increasing at $x = a$ which implies $0 < a_2 = h_{a_1}(a) < a_1 < a_0 < a < b$ and so on, continuing this way we obtain

$$k = f_{a_n}(a_{n+1}) = f_{a_{n+1}}(a_n) = f_{a_{n+1}}(a_{n+2})$$

where $a_{n+2} = h_{a_{n+1}}(a_n)$

and $b^2 > a^2 > a_1^2 > a_2^2 > a_3^2 > a_n^2 > \cdots \cdots \geq k$



Construction of the a_n

Consequently because of we are dealing with integers, we must have $k = a_n^2$ for a certain index n. The desired result follows.

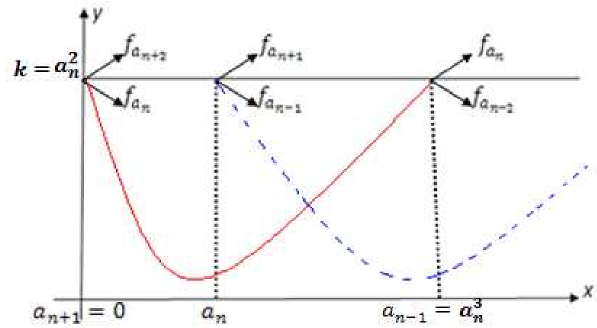
(*) This indeed the proposition (13) given below but stated otherwise.

NOTE: Paragraph (9) gives a third solution which in addition to the two previously known to the author, the first given by the Bulgarian participant in IMO 1988 Emmanuel Atanasiov and the second by the Australian Professor J. Campbell, University of Canberra (see [1], page 65).

The following figure charts the end of the reasoning used in (9) which together with (8) and (1) provides a means of finding integer solutions of the equation $\frac{x^2+y^2}{xy+1} = n^2$

The two curves, f_{a_n} and $f_{a_{n-1}}$ are distorted for practical reasons (the real graphs very quickly stick to the y-axis as can be seen in the figure above where two real graphs are shown).

As $a_n^2 = f_{a_n}(a_{n-1}) = f_{a_n}(0) = f_{a_n}(a_n^3)$ then, by (5), $a_{n-1} = a_n^3$; on the other hand, (8) gives $a_n + a_{n-2} = a_n^3 x\ a_n^2 = a_n^5$, i. e., $a_{n-2} = a_n^5 - a_n$ Continuing in the same way we get integers (by ascent, and not, as in (9), by descent) that are solutions of the proposed equation.



SOLUTIONS OF $\frac{x^2+y^2}{xy+1} = n^2$

10. Thus, given f_n and the trivial point with integer coordinates (n^3, n^2), we consider this point as the intersection of f_n with another curve f_m whose index $m > n$, according to (8), is given by $n + m = n^3\ n^2 = n^5$, i. e. $m = n^5 - n$ (which also goes for the rest solving the equation

$$f_n^3(m) = n^2 \text{ which gives } m = \frac{n^5 + \sqrt{(n^{10} - 4n^6 + 4n^2}}{2} = n^5 - n).$$

The iterated application of the procedure gives the recurrence equation

$$x_{k+2} = n^2 x_{k+1} - x_k, \qquad (x_0, x_1) = (0, n)$$

whose solutions satisfy the condition $f_{x_k}(x_{k+1}) = n^2$ for all $k \geq 1$. The solutions of this equation are given by

$$2^k x_k = \frac{n[(n^2+\alpha)^k - (n^2-\alpha)^k])}{\alpha}$$

where $\alpha = \sqrt{n^4 - 4}$, this is,

$$2^{k-1} x_k = n \sum_i \binom{k}{i} n^{2(k-i)} \alpha^{i-1}$$

where the indexes are the positive odds $i \leq k$.

We finally have

$$2^{k-1} x_k = n \sum_{j=0}^{[k_1]} \binom{k}{i} n^{2(k-2j-1)} (n^4 - 4)^j$$

where $[k_1]$ denotes the integer part of $k_1 = \frac{k-1}{2}$ and moreover

$$F(x_k, x_{k+1}) = \frac{x_k^2 + x_{k+1}^2}{x_k x_{k+1} + 1} = n^2; \qquad k = 1, 2, 3, \ldots\ldots\ldots$$

11. By construction of the integers x_k, the sum in its general definition must be divisible by 2^{k-1} which is clear if n is odd and easily verified in each of the summands if n is even. Therefore each x_k is a multiple of n and moreover, a simple induction using the recurrence equation that defines them proves that n is the greatest common divisor of each pair of consecutive (x_k, x_{k+1}) in that succession.

12. **EXAMPLES:**

$n = 3 \to n^3 = \mathbf{27} \to n^5 - n = \mathbf{240} \to n^7 - 2n^3 = \mathbf{2133} \to n^9 - 3n^5 + n = \mathbf{18957} \to n^{11} - 4n^7 + 3n^3 = \mathbf{168480} \to n^{13} - 5n^9 + 6n^5 - n = \mathbf{1497363} \to \cdots$

$3^2 = 9 = \frac{3^2+27^2}{3*27+1} = \frac{27^2+240^2}{27*240+1} = \frac{240^2+2133^2}{240*2133+1} = \frac{2133^2+18957^2}{2133*18957+1} =$

$= \frac{18957^2+168480^2}{18957*168480+1} = \frac{168400^2+1497363^2}{168480*1497363+1} = \cdots$

13. $f_n(x)$ is not an integer for all integer $x > n^3$.

Proof: Suppose x is an integer with $x > n^3$. If $f_n(x)$ is an integer, by (9) it must be the square of an integer clearly greater than n, then for some integer $h \geq 1$ we have $f_n(x) = (n+h)^2$ which gives the equation $n^2 + x^2 = (nx+1)(n+h)^2$ whose discriminant, $n^2(n+h)^4 + 4(2nh + h^2)$, should be a perfect square. Then there exists an integer $k \geq 1$ such that

$$2n(n+h)^2 k + k^2 = 4(2nh + h^2)$$

$$\text{i.e.} \quad 2kn^3 + k^2 + (kn-2)(4nh + 2h^2) = 0$$

This is clearly impossible if $(kn-2) \geq 0$ and then $kn = 1$, but then we have $2h^2 + 4h - 3 = 0$ which gives h irrational. This completes the proof.

Let $[|n|]$ denotes the infinite set of solutions, generated by n, of the recurrence equation $x_{k+2} = n^2 x_{k+1} - x_k$, $(x_0, x_1) = (0, n)$ solved in (10).

14. If $f_n(x) = b^2$; $b \in \mathbb{N}$; $x \in \mathbb{N}$; $0 < x < n^3$, then $n \in [|b|]$, i.e. n is one of the solutions in (10) generated by b.

Proof: Suppose $a \in \mathbb{N}$; $0 < a < n^3$ and $f_n(a) \in \mathbb{N}$. By (10) we have $f_n(a) = m^2 < n^2$. By the involutive function of (5) we can choose a such that f_n be decreasing in a which means $0 < a < n_0$ (by (3), (5) and (6)). Then there exists, by (7) and (8), a function f_m increasing in a such that $f_m(a) = f_n(a) = k^2$; $k^2 < m^2 < n^2$ and moreover $m = ak^2 - n$

(Note that n, a and m satisfy the recurrence equation of (10) for the coefficient k^2). We repeat the procedure, now with f_m applied to the point $h_m(a)$ making a descent, as in (9), which should end with f_b such that $f_b(0) = f_b(b^3) = k^2 = b^2$ and then $n \in [|b|]$.

15. **Theorem.-** If $p > 0$ is a prime number, then the unique integer solutions (x, z) of the equation $f_p(x) = z$ are the trivial ones $(0, p^2)$ and (p^3, p^2).

    Proof: It is a consequence of (11), (13) and (14).

CONCLUSION.- Let us denote $A = \{m \in \mathbb{N};\ m > n^3\}$. So far we have obtained the following:

► $f_n(A) \cap \mathbb{N} = \emptyset$ for all natural n ◄

► $f_p(\mathbb{N}) \cap \mathbb{N} = \{p^2\}$ for all prime $p > 0$ ◄

more generally, by (14), we can deduce without difficulty

► $f_n(\mathbb{N}) \cap \mathbb{N} = \{n^2\}$ for all n which does not belong to $[|b|]$ for any non trivial divisor b of n ◄

We know $f_n(\mathbb{N}) \cap \mathbb{N}$ trivially contains $\{n^2\}$. The discussion above leads to conjecture it contains at most one non trivial element.

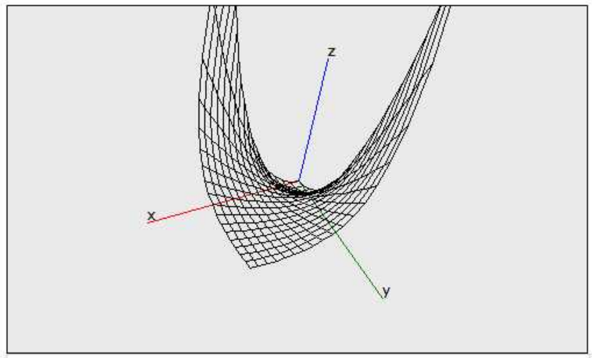
►CONJECTURE◄

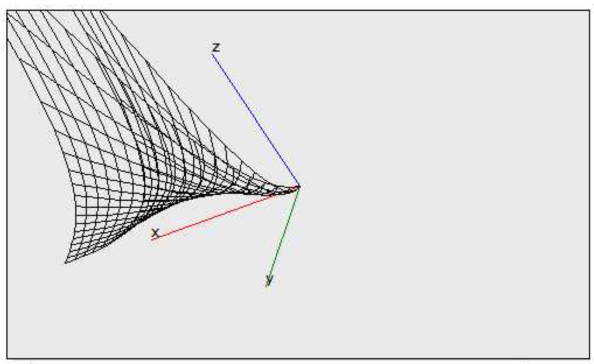
For all $n > 0$, $f_n(\mathbb{N}) \cap \mathbb{N} = \{n^2\}$ or $\{n^2, b^2\}$; ($b < n$ and, by (14), $n \in [|b|]$ therefore, by (11), b divides n).

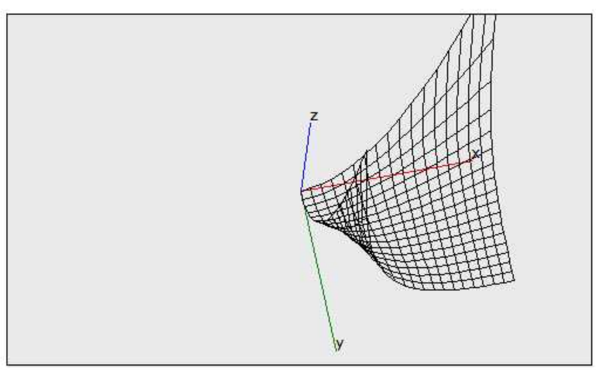
Referencias

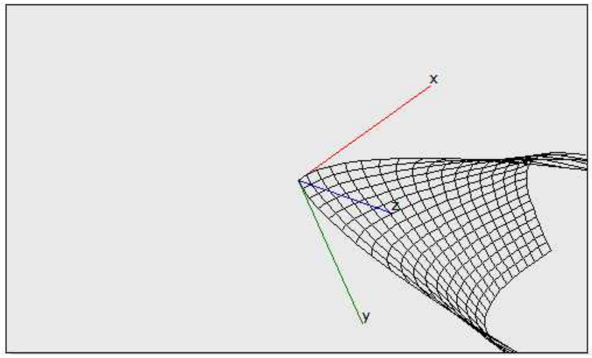
[1] Francisco Bellot Rosado, Ascensión López Ch. *Cien Problemas de Matemáticas.* ICE, Valladolid, 1994

FOUR VIEWS OF THE SURFACE OF EQUATION $z = \frac{x^2+y^2}{xy+1}$









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