Some KKM type, intersection and minimax theorems in spaces with abstract convexities

Luis González Espinoza

Abstract. In this paper we obtain KKM type theorems for G-spaces, M-spaces and L-spaces which are spaces with no linear structure, these theorems are used to obtain some minimax results for these spaces. Also an intersection theorem for M-spaces is presented.

Resumen. En este trabajo obtenemos teoremas de tipo KKM para G-espacios, M-espacios y L-espacios que son espacios sin una estructura lineal, estos teoremas se utilizan para obtener unos resultados minimax para estos espacios. También se presenta un teorema de intersección para M-espacios.

1 Introduction

In this paper we obtain some KKM type theorems for G-spaces. These are Theorems 2.3, 2.6 and 2.11. These latter two results generalize Theorems 1 and Theorem 2 of Bardaro and Cepitelli [1]. We then apply our results to obtain some minimax theorems, including a generalization to G-spaces of an inequality of Fan [4]. This is our Corollary 3.3.

Then, using theorem 3.2 and theorem 3.4 of [2], we obtain a collection of similar results for M-spaces and for L-spaces.

Finally using a theorem of J. Kindler [5] we prove an intersection theorem for M-spaces.

2 Some KKM type theorems for G-spaces

In this section we present some KKM type theorem for G-spaces. KKM type theorems are intersection theorems for multifunctions which satisfy a condition known as the KKM condition. We begin by recalling the definition of a G-space and the concept of a multifunction of KKM type.

Definition 2.1 *We call a triple* (X, D, Γ) *a G-space if* X *is a topological space,* D *is a nonempty subset of* X *and* $\Gamma :< D >:\to 2^X$ *is a multifunction from the set* $< D >$ *of nonempty finite subsets of* D *into* X *such that*

1. $\Gamma(A) \subset \Gamma(B)$ *whenever* $A \subset B$

2. *For each* $A = \{a_1, ..., a_{n+1}\} \in< D >$*, there is a continuous function* $\phi_A : \Delta_n \to \Gamma(A)$ *such that for any subset* $B = \{a_{i1}, ..., a_{1m}\} \subset A$*. we have* $\phi_A([e_{i1}, ..., e_{im}]) \subset B$ *where* Δ_n *denotes the standard closed n-simplex.*

Definition 2.2 *Let* (X, D, Γ) *be a G-space. A multifunction* $F : D \to 2^X$ *such that* $\Gamma(A) \subset F(A)$ *for every* $A \in< D >$ *is called a* **G-KKM multifunction**.

The following theorem was proved in [3]

Theorem 2.3 *Let* (X, D, Γ) *be a compact G-space. Let* $F : D \to 2^X$ *be a closed valued G-KKM multifunction. Then* $\bigcap\{F(x) : x \in D\} \neq \emptyset$.

Next, we generalize Theorem 2.3 to the case where X is not compact; however, before doing so some definitions are required.

Definition 2.4 *Let* (X, D, Γ) *be a G-space. A subset* S *of* X *is G-convex if* $\Gamma(A) \subset S$ *whenever* $A \in< D \cap S >$.

Definition 2.5 *Let* (X, D, Γ) *be an G-space, a set* $K \subset X$ *is* **G-compact** *if for every* $A \in< X >$ *there is a compact, G-convex set* Y *such that* $K \cup A \subset Y$.

To present the following theorem let us recall that a set H is compactly closed if $H \cap B$ is closed in B for every compact set B.

Theorem 2.6 *Let* (X, Γ) *be an G-space, and let* $F : X \to 2^X$ *be a closed valued G-KKM multifunction such that:*

1. *For each* $x \in X$ $F(x)$ *is compactly closed.*

2. *There is a compact set* $L \subset X$ *and an G-compact set* $K \subset X$ *such that for each compact G-convex set* Y *with* $K \subset Y \subset X$ *we have that*

   $\bigcap\{(F(x) \cap Y : x \in Y\} \subset L$.

Then $\bigcap\{F(x) : x \in X\} \neq \emptyset$.

Proof:
It will suffice to show that $\bigcap\{(F(x)\cap L) : x \in X\} \neq \emptyset$. From condition (1) it follows that $\{F(x) \cap L : x \in X\}$ is a family of closed sets in the compact set L. Thus, it suffices to show that this family has the finite intersection property.

Suppose $A \in< X >$. By condition (2) there is a compact, G-convex set Y_0 such that $K \cup A \subset Y_0$ and $\bigcap\{F(x) \cap Y_0 : x \in Y_0\} \subset L$.

But, $\bigcap\{(F(x) \cap Y_0) : x \in Y_0\} \subset \bigcap\{(F(x) \cap L) : x \in Y_0\} \subset \bigcap\{(F(x) \cap L) : x \in A\}$, so, to show that $\bigcap\{(F(x) \cap L) : x \in A\} \neq \emptyset$, it suffices to prove that $\bigcap\{(F(x) \cap Y_0) : x \in Y_0\} \neq \emptyset$.

Now, because Y_0 is G-convex, the pair $(Y_0, \Gamma| < Y_0 >)$ is itself a compact G-space, and the multifunction $H : Y_0 \to 2^{Y_0}$ given by $H(x) = F(x) \cap Y_0$, is a G-KKM multifunction.

Indeed, let $B \in< Y_0 >$. Then,
$\Gamma(B) = \Gamma(B) \cap Y_0$
$\subset (\bigcup\{F(x) : x \in B\}) \cap Y_0$
$= \bigcup\{F(x) \cap Y_0 : x \in B\}$
$= \bigcup\{H(x) : x \in B\} = H(B)$.

Therefore, H is a G-KKM multifunction for the compact G-space $(Y_0, \Gamma| < Y_0 >)$. Thus by Theorem 2.3, it follows that $\bigcap\{(F(x) \cap Y_0) : x \in Y_0\} = \bigcap\{H(x) : x \in Y_0\} \neq \emptyset$. $\diamondsuit$

Now we will introduce a definition which describe a weaker condition for a multifunction than that of G-KKM, and we will use it later. Before doing that we need the following concept.

Definition 2.7 *Let (X, D, Γ) be a G-space. Let A be a subset of X. We define the G-convex hull of A, denoted by $co^G(A)$, as*

$$co^G(A) = \bigcap\{S \subset X : S \text{ is G-convex, and } A \subset S\}$$

Definition 2.8 *Let (X, D, Γ) be a G-space. A multifunction $F : D \to 2^X$ such that $co^G(A) \subset F(A)$ for every $A \in< D >$ is called an* **G*-KKM multifunction**.

The next proposition and its corollary were proved in [3].

Proposition 2.9 *Let (X, D, Γ) be an G-space. Suppose $F : D \to 2^X$ is a G*-KKM multifunction, then it is a G-KKM multifunction.*

Corollary 2.10 *Let (X, D, Γ) be a compact G-space. Let $F : D \to 2^X$ be a closed valued G*-KKM multifunction. Then $\bigcap\{F(x) : x \in D\} \neq \emptyset$.*

Theorem 2.11 *Let (X,Γ) be a G-space, and let $F,H : X \to 2^X$ be two multifunctions such that:*

1. *For all $x \in X$, $H(x)$ is compactly closed, and $F(x) \subset H(x)$;*

2. *$x \in F(x)$ for every $x \in X$;*

3. *For all $x \in X, F^*(x)$ is G-convex;*

4. *H satisfies condition (2) of Theorem 2.6.*

Then $\bigcap\{H(x) : x \in X\} \neq \emptyset$.

Proof:
By Corollary 2.10 it will suffice to show that the multifunction H is a G*-KKM multifunction.

Suppose that H is not a G*-KKM multifunction, then there is a subset $A \in \ < D >$ such that $co^G(A) \not\subset H(A)$.

Thus, there exists $y \in co^G(A)$ such that $y \notin H(A)$, which means that, $y \notin H(x)$ for all $x \in A$, that is, $x \in H^*(y)$ for all $x \in A$. Thus, $A \subset H^*(y)$.

On the other hand, condition (1) implies $H^*(y) \subset F^*(y)$. Thus, $F^*(y)$ is a G-convex subset containing A, which implies that, $co^G(A) \subset F^*(y)$, but $y \in co^G(A)$. Then $y \in F^*(y)$, which is equivalent to $y \notin F(y)$, in contradiction with condition (2).

Hence H is a G*-KKM multifunction and so $\bigcap\{H(x); x \in X\} \neq \emptyset$. $\diamondsuit$

Thus, theorems 2.6 and 2.11 generalize to G-spaces, theorems 1 and 2 in [1].

Corollary 2.12 *Let (X,Γ) be a compact G-space. Let $F : X \to 2^X$ be a multifunction and let $H : X \to 2^X$ be a closed valued multifunction such that:*

1. *For all $x \in X$, $F(x) \subset H(x)$;*

2. *$x \in F(x)$ for every $x \in X$;*

3. *For all $x \in X$, $F^*(x)$ is G-convex.*

Then $\bigcap\{H(x) : x \in X\} \neq \emptyset$.

3 Some Minimax theorems for G-spaces

In this section we present a minimax inequality which is a generalization to G-spaces of an inequality previously proved by K. Fan in [4].

Theorem 3.1 *Let (X,Γ) be a compact G-space, let $f : X \times X \to R$ and $h : X \times X \to R$ be two functions such that:*

1. *$h(x,y) \le f(x,y)$ for all $(x,y) \in X \times X$.*
2. *The function $h_x : X \to R$ given by $h_x(y) = h(x,y)$ is lower semicontinuous.*
3. *Given any $\lambda \in R$ and any $y \in X$ the set $\{x \in X : f(x,y) > \lambda\}$ is G-convex.*

Then for any $\lambda \in R$ either there exists $y_0 \in X$ such that , $h(x,y_0) \le \lambda$ for all $x \in X$, or there exists $y_0 \in X$ such that $f(y_0,y_0) > \lambda$.

Proof:
Let us set $H(x) = \{y \in X : h(x,y) \le \lambda\}$ and $F(x) = \{y \in X : f(x,y) \le \lambda\}$. Since h_x is lower semicontinuous, $H(x)$ is a closed set, so in the terminology of multifunctions, we have a multifunction $F : X \to 2^X$, and a closed valued multifunction $H : X \to 2^X$, such that $F(x) \subset H(x)$ for all $x \in X$ because of condition (1).

Now for the multifunction F, we have two possibilities:

Either there is an $x_0 \in X$, such that $x_0 \notin F(x_0)$, in which case we have that $f(x_0,x_0) > \lambda$, that is, the second part of the alternative is true.

Or, for all $x \in X$, $x \in F(x)$. Now $F^*(y) = \{x \in X : y \notin F(x)\} = \{x \in X : f(x,y) > \lambda\}$ which is an M-convex set for all $y \in X$ because of condition (3).

Therefore F and H are two multifunctions satisfying the hypotheses of Corollary 2.12, so we have that, $\bigcap\{H(x) : x \in X\} \neq \emptyset$.

Thus if $x_0 \in \bigcap\{H(x) : x \in X\}$ we have that $h(x_0,y) \le \lambda$ for all $y \in X$, that is the first part of the alternative is true. $\diamondsuit$

Corollary 3.2 *With the hypotheses of Theorem 3.1 we obtain the following minimax inequality.*

$$min_{y\in X} sup_{x\in X} h(x,y) \le sup_{x\in X} f(x,x).$$

Proof:
Let $\lambda = sup_{x\in X} f(x,x)$, then either $\lambda = \infty$, in which case the inequality is obvious or λ is finite. Then because of definition of λ, the first part of the alternative in Theorem 3.1 is true. Therefore exists $y_0 \in X$ such that:

$$h(x,y_0) \le sup_{x\in X} f(x,x) \quad forall \;\; x \in X.$$

Then

$$sup_{x\in X} h(x,y) \le sup_{x\in X} f(x,x) \quad forall \;\; y \in X$$

that is,

$$sup_{x\in X}h_x(y) \le sup_{x\in X}f(x,x) \quad forall \;\; y \in X.$$

Thus

$$inf_{y\in X}sup_{x\in X}h_x(y) \le sup_{x\in X}f(x,x);$$

but $sup_{x\in X}h_x$ is lower semicontinuous, and it is well known that in this case this infimum is a minimun therefore we have that

$$min_{y\in X}sup_{x\in X}h(x,y) \le sup_{x\in X}f(x,x). \quad \diamondsuit$$

Based on this, the inequality proved by Fan in [4] can be generalized to G-spaces by the following corollary.

Corollary 3.3 *Let (X,Γ) be a compact G-space and let $f: X \times X \to R$ be a function such that:*

1. *The function $f_x : X \to R$ given by $f_x(y) = f(x,y)$ is lower semicontinuous.*
2. *Given any $\lambda \in R$ and any $y \in X$ the set $\{x \in X : f(x,y) > \lambda\}$ is G-convex.*

Then the following inequality is true

$$min_{y\in X}sup_{x\in X}f(x,y) \le sup_{x\in X}f(x,x).$$

Proof:
Take $h(x,y) = f(x,y)$ in Corollary 3.2. $\diamondsuit$

4 Some KKM and Minimax Theorems for M-spaces and L-spaces

Theorem 3.2 of [2], shows that if $(X,\mathbf{M},\mathbf{k})$ is an M-space, and $D \subset X$ is an admissible subset, then there exists the corresponding M-space (X,D,Γ), such that the collection of M-convex subsets with respect to D in $(X,\mathbf{M},\mathbf{k})$ coincides with the collection of G-convex sets in (X,D,Γ). We will use this result to obtain from the KKM and minimax theorems proved for G-spaces, similar results for M-spaces.

On the other hand, Theorem 3.4 of [2] states that given an L-space $(X,D,\mathbf{P})$, there is an M-space $(X,\mathbf{M},\mathbf{k})$ for which D is an admissible subset, and the collection of L-convex subsets in $(X,D,\mathbf{P})$ coincides with the collection of M-convex subsets with respect to D in $(X,\mathbf{M},\mathbf{k})$. Based on this theorem some KKM and minimax theorems for L-spaces will be obtained.

Let us begin by recalling the concepts of M-space and M-convex subset, to introduce next the concept of M*-KKM multifunction.

Notation. Given any integer $m \geq 2$ and $1 \leq i \leq m$, let $\delta_i : R^n \to R^n$ denote the function defined by $\delta_i(x_1, ..., x_n) = (x_1, ..., x_{i-1}, x_{i+1}, ..., x_n)$.

Definition 4.1 *An M-space is a triple* $(X, \mathbf{M}, \mathbf{k})$*, where* X *is a topological space,* $\mathbf{M} = Mn : ninteger, n \leq 1$ *is a collection of sets where* $Mn \subset X^n$ *for all* $n \geq 1$*, and* $\mathbf{k} = kn : ninteger, n \leq 1$ *is a collection of functions satisfying*

1. $k_{n+1} : M_{n+1} \times \Delta_n \to X$.
2. *If* $x \in M_{n+1}(n \geq 1)$ *and* $i \leq n+1$*, then* $\delta_i(x) \in M_n$ *and for any* $t \in \Delta_n$ *with* $t_i = 0, k_{n+1}(x, t) = k_n(\delta_i(x), \delta_i(t))$.
3. *If* $x \in M_{n+1}$*, then the map* $t \to k_{n+1}(x, t)$*, from* Δ_n *to* X*, is continuous.*

Definition 4.2 *Let* $(X, \mathbf{M}, \mathbf{k})$ *be an M-space. A nonempty subset* $D \subset X$ *is said to be admissible if* $D^n \subset M_n$ *for all* n*.*

Definition 4.3 *Let* $(X, \mathbf{M}, \mathbf{k})$ *be an M-space, let* $D \subset X$ *be an admissible subset. We say that a subset* S *of* X *is M-convex with respect to D, if for each subset* $A \in < S \cap D >$ *and any indexing of* $A = \{a_1, ..., a_{n+1}\}$*, we have that*

$$k_{n+1}((a_1, ..., a_{n+1}), \Delta_n) \subset S.$$

If $D = X$ *we say M-convex.*

Definition 4.4 *Let* $(X, \mathbf{M}, \mathbf{k})$ *be an M-space, let* $D \subset X$ *be an admissible subset. Let* K *be subset of* X*. We define the M-convex hull of* K *with respect to* D*, denoted by* co_D^M *as:*

$$co_D^M = \bigcap \{S \subset X : S \text{ is M-convex with respect to } D, K \subset S\}.$$

In case $D = X$*, the M-convex hull of* K *with respect to* X *will be denoted by* co^M*.*

Definition 4.5 *Let* $(X, \mathbf{M}, \mathbf{k})$ *be an M-space and let* $D \subset X$ *be an admissible subset. A multifunction* $F : D \to 2^X$ *is said to be M*-KKM, if for each* $A \in < D >$*,* $co_D^M(A) \subset F(A)$*.*

Proposition 4.6 *Let* $(X, \mathbf{M}, \mathbf{k})$ *be a compact M-space, and let* $D \subset X$ *be an admissible subset. Let* $F : D \to 2^X$ *be a closed valued M*-KKM multifunction. Then* $\bigcap\{F(x) : x \in D\} \neq \emptyset$.

Proof:
By Theorem 3.2 of [2], the collection of M-convex subsets with respect to D in the space $(X, \mathbf{M}, \mathbf{k})$, coincide with the collection of G-convex subsets in the corresponding G-space (X, D, Γ). Therefore $F : D \to 2^X$ is a G*-KKM multifunction in the G-space (X, D, Γ). Thus, by Corollary 2.9 we have that $\bigcap\{F(x) : x \in D\} \neq \emptyset$. ♢

As consequences of our next proposition we obtain minimax results for M-spaces, all these proofs are omitted because they are similar to those corresponding to G-spaces.

Proposition 4.7 *Let* $(X, \mathbf{M}, \mathbf{k})$ *be a compact M-space, such that* X *is admissible. Let* $F : X \to 2^X$ *be a multifunction and let* $H : X \to 2^X$ *be a closed valued multifunction such that:*

1. *For all* $x \in X$, $F(x) \subset H(x)$;
2. $x \in F(x)$ *for every* $x \in X$;
3. *For all* $x \in X$, $F^*(x)$ *is M-convex.*

Then $\bigcap\{H(x) : x \in X\} \neq \emptyset$.

Proposition 4.8 *Let* $(X, \mathbf{M}, \mathbf{k})$ *be a compact M-space, such that* X *is admissible. Let* $f : X \times X \to R$ *and* $h : X \times X \to R$ *be two functions such that:*

1. $h(x, y) \leq f(x, y)$ *for all* $(x, y) \in X \times X$.
2. *The function* $h_x : X \to R$ *given by* $h_x(y) = h(x, y)$ *is lower semicontinuous.*
3. *Given any* $\lambda \in R$ *and any* $y \in X$ *the set* $\{x \in X : f(x, y) > \lambda\}$ *is M-convex.*

Then for any $\lambda \in R$ *either there exist* $y_0 \in X$ *such that for all* $x \in X$, $h(x, y_0) \leq \lambda$, *or there exists* $y_0 \in X$ *such that* $f(y_0, y_0) > \lambda$.

Proposition 4.9 *With the hypotheses of Proposition 4.8 we obtain the following minimax inequality.*

$$min_{y \in X} sup_{x \in X} h(x, y) \leq sup_{x \in X} f(x, x).$$

Proposition 4.10 *Let $(X, \mathbf{M}, \mathbf{k})$ be a compact M-space, such that X is admissible and let $f : X \times X \to R$ be a function such that:*

1. *The function $f_x : X \to R$ given by $f_x(y) = f(x, y)$ is lower semicontinuous.*
2. *Given any $\lambda \in R$ and any $y \in X$ the set $\{x \in X : f(x, y) > \lambda\}$ is M-convex.*

Then the following inequality is true

$$min_{y \in X} sup_{x \in X} f(x, y) \leq sup_{x \in X} f(x, x).$$

This proposition generalizes to M-spaces an inequality proved by Fan in [4].

Now, we give the definition of an L*-KKM multifunction, and then by employing of Theorem 3.4 of [2], we state some KKM and minimax theorems for L-spaces. We begin by recalling the concepts of an L-space, an L-convex subset and the L-convex hull of a subset.

Definition 4.11 *An L-space is a triple $(X, D, \mathbf{P})$, where X is a topological space, D is a nonempty subspace of X and $\mathbf{P} = \{P_a : a \in X\}$ is a collection of functions $P_a : D \times [0, 1] \to D$, such that $P_a(x, 0) = x$, $P_a(x, 1) = a$, and P_a is continuous respect to $t \in [0, 1]$. When $D = X$, we write (X, P).*

Definition 4.12 *Suppose $(X, D, \mathbf{P})$ is an L-space. Given $A \in < D >$, let $A = \{a_0, ..., a_n\}$ be any indexing of A by $\{0, ...n\}$. Define the multifunction $G_A : [0, 1]^n \to D$ by*

$$G_A(t_0, ..., t_n) = P_{a_0}(P_{a_1} ...(P_{a_{n-1}}(a_n, t_{n-1})..., t_1), t_0)$$

. For $A = \{a\}$, we define $G_{\{a\}} = \{a\}$. We say that a subset $S \subset X$ is L-convex if for every $A \in < A \cap D >$, and every indexing of $A = \{a_0, ...a_n\}$, it follows that $G_A([0, 1]^n) \subset S$.

Definition 4.13 *Let $(X, D, \mathbf{P})$ be an L-space. Let A be a subset of X. We define the L-convex hull of A by*

$$co^L(A) = \bigcap \{S \subset X : S \text{ is L-convex and } A \subset S\}$$

.

Definition 4.14 *Let $(X, D, \mathbf{P})$ be an L-space. A multifunction $F : D \to 2^X$ such that $co^L(A) \subset F(A)$ for every $A \in < D >$ is called an* **L*-KKM multifunction**.

Proposition 4.15 *Let $(X, D, \mathbf{P})$ be a compact L-space. Let $F : D \to 2^X$ be a closed valued L*-KKM multifunction. Then $\bigcap\{F(x) : x \in D\} \neq \emptyset$.*

Proof:
The proof follows from Theorem 3.4 of [2] and Proposition 4.6 in similar way to the proof of Proposition 4.6.

The followings propositions together with Proposition 3.4 of [2] allow us to present some minimax results for L-spaces, whose proofs are omitted because of their similarities with the corresponding for M-spaces.

Proposition 4.16 *Let $(X, \mathbf{P})$ be a compact L-space. Let $F : X \to 2^X$ be a multifunction and let $H : X \to 2^X$ be a closed valued multifunction such that:*

1. *For all $x \in X$, $F(x) \subset H(x)$;*
2. *$x \in F(x)$ for every $x \in X$;*
3. *For all $x \in X$, $F^*(x)$ is L-convex.*

Then $\bigcap\{H(x) : x \in X\} \neq \emptyset$.

Proposition 4.17 *Let $(X, \mathbf{P})$ be a compact L-space , let $f : X \times X \to R$ and $h : X \times X \to R$ be two functions such that:*

1. *$h(x, y) \leq f(x, y)$ for all $(x, y) \in X \times X$.*
2. *The function $h_x : X \to R$ given by $h_x(y) = h(x, y)$ is lower semicontinuous.*
3. *Given any $\lambda \in R$ and any $y \in X$ the set $\{x \in X : f(x, y) > \lambda\}$ is L-convex.*

Then for any $\lambda \in R$ either there exist $y_0 \in X$ such that for all $x \in X$, $h(x, y_0) \leq \lambda$, or there exists $y_0 \in X$ such that $f(y_0, y_0) > \lambda$.

Corollary 4.18 *With the hypotheses of Proposition 4.17 we obtain the following minimax inequality.*

$$min_{y \in X} sup_{x \in X} h(x, y) \leq sup_{x \in X} f(x, x).$$

Corollary 4.19 *Let $(X, \mathbf{P})$ be a compact L-space and let $f : X \times X \to R$ be a function such that:*

1. *The function $f_x : X \to R$ given by $f_x(y) = f(x, y)$ is lower semicontinuous.*
2. *Given any $\lambda \in R$ and any $y \in X$ the set $\{x \in X : f(x, y) > \lambda\}$ is L-convex.*

Then the following inequality is true

$$min_{y \in X} sup_{x \in X} f(x, y) \leq sup_{x \in X} f(x, x).$$

5 An intersection Theorem for M-spaces

In this section, by employing an intersection theorem due to J. Kindler [5], proved without using the Theorem of Knaster-Kuratowski-Mazurkiewicz, we show another type of intersection theorem for M-spaces.

Theorem 5.1 *For a multifunction $F : X \to 2^Y$ the following are equivalent.*

1. $\bigcap\{F(x) : x \in X\} \neq \emptyset$.
2. *There exist topologies on X and Y such that*
   - (a) *Y is compact.*
   - (b) *Every value $F(x), x \in X$ is closed.*
   - (c) *For all $A \in < X >$ the subset $\bigcap\{F(x) : x \in A\}$ is connected.*
   - (d) *For all $B \subset Y$ the subset $\bigcap\{F^*(y) : y \in B\}$ is connected.*

Theorem 5.2 *Let $(X, \mathbf{M}, \mathbf{k})$ be an M-space such that X is admissible, and such that $k_1(x, 1) = x$ for all $x \in X$. Let Y be a compact topological space and $F : X \to 2^Y$ an upper semicontinuous multifunction such that*

1. $F(\Gamma_{\{x_1,x_2\}}) = F(x_1) \cup F(x_2)$ *for all* $x_1, x_2 \in X$.
2. $\bigcap\{F(x) : x \in A\}$ *is connected for all* $A \in < X >$.

Then $\bigcap\{F(x) : x \in X\} \neq \emptyset$.

Proof:
Due to Theorem 5.1 it suffices to prove that for all $B \subset Y$ the subset $\bigcap\{F^*(y) : y \in B\}$ is connected, so let $B \subset Y$ and let us prove that $\bigcap\{F^*(y) : y \in B\}$ is connected.

To this end we will show that given $x_1, x_2 \in \bigcap\{F^*(y) : y \in B\}$ there is a connected set C such that $\{x_1, x_2\} \subset C \subset \bigcap\{F^*(y) : y \in B\}$.

Now $x_1, x_2 \in \bigcap\{F^*(y) : y \in B\}$ means that $B \cap F(x_1) = \emptyset$ and $B \cap F(x_2) = \emptyset$, then $B \cap (F(x_1) \cup F(x_2)) = B \cap \ F(\Gamma_{\{x_1,x_2\}}) = \emptyset$. Therefore $x_1, x_2 \in \Gamma_{\{x_1,x_2\}} \subset \bigcap\{F^*(y) : y \in B\}$. On the other hand $\Gamma_{\{x_1,x_2\}} = \{\bigcup\{k_2((x_1, x_2), t) : t \in \bar{\Delta}_1\}\} \cup \{\bigcup\{k_2((x_2, x_1), t) : t \in \bar{\Delta}_1\}\}$ is path-connected.

In fact, let $x, y \in \Gamma_{\{x_1,x_2\}}$. We will show that there is a path joining x and y. Assume that $x = k_2((x_1, x_2), (t_1, t_2))$ with $(t_1, t_2) \in \bar{\Delta}_1$ and consider the path $\phi : [0, 1] \to X$ defined by $\phi(t) = k_2((x_1, x_2), (t_1 + t - tt_1, t_2 - tt_2))$. By definition of M-space it follows that ϕ is continuous function such that $\phi(0) = k_2((x_1, x_2), (t_1, t_2))$ and $\phi(1) = k_2((x_1, x_2), (1, 0)) = k_1(x_1, 1) = x_1$. Therefore ϕ is a path joining x and x_1.

In a similar way we can construct a path joining y and x_1. Thus any pair $x, y \in \Gamma_{\{x_1,x_2\}}$ can be joined by a path, which means that, $\Gamma_{\{x_1,x_2\}}$ is path connected.

Therefore, given two points $\{x_1, x_2\} \in \bigcap\{F^*(y) : y \in B\}$ we have found a connected set $C = \Gamma_{\{x_1,x_2\}}$ containing these two points and contained in $\bigcap\{F^*(y) : y \in B\}$, this means that $\bigcap\{F^*(y) : y \in B\}$ is connected. $\diamondsuit$

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Luis González Espinoza.
Departamento de Matemáticas
Facultad de Ciencias,
Universidad de los Andes
Mérida, Venezuela.