Some remarks on generalized Cohen-MacAulay rings

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Abstract

We consider the possibility of characterizing Buchsbaum and some special generalized Cohen-Macaulay rings by systems of parameters having certain properties of regular sequences. As an application, we give a bound on Castelnuovo-Mumford regularity of so-called (k, d)-Buchsbaum graded K-algebras.

1 Introduction

Let A be a noetherian local ring (resp. K-algebra) of $dimA = d$ and $\underline{m}$ the maximal (resp. homogeneous maximal) ideal of A. A is called a generalized Cohen-Macaulay (abbr. C-M) ring if all local cohomology modules $H^i_{\underline{m}}(A)$, $i < d$, are of finite length [19]. The class of generalized C-M rings are rather large. The most important subclass among them form Buchsbaum rings [20]. In order to have a unified approach in studying Buchsbaum, quasi-Buchsbaum, and other generalized C-M rings, the notion of (k, r)-Buchsbaum rings was recently introduced, where $k \geq 0$ and $1 \leq r \leq d$ are some integers (see [6, 10, 14, 15]). With this new notion we have a refined classification of generalized C-M rings. Buchsbaum rings are exactly $(1, d)$-Buchsbaum rings.

Our remarks are related to the possibility of characterizing Buchsbaum rings by systems of parameters (abbr. s.o.p.'s) having certain generalized properties of

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regular sequences. We shall do it in a more general context by considering also (k,d)-Buchsbaum rings.

In Section 2 we will give a characterization of Buchsbaum rings with $H^i_{\underline{m}}(A)=0$ for $i\neq depthA, dimA$ by the help of only one s.o.p.. This is related to a problem posed by J. Stückrad and W. Vogel [20, p. 87]. Such a ring is important because it is related to the C-M property of Rees algebras of certain powers of parameter ideals of A [2].

A relative regular sequence (abbr. r.r.s.) [5], or equivalently, a d-sequence [12] is a kind of (I,J)-r.r.s.'s of A, where $J\subseteq I$ are ideals of A (see Definition 3.2). Buchsbaum rings can be good characterized by d-sequences [12], whereas there exists a local ring which is not a Buchsbaum ring, although every s.o.p. $x_1,...,x_d$ of A is a r.r.s. with respect to the ideal $(x_1,...,x_d)A$ in the sense of [5]. Such an example will be given in Section 3. In this example every s.o.p. of A is also a so-called sequence of linear type [3]. Some properties of sequences of linear type will be also given.

In the last section 4 we study Castelnuovo-Mumford regularity [4, 10, 21] of a (k,d)-Buchsbaum K-algebra A. Using the property that some s.o.p. of A is an $(A,\underline{m}^p)$-r.r.s. for some well-defined p, we can give a bound on Castelnuovo-Mumford regularity of A in terms of reduction exponent of $\underline{m}$ and k,d, which extends a result of [23] in the case of Buchsbaum rings.

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2 Buschbaum rings with $H^i_{\underline{m}}(A)=0$ for $i\neq depthA, dimA$

In general, Buchsbaum rings are hardly characterized using only one fixed s.o.p. (see [20], Problem on p. 87). In a special situation, when $H^i_{\underline{m}}(A)=0$ for $i\neq depthA, dimA$, Proposition I.2.12 of [20] gives such a characterization. Such rings deserve our attention because of the following result of M. Brodmann, which generalizes a result of S. Goto and Y. Shimoda:

Proposition([2], Proposition 6.1) *Assume that $(A,\underline{m})$ is a local ring of dimension $d\geq 1$ and $depthA\geq r>0$. The following conditions are equivalent:*

(i) A is a Buchsbaum ring and $H^i_{\underline{m}}(A)=0, i\neq r,d$.

(ii) For every parameter ideal $\underline{q}$ of A the Rees algebra $R(\underline{q}^r;A)$ is C-M (recall that $R(I,A)=\oplus_{n\geq 0}I^n$).

The above mentioned result of [20] depends still on the knowledge of local cohomology of A. Here we shall give a characterization free of local cohomology. Let us recall some definitions.

Let $x_1,...,x_n$ be a sequence of elements of a commutative ring R. Throughout this paper we will use the notations:

$\underline{q}_0=0; \underline{q}=(x_1,...,x_n)R,$ and

$\underline{q}_i=(x_1,...,x_i)R$ for $i=1,...,n-1.$

Let $(A,\underline{m})$ be a local ring. Let M be a noetherian A-module of dimension d and

$\underline{a}$ an ideal of A. A sequence $x_1, ..., x_n \in \underline{m}$ is said to be an *$\underline{a}$-weak M-sequence* if

$$\underline{q}_{i-1}M : x_i \subseteq \underline{q}_{i-1}M : \underline{a} \quad \text{for} \quad : i = 1, ..., n.$$

If $x_1, ..., x_d$ is a s.o.p. of M, we put

$$I(x_1, ..., x_d; M) = I(\underline{q}; M) = \ell(M/\underline{q}M) - e(\underline{q}; M),$$

the difference between the length of $M/\underline{q}M$ and the multiplicity of M relative to $\underline{q}$. A s.o.p. $x_1, ..., x_d$ of M is called a *standard s.o.p.* of M if

$$I(x_1^2, ..., x_d^2; M) = I(x_1, ..., x_d; M).$$

This notion was introduced by M. Brodmann and N. V. Trung. For properties of standard s.o.p.'s see [22]. We quote here from that paper only Proposition 3.1 needed later.

Lemma 2.0 *Assume that $\underline{a}$ is an ideal with $\ell(M/\underline{a}M) < \infty$. If every s.o.p. of M contained in $\underline{a}$ is an $\underline{a}$-weak M-sequence then it is also a standard s.o.p. of M.*

Definition 2.1 ([19], 3.3; [22], Theorem 2.1) *M is called a generalized C-M module if one of the following equivalent conditions holds:*
(i) $\ell(H^i_{\underline{m}}(M)) < \infty$ for $i < d$.
(ii) There exists a positive integer k such that every s.o.p. of M is an $\underline{m}^k$-weak M-sequence.
(iii) There exists a standard s.o.p. of M.

A is called a generalized C-M ring if it is a generalized C-M module as a module over itself.

Definition 2.2 ([6], [10]) *Let k be any non-negative integer and $1 \leq r \leq d$. M is called a (k, r)-Buchsbaum module if for every s.o.p. $x_1, ..., x_d$ of M we have*

$$\underline{m}^k H^i_{\underline{m}}(M/\underline{q}_j M) = 0,$$

for all non-negative integers i, j with $j \leq r - 1$ and $i + j < d$.

Similarly, we define the notion of (k, r)-*Buchsbaum rings*. $(k, 1)$-Buchsbaum modules are also called k-Buchsbaum modules.

Remark 2.3. $(0, r)$-, $(1, d)$- and 1-Buchsbaum modules are exactly the Cohen-Macaulay, Buchsbaum and quasi-Buchsbaum modules, respectively (see [20]). Every generalized C-M module is a (k, r)-Buchsbaum module for some k and r, $1 \leq r \leq d$. Moreover, it is easy to show that M is (k, d)-Buchsbaum if and only if every s.o.p. of M is an $\underline{m}^k$- weak M-sequence (cf. Definition 2.1(ii)). Hence, by Lemma 2.0, every s.o.p. of M contained in $\underline{m}^k$ is a standard s.o.p. of M.

The following lemma is easy and well-known:

Lemma 2.4 *Suppose that M is a generalized C-M module. Let x be a parameter element of M. Then $0_M : x \subseteq H^0_{\underline{m}}(M)$. In particular, if depth$M = r$ then every sub-s.o.p. $x_1, ..., x_r$ of M forms a regular M-sequence.*

From now on let A be a local ring of dimension d. One can characterize k-Buchsbaum rings by one fixed s.o.p. (see [20], Proposition I.2.1 and Proposition 13 in the appendix). Here we give another one.

Lemma 2.5 *A is a k-Buchsbaum ring if and only if there exists a s.o.p. of A which is a standard s.o.p. of A as well as an $\underline{m}^k$-weak sequence.*

Proof. Let $\underline{a} = \underline{m}^k$. To prove $(\Rightarrow)$ let us take any s.o.p. $x_1, ..., x_d$ of A contained in $\underline{a}^2$. This s.o.p. is an $\underline{a}$-weak sequence by [20], Proposition 13. Moreover, by that proposition and by Lemma 2.0 it follows that $x_1, ..., x_d$ is also a standard s.o.p..

$(\Leftarrow)$: Let $x_1, ..., x_d$ be a standard s.o.p. of A which is also an $\underline{a}$-weak sequence. We consider the following exact sequence:

$$0 \to (0 : x_i)_{A/\underline{q}_{i-1}} \to A/\underline{q}_{i-1} \to A/(\underline{q}_{i-1} : x_i) \to 0, \tag{1}$$

and

$$0 \to A/(\underline{q}_{i-1} : x_i) \xrightarrow{\cdot x_i} A/\underline{q}_{i-1} \to A/\underline{q}_i \to 0, \tag{2}$$

where $1 \le i \le d$. Note that $x_1, ..., x_d$ is a standard s.o.p. of A if and only if $\underline{q}H^i_{\underline{m}}(A/\underline{q}_j) = 0$ for all non-negative integers i, j with $i + j < d$ ([22], Theorem 2.5). Hence $x_i, ..., x_d$ is a standard s.o.p. of $A/\underline{q}_{i-1}$. Therefore

$$H^0_{\underline{m}}(A/\underline{q}_{i-1}) = (0 : x_i)_{A/\underline{q}_{i-1}}, \tag{3}$$

and this module is of finite length. From (1) we then get

$$H^j_{\underline{m}}(A/\underline{q}_{i-1}) \cong H^j_{\underline{m}}(A/(\underline{q}_{i-1} : x_i)) \ \text{ for } \ j \ge 1.$$

Since $x_i H^j_{\underline{m}}(A/\underline{q}_{i-1}) = 0$ for $j \le d - i$, we get from the above isomorphisms and the exact sequence (2) epimorphisms:

$$H^{j-1}_{\underline{m}}(A/\underline{q}_i) \to H^j_{\underline{m}}(A/\underline{q}_{i-1}) \to 0,$$

for $1 \le j \le d - i$. The composition

$$H^0_{\underline{m}}(A/\underline{q}_{i-1}) \to H^1_{\underline{m}}(A/\underline{q}_{i-2}) \to \cdots \to H^{i-1}_{\underline{m}}(A),$$

yields an epimorphism

$$H^0_{\underline{m}}(A/\underline{q}_{i-1}) \to H^{i-1}_{\underline{m}}(A), \ \text{ for } \ 1 \le i \le d.$$

Since $x_1, ..., x_d$ is an $\underline{a}$-weak sequence, from (3) we then get $\underline{a}H^{i-1}_{\underline{m}}(A) = 0$ for $i \le d$, i. e. A is a k-Buchsbaum ring.

Example 2.6 ([7], Example 1.10). Let $R = K[[x_1, ..., x_d, y_1, ..., y_d]], d \ge 3$. Put

$$\begin{aligned} \underline{a} &= (x_1,...,x_d)R \cap (y_1,...,y_d)R, \\ \underline{q} &= (x_1^2, x_2,...,x_d, y_1^2, y_2,...,y_d)R, \\ F_i &= x_i + y_i, \ i = 1,...,d, \end{aligned}$$

and

$$A = R/((\underline{a} \cap \underline{q}) + F_1^n R) \ \text{ with } n \geq 3.$$

Then $dimA = d-1$, A is not a Buchsbaum ring and the images $\bar{F}_2,...,\bar{F}_d$ of $F_2,...,F_d$ in A satisfy the condition (ii) of Lemma 2.5 with $k=1$. Hence, by that lemma, A is a quasi-Buchsbaum ring. Note that $\bar{F}_2,...,\bar{F}_d \in \underline{m} \backslash \underline{m}^2$.

Proposition 2.7 *The following conditions are equivalent:*
1) A is a (k,d)-Buchsbaum ring with $H^i_{\underline{m}}(A) = 0$ for $i \neq r, d$ $(0 \leq r \leq d)$.
2) There exists a s.o.p. $x_1,...,x_d$ of A which satisfies
   i) $x_1,...,x_d$ is a standard s.o.p. of A,
   ii) $x_1,...,x_d$ is an $\underline{m}^k$-weak sequence,
   iii) $x_1,...,x_r$ is a regular sequence, and
   iv) If $r < d$, then $I(x_1,...,x_d; A) = \binom{d-1}{r}\ell((\underline{q}_r : x_{r+1})/\underline{q}_r)$.
3) There exists a s.o.p. of A contained in $\underline{m}^k$ which satisfies the above conditions ii), iii) and iv).

Proof. Note that if $x_1,...,x_d$ is a standard s.o.p. of A and if $depthA \geq i$ for some $0 \leq i < d$, then A is a generalized C-M ring with

$$\ell(H^i_{\underline{m}}(A)) = \ell(H^0_{\underline{m}}(A/\underline{q}_i)) = \ell((\underline{q}_i : x_{i+1}/\underline{q}_i),$$

(by [22], Proposition 2.9), and

$$I(x_1,...,x_d; A) = \sum_{j=0}^{d-1} \binom{d-1}{j} \ell(H^j_{\underline{m}}(A)).$$

Hence 1) $\Rightarrow$ 3) is immediate from Remark 2.3 and Lemma 2.4.
3) $\Rightarrow$ 2): If $x_1,...,x_d$ satisfies (3), then by [20], Proposition 13, every s.o.p. of A contained in $\underline{m}^{2k}$ is an $\underline{m}^k$-weak sequence. Hence, by Lemma 2.0, $x_1,...,x_d$ is also a standard s.o.p. of A.
2) $\Rightarrow$ 1): Condition (iii) gives $depthA \geq r$. Hence, from the above remark we get that $H^i_{\underline{m}}(A) = 0$ for $i \neq r, d$. By Lemma 2.5, A is a k-Buchsbaum ring. Therefore, by [11] Corollary 2.4, A must be a (k,d)-Buchsbaum ring (the fact that a quasi-Buchsbaum ring with $H^i_{\underline{m}}(A) = 0$ for $i \neq r, d$ must be a Buchsbaum ring is well known).

Remark 2.8. From Theorem 3.1 of [18] one can get some another characterizations of Buchsbaum rings with $H^i_{\underline{m}}(A) = 0$ for $i \neq 1, d$ by one fixed s.o.p. of A.

3 Sequence of linear type

Let A be a commutative ring, I an ideal of A, $S_A(I)$ the symmetric algebra of I, and $R_A(I)$ the Rees algebra of I. Then I is called *an ideal of linear type* if the canonical surjection $S_A(I) \to R_A(I)$ is an isomorphism [9]. In order to employ an inductive method in studying ideals of linear type, D. L. Costa has introduced:

Definition 3.1 *$x_1, ..., x_n$ is called a sequence of linear type if all ideals $\underline{q}_i$, $i = 1, ..., n$, are ideals of linear type.*

Definition 3.2 (cf. [5]) *Let A be a commutative ring, M an A-module and $N_2 \subseteq N_1$ submodules of M. A sequence $x_1, ..., x_n$ is said to be a relative regular sequence (abbr. r.r.s.) of M with respect to (N_1, N_2), or shortly (N_1, N_2)-r.r.s., if*

$$(\underline{q}_{i-1} N_1 : x_i) \cap N_2 \subseteq \underline{q}_{i-1} M, \quad i = 1, ..., n.$$

If $N_1 = N_2 = N$, it is also called an N-r.r.s. of M.

This definition includes various generalizations of regular sequences. If we set $N_1 = M, N_2 = (x_1, ..., x_n)M$ then an (N_1, N_2)-r.r.s. is exactly a r.r.s. introduced by the first named author in [5], or equivalently, a d-sequence introduced by C. Huneke [12], [13]. For a graded ring $A = \oplus_{n \geq 0} A_n$ with $\underline{m} = \oplus_{n \geq 1} A_n$, P. Bouchard has considered (N_1, N_2)-r.r.s.'s with $N_1 = N_2 = \underline{m}^k$ for some $k > 0$ [1]. Weakly regular sequences considered by M. Herrmann et al. [8], [17] are (N_1, N_2)-r.r.s.'s in the graded case with $N_1 = A$ and $N_2 = \underline{m}^k$ for some k. In each case, a suitable choice of N_1 and N_2 turns out to be a good tool in studying various quetions of commutative algebra (see also the next section for an application). The following result of [3], Theorem 3 extends a result of C. Huneke and G. Valla.

Lemma 3.3 *Let $x_1, ..., x_n$ be a sequence of elements in a commutative ring A. If $x_1, ..., x_n$ is a $\underline{q}^t$-r.r.s. of $\underline{q}^{t-1}$, i. e. if*

$$(*) \qquad (\underline{q}_{i-1} \underline{q}^t : x_i) \cap \underline{q}^t = \underline{q}_{i-1} \underline{q}^{t-1},$$

for all $t \geq 1$, then $x_1, ..., x_n$ is a sequence of linear type. In particular, a d-sequence is a sequence of linear type.

From this lemma, we get the following diagram (see [9], 6.1 and [13], Lemma 2.3):

$$\begin{array}{c}
\text{d-sequence (or r.r.s.)} \\
\Updownarrow \\
(\underline{q}_{i-1} \underline{q}^t : x_i) \cap \underline{q}^t = \underline{q}_{i-1} \underline{q}^{t-1} \ \text{ for all } t \geq 0 \\
\Downarrow \\
(\underline{q}_{i-1} \underline{q}^t : x_i) \cap \underline{q}^t = \underline{q}_{i-1} \underline{q}^{t-1} \ \text{ for all } t \geq 1 \\
\Downarrow \\
\text{sequence of linear type}
\end{array}$$

Inspired with the fact that Buchsbaum rings can be characterized by various generalizations of regular sequences (see [20], Proposition I.1.17), and from the above diagram it is raised for us a quesion : Can we characterize Buchsbaum rings by sequences of linear type and by s.o.p. satisfying (*), respectively? We shall give a counter-example to it. We need the following auxiliary result which is the converse of Lemma 3.3 in a particular case.

Proposition 3.4 *Assume that A is a domain or a generalized C-M local ring. In the last case, we assume that x, y be two parameter elements of A. Then x, y is a sequence of linear type if and only if it satisfies the condition (*) for all $t \geq 1$.*

Proof. By Lemma 3.3 one has only to prove the implication ($\Rightarrow$).
(1) To prove $(0 : x) \cap (x, y)A = 0$.

One has only to consider the case $(A, \underline{m})$ is a generalized C-M local ring and $x, y \in \underline{m}$ are parameter elements of A. Let $u \in (0 : x) \cap (x, y)A$. By Lemma 2.4, $0 : x \subseteq H^0_{\underline{m}}(A)$. Hence, there exists a positive integer n such that $y^n u = 0$. Then we get

$$\begin{aligned} u &\in (0 : x) \cap (x, y)A \cap (0 : y^n) \\ &= (0 : x) \cap yA \cap (0 : y^n) \quad (\text{by [3], Lemma 2}) \\ &= 0 \quad (\text{by [3], Lemma 4}). \end{aligned}$$

(2) To prove $(x(x, y)^t A : y) \cap (x, y)^t A = x(x, y)^{t-1}A,\ t \geq 1$.

Let $u \in (x(x, y)^t A : y) \cap (x, y)^t A$. We may assume that $u = y^t a$ for some $a \in A$. Then $a \in x(x, y)^t A : y^{t+1} = xA : y$ (by [3], Theorem 4). Hence $ya \in xA$, so $u = y^{t-1}(ya) \in x(x, y)^{t-1}A$, as required.

Example 3.5 This example shows that Proposition 3.4 is not true without any assumption on A. Let $A = K[[X, Y, Z]]/(X) \cap (Y, Z) = K[[X, Y, Z]]/(XY, XZ) = K[[x, y, z]]$. Since $x + z$ is a non-zero divisor, using Lemma 3.3 one can show that $x + z, y$ is a sequence of linear type. On the other hand, y is a d-sequence because $0 : y = 0 : y^2 = xA$. Hence $y, x + z$ is also a sequence of linear type. But

$$(0 : y) \cap (y, x + z)A = xA \cap (y, x + z)A \ni x(x + z) = x^2 \neq 0.$$

That means, $y, x + z$ does not satisfy (*) even for $t = 1$. Otherwords, $y, x + z$ is not a r.r.s. w.r.t. $(y, x + z)A$ in the sense of Definition 3.2.

Example 3.6 There exists a local ring A which is not Buchsbaum, but every s.o.p. of A satisfies the condition (*) for all $t \geq 1$, and hence, by Lemma 3.3, is also a sequence of linear type. Indeed, let $A = K[[s^2, s^5, st, t]]$. It is easy to see that A is a two-dimensional generalized C-M, but not Buchsbaum local domain with $\underline{m} = (s^2, s^5, st, t)A$. We will show that A has the above property.

Proof. A has the following property: $s^2 H^1_{\underline{m}}(A) \neq 0$ (so A is even not a quasi-Buchsbaum ring) and $\underline{n} H^1_{\underline{m}}(A) = 0$, where $\underline{n} = (s^4, s^5, st, t)A$ (see [8], Example

42.8). From this and $H^0_{\underline{m}}(A) = 0$ it is easily seen that every s.o.p. of A contained in $\underline{n}$ is an $\underline{n}$-weak sequence. Hence every s.o.p. of A contained in $\underline{n}$ is a d-sequence ([22], Proposition 3.1).

Now let x, y be a s.o.p. of A. If both x, y are contained in $\underline{n}$, from the above observation and from Lemma 3.3, x, y is a sequence of linear type. Let $x \notin \underline{n}$. Then one can write $x = s^2a + s^5b + stc + td$, for some $a \in K$ and $b, c, d \in A$. Since $(x, y)A = (x, y - xu)A$ for an arbitrary element $u \in A$, we may assume that $y = s^5b' + stc' + td'$ for some $b', c', d' \in A$, i. e. $y \in \underline{n}$. From the exact sequence

$$0 \to H^0_{\underline{m}}(A/xA) \to H^1_{\underline{m}}(A) \xrightarrow{\cdot x} H^1_{\underline{m}}(A),$$

$\underline{n}H^1_{\underline{m}}(A) = 0$, we see that $0_{\bar{A}} : y = 0_{\bar{A}} : y^2$, where $\bar{A} = A/xA$. By [12], Proposition 1.7, $(0_{\bar{A}} : y) \cap y\bar{A} = 0_{\bar{A}}$, or equivalently, $(xA : y) \cap (x, y)A = xA$. Since x is a non zero-divisor, x, y is a r.r.s. (d-sequence). Hence $(x, y)A$ is also an ideal of linear type in the case $x \notin \underline{n}$. By Proposition 3.4, the s.o.p. x,y satisfies the condition (*) for all $t \geq 1$, as required.

Finally, we want to give a remark about a question of D. L. Costa in [3], p. 261: For any sequence $a_1, ..., a_n$ of linear type and any integer $s \geq 2$ is $a_1^s, ..., a_n^s$ also a sequence of linear type? In the mentioned paper, a positive answer was given for $n \leq 2$. The following simple example shows that in general this is not true already for $n = 3$.

Example 3.7 Let a, b be any regular sequence in a commutative ring A with $1/2 \in A$. Then $a, b, a + b$ and a^2, b^2 are sequences of linear type. Using the relation

$$(a^2, b^2)(a^2, b^2, ab)A : (ab)^2 = A \neq (a^2, b^2)A : ab = (a, b)A,$$

we deduce from [3], Theorem 4 that the ideal $(a^2, b^2, (a+b)^2)A = (a^2, b^2, ab)A$ is not of linear type. Hence, $a^2, b^2, (a + b)^2$ is not a sequence of linear type.

4 Castelnuovo-Mumford regularity

In this section we will consider K-algebra, i. e. noetherian graded rings $A = \oplus_{n\geq 0} A_n$, where $A_0 = K$ is an infinite field, and A is generated over K by A_1. Let $\underline{m} = \oplus_{n>0} A_n$ be the unique homogeneous maximal ideal of A. Analogy to Section 2, one can define the notion of generalized C-M rings, etc. for the graded case. We define Castelnuovo-Mumford regularity $regA$ of A by

$$regA = inf\{n \in \mathbf{Z}; [H^i_{\underline{m}}(A)]_j = 0 \text{ for all } i, j \text{ such that } i + j > n\}.$$

The significance of this notion stems from the fact that $regA$ governs the complexity of computing the syzygies and other invariants of A [4]. For further informations, especially for Castelnuovo-Mumford regularity of generalized C-M K-algebras, see [10], [11], and [21].

To get the main theorem of this section, let us recall the notion of minimal reductions of an ideal [16]. If I, J are ideals of A, Then J is called a *reduction* of

I if $J \subseteq I$ and $I^{r+1} = JI^r$ for at least one non-negative integer r. If J is minimal with respect to being a reduction, then it is called a *minimal reduction*. Since K is infinite, any homogeneous ideal of A has a minimal reduction. The *reduction exponent* $r(I)$ of I is just the least non-negative integer n such that there exists a minmal reduction J with $I^{n+1} = JI^n$. There are some relations between $r(\underline{m})$ and $regA$ in [23]. The following theorem extends [23], Corollary 3.5 which states that $regA = r(\underline{m})$ if A is a Buchsbaum ring.

Theorem 4.1 *Assume that A is a (k,d)-Buchsbaum K-algebra, where $d = dimA$ and $k \geq 1$. Then*

$$r(\underline{m}) \leq regA \leq r(\underline{m}) + d(k-1).$$

Proof. The first inequality was proven in [23], Proposition 3.2. We shall prove the second one. Let $r = r(\underline{m})$ and J be any minimal reduction of $\underline{m}$ such that $\underline{m}^{r+1} = J\underline{m}^r$. It is well-known that J is generated by d linear forms $x_1, ..., x_d$. Let $p = r + d(k-1) + 1$. Then we have

$$\begin{aligned} \underline{m}^p &= (x_1, ..., x_d)^{d(k-1)+1}\underline{m}^r \\ &= (x_1^{d(k-1)+1}, x_1^{d(k-1)}x_2, ..., x_d^{d(k-1)+1})\underline{m}^r \\ &\subseteq (x_1^k, ..., x_d^k)A. \end{aligned}$$

Let $1 \leq i \leq d$ be any integer. Put $\bar{A} = A/\underline{q}_{i-1}$ (recall that $\underline{q}_{i-1} = (x_1, ..., x_{i-1})A$). By Definition 2.2, $\bar{A}$ is a $(k, d-i+1)$-Buchsbaum A-module. Hence, by Remark 2.3, $x_i^k, ..., x_d^k$ is a standard s.o.p. of $\bar{A}$. By [22], Corollary 2.3 we have

$$H^0_{\underline{m}}(\bar{A}) \cap \underline{m}^p\bar{A} = H^0_{\underline{m}}(\bar{A}) \cap (x_i^k, ..., x_k^d)\bar{A} = 0_{\bar{A}}.$$

Note that, by Lemma 2.4, $0_{\bar{A}} : x_i = H^0_{\underline{m}}(\bar{A})$. Therefore

$$(\underline{q}_{i-1} : x_i) \cap \underline{m}^p \subseteq \underline{q}_{i-1}, \quad i = 1, ..., d.$$

That means, $x_1, ..., x_d$ form an $(A, \underline{m}^p)$-r.r.s. (see Definition 3.2). Note that a sequence is an $(A, \underline{m}^p)$-r.r.s. if and only if it is a t-regular sequence for all $t \geq p$ in the sense of [23]. Hence, if we denote by

$$a(\underline{x}) = inf\{n;\ \underline{x} = x_1, ..., x_d \text{ is : an : } (A, \underline{m}^n) - \text{r.r.s.}\},$$

then by [23], Corollary 3.3 and Proposition 2.2 we have

$$regA + 1 = max\{r + 1, a(\underline{x})\}.$$

Obviously, $a(\underline{x}) \leq p$ and $r + 1 \leq p$. So $regA \leq p - 1$, as required.

Note that in Theorem 4.1 we need no assumption that A is to be a domain (cf. [21] and [10]). Unfortunately, in general we do not know how large is $r(\underline{m})$. On the other hand, if $r(\underline{m})$ is small then we can have a good bound on $regA$. For example, if A is a Buchsbaum ring of maximal embedding dimension, then $r(\underline{m}) = 1$, hence $reg A = 1$.

As we have remarked, $(A, \underline{m}^k)$-r.r.s.'s for some k are just weakly regular sequences in [8], or t-regular sequences for all $t \geq k$ in [23]. Because of the importance of $(A, \underline{m}^k)$-r.r.s.'s, we shall give them a characterization by Koszul homology $H_p(\underline{x}; A)$, where $\underline{x}$ denotes the sequence $x_1, ..., x_n$. In the following proposition A is not necessary a K-algebra.

Proposition 4.2 *Let $A = \oplus_{i \geq 0} A_i$ be a graded ring generated by A_1 over A_0. Let $\underline{m} = \oplus_{i \geq 1} A_i$ and $x_1, ..., x_n$ be homogeneous elements of A such that $deg\, x_1 \leq \cdots \leq deg\, x_n$. Assume that $\underline{x} = x_1, ..., x_n$ is an $(A, \underline{m}^k)$-r.r.s. for some $k \geq 0$. Then*

$$\underline{m}^k H_p(\underline{x}; A) = 0 \quad for\ p > 0.$$

Proof. Let

$$K.(\underline{x}; A): \quad K_{p+1} \xrightarrow{d_{p+1}} K_p \xrightarrow{d_p} K_{p-1} \to \cdots,$$

denote the Koszul complex generated by $\underline{x}$ over A. It is enough to show that

$$Ker\, d_p \cap \underline{m}^k K_p(\underline{x}; A) \subseteq Im\, d_{p+1} \text{ for all } p > 0.$$

Moreover, we will show by induction on n that for every $a \in Ker\, d_p \cap \underline{m}^k K_p(\underline{x}; A)$ there exists an element $a' \in \underline{m}^{k - deg\, x_n} K_{p+1}$ such that $a = d_{p+1}(a')$ (we set $\underline{m}^t = A$ if $t \leq 0$).

If $n = 1$, one has only to consider the case $p = 1$. Since

$$Ker\, d_1 \cap \underline{m}^k K_1(x_1; A) \subseteq (0 : x_1) \cap \underline{m}^k = 0,$$

the above assertion is trivial in this case.

Let $n > 1$. We denote by (L_p, e_p) the Koszul complex generated by $x_1, ..., x_{n-1}$ over A. It is well-known that we can consider K_p as the direct sum $L_{p-1} \oplus L_p$ and

$$d_p((u, v)) = (e_{p-1}(u), (-1)^{p-1} x_n u + e_p(v)),$$

for $u \in L_{p-1}, v \in L_p$. Hence, if $(u, v) \in Ker\, d_p \cap \underline{m}^k K_p(\underline{x}; A)$ we have

$$\begin{aligned} u &\in Ker\, e_{p-1} \cap \underline{m}^k L_{p-1}, \text{ and} && (1)\\ v &\in \underline{m}^k L_p \text{ and } (-1)^{p-1} x_n u + e_p(v) = 0. && (2) \end{aligned}$$

If $p \geq 2$ then (1) and the induction hypothesis imply that $u = e_p(u')$ for some $u' \in \underline{m}^{k - deg\, x_{n-1}} L_p$. If $p = 1$ this also holds. Indeed, in this case $e_1(v) \in (x_1, ..., x_{n-1})A = Im\, e_1$. From the equality of (2) and from (1) we get by Definition 3.2 that

$$u \in ((x_1, ..., x_{n-1})A : x_n) \cap \underline{m}^k \subseteq (x_1, ..., x_{n-1})A.$$

Writting $u = x_1 u_1 + ... + x_{n-1} u_{n-1}$, where $u_i = 0$ or u_i is a homogeneous element with $deg\, u_i \geq k - deg\, x_i \geq k - deg\, x_{n-1}$, we obtain $u = e_1(u')$ with $u' \in \underline{m}^{k - deg\, x_{n-1}} L_1$. The existence of u' is proven. Now we have for all $p \geq 1$:

$$(**) \qquad e_p((-1)^{p-1} x_n u' + v) = (-1)^{p-1} x_n u + e_p(v) = 0.$$

On the other hand, since $deg\, x_n \geq deg\, x_{n-1}$,

$$x_n u' \in \underline{m}^{deg\, x_n} \underline{m}^{k-deg\, x_{n-1}} L_p \subseteq \underline{m}^k L_p.$$

Hence we get from (**) and (2)

$$(-1)^{p-1} x_n u' + v \in Ker\, e_p \cap \underline{m}^k L_p.$$

By the induction hypothesis, there exists an element $v' \in \underline{m}^{t-deg\, x_{n-1}} L_{p+1}$ such that $(-1)^{p-1} x_n u' + v = e_{p+1}(v')$. Thus $(u, v) = d_{p+1}((u', v'))$ and

$$(u', v') \in \underline{m}^{t-deg\, x_{n-1}} K_{p+1} \subseteq \underline{m}^{t-deg\, x_n} K_{p+1},$$

as required.

Remark. The following result suggests our Proposition 4.2. Let $N_2 \subseteq N_1$ be submodules of M such that $(x_1, ..., x_n)M \subseteq N_2$. Let $N_2 : M = \{a \in A; aM \subseteq N_2\}$. Assume that $x_1, ..., x_n$ is an (N_1, N_2)-r.r.s. of M. Then

$$(N_2 : M) H_p(x_1, ..., x_n; M) = 0 : \quad \text{for all } p > 0.$$

This can be easily deduced from [5] and an exact sequence of Koszul homology derived from the exact sequence

$$0 \to N_2 \to M \to M/N_2 \to 0,$$

(cf. also [9], 6.1.2). If $(x_1, ..., x_n)M \nsubseteq N_2$ we cannot find in references any similar result.

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