A Characterization of the Grassmanian of points and lines for $C_{3,2}$-buildings

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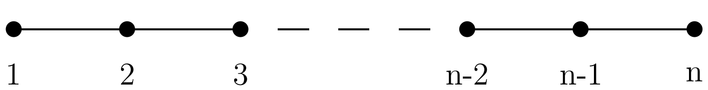
Abstract

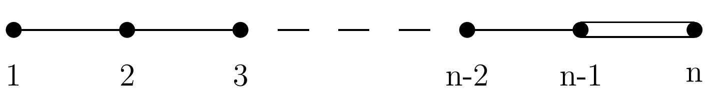
We give necessary and sufficient conditions for a line space to be the shadow space of a $C_{3,2}$-building.

1 Introduction

Consider a Coxeter diagram of spherical type A_n, C_n, D_n, $\ldots$,F_4 with a natural labelling of its nodes as in Bourbaki [1]. The following examples will be considered in the present paper:
A_n



and C_n



*Supported by a grant of the "Fonds National Belge de la Recherche Scientifique"

Received by the editors November 1993
Communicated by J.Doyen
AMS Mathematics Subject Classification : 51A.
Keywords : incidence geometry, building, polar space and line space.

Each of these diagrams corresponds to a class of buildings (Tits [16]). If the node labelled i is singled out, we get a diagram called $A_{n,i}$ or $C_{n,i}$ respectively. Geometrically, this amounts to the construction of a *line space* (the definition is given below) with one type (namely i) of vertices of a building Δ of type X_n, where X_n is a Coxeter diagram of spherical type. The point-set P of this *line space* is the set of all i-elements; a subset l of P is a line if and only if there exists a flag F of cotype i such that l is the set of all i-elements incident with F. We call this line space an $X_{n,i}$-*building space* and we denote it by $S(\Delta, i)$. We can make a similar construction from a geometry.

The $A_{n,1}$-building spaces correspond exactly to projective spaces (Tits [16]) and the classical work of Veblen and Young [17] characterizes the latter in terms of points and lines. Buekenhout-Shult's characterization of polar spaces [4] gives an analogous result for $C_{n,1}$-building spaces. It seems reasonable to try to find a similar characterization for all building spaces $A_{n,i}$, $C_{n,i}$, ..., $F_{4,i}$. Many authors (e.g. Cameron [5], Cooperstein [10], Cohen [6] [7], Buekenhout [3], Cohen-Cooperstein [9], Hanssens[12] [13], Hanssens-Thas [14]) have worked on this problem. For a recent survey, see Cohen [8].

The first open case (in alphabetical order) is $C_{n,n-1}$; it is a difficult one in view of earlier approaches. In order to deal with this case, it seems appropriate to study $C_{3,2}$ first. This is the purpose of the present paper. The general case $C_{n,n-1}$ is discussed in Lehman [15].

A *line space* Γ is a pair (P, L) where P is a set whose elements are called *points* and L is a set of subsets of P called *lines* such that each line contains at least two points. Two distinct points p and q are called *collinear* if there exists a line which contains these two points; we denote this fact by $p \sim q$ and the fact that p and q are not collinear by $p \not\sim q$. A line space Γ is called a *partial linear space* if any two distinct points are contained in at most one line. A *subspace* Γ' of a partial linear space Γ is a pair (P', L') such that $P' \subset P$, $L' \subset L$ and two points of P' are collinear in Γ if and only if they are collinear in Γ'. If Γ is a partial linear space and if $p \sim q$ then we denote by pq the unique line containing these two points. A *path* between two points p_0 and p_k is a sequence of points $p_0, p_1, p_2, \ldots, p_k$ such that $p_i \sim p_{i+1}$ for each $i = 0, ..., k-1$. A *circuit* is a path such that p_0 and p_k are equal. A line space Γ is called *connected* if there exists a path between any two points.

Let Γ be a connected partial linear space. By definition, a *plane* (resp. a *quad*) of Γ is a subspace of Γ which is a projective plane (resp. a maximal (with respect to inclusion) generalized quadrangle).

We are interested in the following properties:

CO1. Any two distinct planes intersect in at most one point.

CO2. Any two distinct quads intersect in at most one point.

CO3. Every line is contained in at least one plane.

CO4. Every line is contained in at least one quad.

CO5. The intersection of a plane and a quad is a line or the empty set .

A connected partial linear space with these 5 properties will be called a *cuboctahedral space*. The term cuboctahedral has been chosen on purpose. Indeed, the prototype of a building is one of its apartments, namely a Coxeter complex. For

$C_{3,1}$ this is an octahedron. For $C_{3,3}$ it is a cube. For $C_{3,2}$ the prototype is a cuboctahedron (see for instance Coxeter [11]).

We will prove that these 5 properties characterize buildings of type $C_{3,2}$; more precisely, we will prove the following results.

Theorem 1. *Let Δ be a building of type C_3. Then the building space $S(\Delta, 2)$ is a cuboctahedral space.*

Theorem 2. *If S is a cuboctahedral space, then there is a building Δ of type C_3 such that S is isomorphic to $S(\Delta, 2)$.*

2 Lemmas

In a partial linear space, let us define a *quadrangle* as a set of four points $\{p_1, p_2, p_3, p_4\}$ such that $p_1 \sim p_2 \sim p_3 \sim p_4 \sim p_1 \not\sim p_3$ and $p_2 \not\sim p_4$. We also define a *triangle* as a set of three points any two of which are collinear.

2.1 The quadrangle lemma

In a cuboctahedral space, every quadrangle is contained in exactly one quad.

Proof. Let p_1, p_2, p_3, p_4 be the four points of a quadrangle. The line p_2p_3 (resp. p_4p_1) is contained in a plane P_1 (resp. P_2). The two planes P_1 and P_2 are distinct, otherwise $p_1 \sim p_3$ or $p_2 \sim p_4$, contradicting the fact that $\{p_1, p_2, p_3, p_4\}$ is a quadrangle. The line p_1p_2 (resp. p_3p_4) is contained in a quad Q_1 (resp. Q_2). The intersection of P_1 and Q_1 (resp. P_2 and Q_1, P_1 and Q_2, P_2 and Q_2) is a line D_1 (resp. D_2, D_3, D_4). As the two lines D_1 and D_3 (resp. D_2 and D_4) are contained in a projective plane, they meet in a point q_1 (resp. q_2). The points q_1 and q_2 are distinct, otherwise there would be a triangle (namely $\{p_1, p_2, q_1\}$) in the generalized quadrangle Q_1. The quads Q_1 and Q_2 are equal, otherwise their intersection would contain at least two points (q_1 and q_2). The quadrangle $\{p_1, p_2, p_3, p_4\}$ is contained in the quad $Q = Q_1 = Q_2$. The unicity of this quad follows from property CO2.

■

2.2 Remark

We can prove a similar property for triangles (the triangle lemma) : *In a cuboctahedral space, every triangle is contained in exactly one plane.*

2.3 Lemma

In a cuboctahedral space, every line is contained in exactly one plane and exactly one quad.

Proof.If a line is contained in two distinct planes (resp. quads), this contradicts property CO1 (resp. CO2). The existence of such a plane (resp. such a quad) follows from property CO3 (resp. CO4).

■

3 Cuboctahedral geometry

We first recall some definitions and notations about geometries.

Let V and I be two sets, let t be a function from V to I called the *type function* and let * be a reflexive and symmetric relation on V called *incidence*. The quadruple $(V, I, t, *)$ is an *incidence system*. A *flag* is a subset F of V such that any two elements of F are incident. A *chamber* is a flag C such that $t(C) = I$. An incidence system $(V, I, t, *)$ is called a *geometry* if and only if the following two conditions are satisfied:
a) Two distinct elements of V of the same type are never incident.
b) Every comaximal flag is contained in at least two chambers.

The *rank* of a geometry is the cardinality of I. Let $\Gamma = (V, I, t, *)$ be a geometry. Let a and b be two elements of V. A *path of length* n between a and b is a sequence $(p_1, p_2, \ldots, p_{n-1}, p_n)$of elements of V such that $p_1 = a, p_n = b$ and, for every $i < n$, p_i is incident with p_{i+1}. The *residue* Γ_F of a flag F is the geometry $(V', I', t', *')$ where V' is the set of all elements of $V \backslash F$ which are incident to every element of F, $I' = I \backslash t(F)$, $t' = t_{|V'}$ and $*' = *_{|V'}$. A geometry is called *connected* if for any two elements of V, there is a path between them. A geometry is *residually connected* if every residue of rank at least 2 is connected. The following result is well-known [2]: *Let $\Gamma = (V, I, t, *)$ be a residually connected geometry. Let i and j be two distinct elements of I. Let a and b be two elements of V. Then there exists an integer n such that there exists a path of length n between a and b with $t(\{p_2, \ldots, p_{n-1}\}) \in \{i, j\}$. If $t(a)$ and $t(b) \in \{i, j\}$, we call such a path an (i-j)-path of length n.*

3.1 Definition

Given a cuboctahedral space Γ, we define an incidence system of rank 3, denoted by $G(\Gamma)$, whose types are called point, plane and quad, as follows: the elements of $G(\Gamma)$ of type point (resp. plane, quad) are the points (resp. planes, quads) of Γ. The incidence is the natural one: for the points the incidence is defined by inclusion and a plane is incident with a quad if and only if their intersection is a line. We call such an incidence system a *cuboctahedral geometry*. We shall prove that it is indeed a geometry (see proposition 3.3).
Remark. We often identify a plane (resp. a quad) of a cuboctahedral geometry with the set of all points contained in it.

3.2 Lemma

1) Each flag of type plane-quad has at least two points incident with it.
2) Each flag of type point-quad has at least two planes incident with it.
3) Each flag of type point-plane has at least two quads incident with it.
Proof. 1) If a plane is incident with a quad, then by definition of the incidence they contain a line and, so at least two points.
2) Given a point of a quad, there are at least two distinct lines of this quad containing this point. Property CO3 shows that each of these two lines is included in a plane,

and by CO5, these planes are distinct.
3) Given a point of a plane, there are at least two distinct lines of this plane containing this point. Property CO4 shows that each of these two lines is included in a quad and, by CO5, these quads are distinct.

■

3.3 Proposition

A cuboctahedral geometry is a geometry.
Proof. Two distinct elements of the same type are never incident by definition of the incidence. Moreover lemma 3.2 implies that every comaximal flag is contained in at least two chambers.

■

3.4 Lemma

1) The residue of a point in a cuboctahedral geometry is a generalized digon.
2) The residue of a plane (resp. quad) in a cuboctahedral geometry is a projective plane (resp. a generalized quadrangle).
3) A cuboctahedral geometry is residually connected.
Proof. 1) By CO5, a quad and a plane both incident with a point p contain a common line and so are incident.
2) By lemma 2.2, there is an obvious bijection between the set of quads incident with a plane P (resp. between the set of planes incident with a quad Q) and the set of lines of P (resp. Q).
3) The geometry $G(\Gamma)$ is connected because Γ is connected. Moreover, the residue of a point (resp. a quad, a plane) is a generalized digon (resp. a generalized quadrangle, a projective plane) and so is connected. Therefore $G(\Gamma)$ is residually connected.

■

3.5 Corollary

A cuboctahedral geometry is a geometry of type C_3.
Proof. This follows immediately from 1) and 2) in lemma 3.4.

■

4 Proof of theorem 1

We will often identify the lines of $S(\Delta, 2)$ and the flags of type $\{1, 3\}$ of Δ. We will denote the 1-, 2- and 3-elements of Δ by symbols such that p, D and π.

First part: *The building space $S(\Delta, 2)$ is a partial linear space.*
Proof. Since Δ is a building and since a building is firm, each line of $S(\Delta, 2)$ contains at least two points. Thus $S(\Delta, 2)$ is a line space. Moreover $S(\Delta, 2)$ is a partial linear space because two distinct 2-elements of Δ are incident with at most one 1-element

of Δ and with at most one 3-element of Δ.

Second part *A subspace $\gamma = (P', L')$ of $S(\Delta, 2)$ is a projective plane if and only if there is a 3-element of Δ such that P' is the set of 2-elements of Δ incident with it.*
Proof. If there is a 3-element of Δ such that P' is the set of 2-elements of Δ incident with it, then γ is obviously a projective plane because the residue of a 3-element in a building of type C_3 is a projective plane.

Let $\gamma = (P', L')$ be a subspace of $S(\Delta, 2)$ which is a projective plane. We claim that there is a 3-element of Δ incident with every point of P'. Indeed, let D and D' be two distinct points of γ. Since γ is a projective plane, these two points are collinear. Let $p\pi$ be the line of γ containing these two points. We know that D and D' are both incident with π (which is a 3-element of Δ). We shall prove that all points of γ are incident with π. Let D" be a point of γ different from D and D'. If D" is contained in $p\pi$, then D" is obviously incident with π. Hence we may assume that D" is not contained in $p\pi$. Since γ is a projective plane, $D \sim D$" and $D' \sim D$". Let $p'\pi'$ (resp. p"π") be the line containing D and D" (resp D' and D"). We can easily see that the 1-elements p, p' and p" are all incident with each of the 3-elements π, π' and π" (because the residue of a 2-element is a generalized digon).

Note that $(D, p\pi, D', p$"π"$, D$"$, p'\pi')$ is a circuit of length 6 in γ. Let Π denote the set of 1-elements of Δ which are incident with each of the elements π, π' and π". There is an element of Δ such that the set Π is the set of all 1-elements incident with it (we can prove this property in $S(\Delta, 1)$ which is a polar space; there the property amounts to the fact that the intersection of three planes is either a point, a line or a plane). Obviously Π cannot be a unique 1-element, otherwise $p = p' = p$" and $(D, p\pi, D', p\pi$"$, D$"$, p\pi')$ would be a circuit of length 6 completely contained in the residue of p which is a generalized quadrangle. Using $S(\Delta, 1)$, we can also prove that there is no 2-element of Δ such that Π is the set of 1-elements incident with it. Indeed, in that case, at most two of the 3-elements π, π' and π" are equal. This means that in $S(\Delta, 1)$ two of the three subspaces $\pi \cap \pi'$, $\pi \cap \pi''$ and $\pi' \cap \pi''$ are lines and are equal to $\pi \cap \pi' \cap \pi''$. Moreover if $\pi \cap \pi'$ (resp. $\pi \cap \pi''$,$\pi' \cap \pi''$) is a line, it must be D (resp. D', D") and then two of those three 2-elements are equal, contradicting the fact that we have assumed them to be distinct. Then there is a 3-element of Δ such that Π is the set of all 1-elements incident with it. This 3-element must be equal to $\pi = \pi' = \pi$" (indeed, in $S(\Delta, 1)$, if the intersection of three planes is a plane then the three planes are equal) and then D" is incident with π.

Since no projective plane contains a proper subspace isomorphic to a projective plane and since the set of all 2-elements incident with a 3-element π of Δ is itself a projective plane, we know that for each projective plane γ in $S(\Delta, 2)$ there exists a 3-element π of Δ such that γ is the set of all 2-elements incident with π.

Third part: *A subspace $\gamma = (P', L')$ of $S(\Delta, 2)$ is a maximal generalized quadrangle if and only if there is a 1-element of Δ such that P' is the set of 2-elements of Δ incident with it.*
Proof. We shall reduce the proof to the proof of the following three statements:

A. If there is a 1-element of Δ such that P' is the set of 2-elements of Δ incident with it, then γ is a generalized quadrangle.

B. If γ is a generalized quadrangle, then for each pair of points of P' there is a 1-element of Δ which is incident with these two points.

C. If γ is a generalized quadrangle, then there is a 1-element of Δ which is incident with every point of P'.
Indeed, if γ is a generalized quadrangle, then by statement **C** there exists a 1-element p of Δ such that every point of γ is a 2-element incident with p. Moreover, if p is a 1-element of Δ, the set of all 2-elements incident with p is itself a generalized quadrangle, and so γ can be a maximal generalized quadrangle only if there is a 1-element p of Δ such that P' is the set of all 2-elements incident with p. The converse is also true. Indeed, if p is a 1-element of Δ and if P' is the set of all 2-elements incident with p, then γ is a generalized quadrangle. Moreover, for each subspace γ' of $S(\Delta, 2)$ which is a generalized quadrangle, there is a 1-element p' such that every point of γ' is a 2-element incident with p'. Thus γ is a maximal generalized quadrangle. Let us now prove statements **A**, **B** and **C**.
A. If there is a 1-element of Δ such that P' is the set of 2-elements of Δ incident with it, then γ is clearly a generalized quadrangle because the residue of a 1-element of Δ is a generalized quadrangle.
B. Let $\gamma = (P', L')$ be a subspace of $S(\Delta, 2)$ which is a generalized quadrangle. We claim that for any two points of P' there exists a 1-element of Δ such that the two chosen points are two 2-elements of Δ both incident with it. Indeed, let D_1 and D_2 be two points of P'. If these two points are collinear, then there is a flag $p\pi$ of type $\{1, 3\}$ such that D_1 and D_2 are incident with $p\pi$ and so with p. If $D_1 \not\sim D_2$ then there is a point-line circuit of length 8 in γ including D_1 and D_2 (because γ is a generalized quadrangle). Let $(D_1, p_1\pi_1, D_3, p_3\pi_3, D_2, p_2\pi_2, D_4, p_4\pi_4)$ be such a circuit. We get the following relations:
- The 2-element D_1 of Δ is incident with the 3-elements π_1 and π_4 of Δ.
- The 1-element p_2 of Δ is incident with the 2-elements D_4 and D_2 of Δ which are incident with the 3-elements π_4 resp. π_3 of Δ. Then, since the residue of a 2-element is a generalized digon, p_2 is incident with π_3 and π_4. It is trivial that π_1 and π_4 are different, otherwise, since the residue of a 3-element is a projective plane, there would be a 1-element of Δ (say p) such that D_3 and D_4 are both incident with p. As the line $p\pi_1$ of $S(\Delta, 2)$ contains each of the points D_3 and D_4 of γ, it is a line of γ. The contradiction follows from the fact that $(D_1, p_1\pi_1, D_3, p\pi_1, D_4, p_4\pi_1)$ is a point-line circuit of length 6 in a generalized quadrangle. We also know that p_2 is incident with π_1; otherwise, we can prove that in $S(\Delta, 1)$ the point p_2 is collinear with all points of the plane π_1, which is impossible in a polar space of rank 3. Indeed, let $p \neq p_1$ be a point of $S(\Delta, 1)$ contained in the plane π_1 of $S(\Delta, 1)$ and let D be a line of $S(\Delta, 1)$ on p which does not contain $D_1 \cap D_3$ and which is included in π_1. Let p' (resp. p") be the point of $S(\Delta, 1)$ common to D and D_1 (resp. to D and D_3). Since p_2 and p' (resp. p_2 and p") are included in the plane π_4 (resp. π_3) of $S(\Delta, 1)$, they are collinear and as p_2 is collinear with two points of D, it is collinear with all points of D (because $S(\Delta, 1)$ is a polar space) and so with p.

Finally, we see that p_2 is incident with both D_2 and D_1; indeed, in $S(\Delta, 1)$, the line D_1 is the intersection of the planes π_1 and π_4; and since p_2 is contained in π_1 and π_4, p_2 is contained in D_1.

C. Let D and D' be two distinct collinear points of γ. Since D and D' are collinear, there is a flag of type $\{1,3\}$ (denoted by $p\pi$) such that D and D' are two 2-elements of Δ incident with $p\pi$. Let D" be a point of γ distinct from D and D'. We shall prove that p is incident with D". If the points D, D' and D" of γ are collinear, then the point D" of γ is contained in the line $p\pi$ of γ, and so D" is incident with p. Therefore we may assume that the points D, D' and D" of γ are not collinear. By **B**, we know that there exists a 1-element p' (resp. p") of Δ such that D and D" (resp. D' and D") are both incident with p' (resp. p"). The 2-element D" of Δ is not incident with the 3-element π of Δ, otherwise, since each of the line $p'\pi$ and p"π of $S(\Delta,2)$ has two of its points in P', it would be a line of γ and there would exist a point-line circuit of length 6 $(D, p\pi, D', p"\pi, D", p'\pi)$ in a generalized quadrangle. Moreover, the 1-elements p' and p" of Δ are equal, otherwise, in $S(\Delta,1)$, the line D" would be the only one containing these two points and the points p' and p" of $S(\Delta,1)$ are contained respectively in the line D and D' of $S(\Delta,1)$ themselves contained in the plane π of $S(\Delta,1)$. The contradiction follows from the fact that the 2-element D" of Δ is included in the 3-element π of Δ. Moreover p is equal to p' because in $S(\Delta,1)$ they belong to D and D' and the intersection of two distinct lines cannot contain more than one point. Finally, the last equality proves that the 2-element D" of Δ is incident with the 1-element p of Δ.

Fourth part: *The building space $S(\Delta,2)$ is a cuboctahedral space.*
We already know that $S(\Delta,2)$ is a partial linear space and that the quads (resp. the planes) can be identified with the 1-(resp. 3-)elements of Δ.We only need to check the five properties CO1, ..., CO5.
In $S(\Delta,1)$ property CO1 becomes: "The intersection of any two distinct planes contains at most one line". This is straightforward because $S(\Delta,1)$ is a polar space of rank three.
In $S(\Delta,1)$ property CO2 becomes: "There is at most one line on two distinct points". This is straightforward because $S(\Delta,1)$ is a line space.
In Δ properties CO3 and CO4 become: "For every flag of type $\{1,3\}$, the set of 2-elements incident with this flag is contained in the set of all 2-elements incident with at least one 3-element (for CO3) or one 1-element (for CO4)".
In Δ property CO5 becomes: "Given one 1-element p and one 3-element π, either p is incident with π and then the intersection of the set of 2-elements incident with p and the set of 2-elements incident with π is the set of 2-elements incident with the flag $p\pi$, or p is not incident with π; then there is no 2-element incident with both p and π (because the residue of a 2-element in Δ is a generalized digon)".

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5 Proof of theorem 2

We will denote the points, quads and planes of $G(\Gamma)$ by symbols such that p, Q and P.

First part. *Let Γ be a cuboctahedral space. Let p be a point of $G(\Gamma)$ and Q be*

a quad of $G(\Gamma)$ non incident with p. Then there is a point-quad path of length 4 between p and Q.
Proof. We reduce the proof of this part to the proof of the following statement: If $(p, Q', p', Q", p", Q)$ is a point-quad path between p and Q, then there is a point-quad path of length 4 between p and Q.
Indeed, by lemma 3.4, Γ is residually connected, and so there is an integer $n > 1$ and a point-quad path of length $2n$ between p and Q. We can prove this part using the statement $(n-2)$ times.

To prove the statement, we distinguish three cases:
(1) p' and $p"$ are collinear.
The line $p'p"$ is contained in a plane P_0. The intersection of P_0 and Q (resp. Q') is a line D_1 (resp. D_2). Since the lines D_1 and D_2 are both contained in the projective plane P_0, they meet in a point q. As the point q is included in D_1 (resp. D_2), it is incident to Q (resp. Q').
The path (p, Q', q, Q) is a point-quad path of length 4 between p and Q.
(2) p and p' (resp. p' and $p"$) are collinear (resp. non-collinear).
The line pp' is contained in a plane P_0. The intersection ofP_0 and $Q"$ is a line D_1. Since $Q"$ is a generalized quadrangle, there is a point-line path of length 4 between D_1 and $p"$ in $Q"$, namely $(D_1, q, D_2, p")$. The line D_2 is contained in a plane P_1. The intersection of P_1 and Q is a line D_3. The line on p and q is contained in the quad Q_1 whose intersection with P_1 is a line D_4. Since the lines D_3 and D_4 are both contained in the projective plane P_1, they meet in a point q'.
The path (p, Q_1, q', Q) is a point-quad path of length 4 between p and Q.
(3) p and p' (resp. p' and $p"$) are not collinear.
Since Q' is a generalized quadrangle, there is a point-line path of length 5 between p and p' in Q', namely (p, D_1, p_1, D_2, p'). The line D_2 is contained in a plane P_0 whose intersection with $Q"$ is a line D_3. Since $Q"$ is a generalized quadrangle, there exists a point-line path of length 4 between D_3 and $p"$ in $Q"$, namely $(D_3, p_2, D_4, p")$. The line D_4 is contained in a plane P_1 whose intersection with Q is a line D_5. The line on p_1 and p_2 is contained in a quad Q_1. The intersection of Q_1 and P_1 is a line D_6. As the lines D_5 and D_6 are both contained in the projective plane P_1, they meet in a point q. The path (p, Q', p_1, Q_1, q, Q) is a path satisfying the hypotheses of the second case. The second case shows that there is a point-quad path of length 4 between p and Q.

Second part. *Let Γ be a cuboctahedral space. Let p be a point of $G(\Gamma)$ and Q be a quad of $G(\Gamma)$ non incident with p. If there exist two distinct (point-quad)-paths of length 4 between p and Q, namely (p, Q', p', Q) and $(p, Q", p", Q)$, then every quad incident with p is incident with a point q incident with Q.*
Proof. We distinguish two cases.
(1) Either the points p and p' or the points p and $p"$ are collinear.
Without loss of generality, we may assume that p and p' are collinear. Let Q_0 be a quad distinct from Q' and incident with p. The line on p and p' is contained in a plane P_0 whose intersection with Q (resp. Q_0) is a line D_1 (resp. D_2). Since the lines D_1 and D_2 are both included in the projective plane P_0, they meet in a point

q. Then Q_0 is incident with q, which is itself incident with Q.
(2) Neither the points p and p', nor the points p and p" are collinear.
Since Q' is a generalized quadrangle, there is a point-line path of length 5 between p and p' in Q', namely (p, D_1, p_1, D_2, p'). The line D_1 is contained in a plane P_0 whose intersection with Q" is a line D_3. Since Q" is a generalized quadrangle, there is a point-line path of length 4 between D_3 and p" in Q", namely $(D_3, p_2, D_4, p")$.

If p' and p" are collinear, then $\{p', p", p_2, p_1\}$ is a quadrangle and so, by the quadrangle lemma, these four points are contained in a unique quad which is Q. Since $(p, Q", p_2, Q)$ is a point-quad path of length 4 where p and p_2 are collinear, the hypotheses of the first case are satisfied, and so every quad incident with p is incident with a point q which is itself incident with Q.

If p' and p" are not collinear, then the line D_4 is contained in a plane P_1 whose intersection with Q is a line D_5. Since Q is a generalized quadrangle, there is a point-line path of length 4 between D_5 and p' in Q, namely (D_5, p_3, D_6, p). The quadrangle $\{p_3, p', p_1, p_2\}$ is contained in a unique quad which is Q. As (p, Q', p_1, Q) is a point-quad path of length 4 with p and p_1 collinear, the hypotheses of the first case are satisfied, and so every quad incident with p is incident with a point q which is itself incident with Q.

Third part. *Conclusion.*
Proof. The first two parts show that $S(G(\Gamma)$,quad) is a polar space. Moreover, corollary 3.5 shows that $G(\Gamma)$ is a geometry of rank three. Therefore the geometry $G(\Gamma)$ is a building of type C_3 such that $S(G(\Gamma)$,point) is equal to Γ.

■

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