Locally C_n^k graphs

Dominique Buset

Abstract

We completely classify the graphs all of whose neighbourhoods of vertices are isomorphic to C_n^k ($2 \leq k < n$), where C_n^k is the k-th power of the cycle C_n of length n.

1 Introduction

All graphs considered in this paper are undirected, without loops or multiple edges. K_n denotes the complete graph on n vertices, P_n the path of length $n-1$, C_n the cycle of length n and $\sim$ the adjacency relation. If v is a vertex of a graph Γ, we denote by $\Gamma(v)$ the *neighbourhood* of v, that is the subgraph induced by Γ on the set of vertices adjacent to v in Γ. Given a positive integer k and a graph Γ, the *k-th power* Γ^k of Γ is the graph whose vertices are those of Γ, two vertices being adjacent in Γ^k iff their distance in Γ is at most k. Obviously $\Gamma^1 \simeq \Gamma$.

Given a graph Γ', a connected graph Γ is said to be *locally* Γ' if, for every vertex v of Γ, the subgraph $\Gamma(v)$ is isomorphic to Γ'. There is an extensive literature on the classification of all graphs which are locally a given graph (see for example the bibliography at the end). The purpose of this paper is to classify the graphs which are locally C_n^k for $2 \leq k < n$. When $k = 1$, it is already known (Brown and Connelly [5] [6], Hell [13] and Vince [17]) that for any given $n \geq 6$, there are infinitely many non isomorphic graphs which are locally C_n and that the only locally C_3, C_4 and C_5 graphs are respectively the 1-skeletons of the tetrahedron, octahedron and icosahedron.

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Our main result is the following :

Theorem *Let k and n be integers such that $2 \leq k < n$ and let Γ be a locally C_n^k graph.*

(i) If $k+1 \leq n \leq 2k+1$, then $\Gamma \simeq K_{n+1}$.

(ii) If $n = 2k+2$, then Γ is isomorphic to the complete $(k+2)$ - partite graph $K_{2,\ldots,2}$.

(iii) If $n \geq 2k+3$, there is no locally C_n^k graph.

Topp and Volkmann [15] have already considered the particular case where $k = 2$, and so we may assume $k \geq 3$ in our proof.

If $a_0, \ldots, a_{n-1}$ are the vertices and if $[a_i, a_{i+1}]$ are the edges of C_n ($i = 0, \ldots, n-1$; the indices being computed modulo n), we shall say that $a_0 \sim a_1 \sim \ldots \sim a_{n-1} \sim a_0$ is a *basic cycle* of C_n^k.

2 Lemmas

The following properties will be used to establish the theorem. The proofs of the first three lemmas are straightforward and will be omitted.

Lemma 1. If $n \geq 2k+1$, C_n^k is a regular graph of degree $2k$.

Lemma 2. If $n \geq 3k+1$ and $k \geq 2$, the neighbourhood of any vertex of C_n^k is isomorphic to P_{2k}^{k-1}.

Lemma 3. If $k \geq 2$, P_{2k}^{k-1} has exactly two vertices of degree $k+j-1$ for every $j \in \{0, \ldots, k-1\}$.

If v and w are two adjacent vertices of a graph Γ, we shall denote by N_w^v the set of all common neighbours of v and w in Γ, by A_w^v the set of all vertices of Γ (distinct from w) adjacent to v but not to w, and by M_w^v the set of all vertices of N_w^v adjacent to every vertex of A_w^v. Obviously $N_w^v = N_v^w$ and $A_w^v \cap A_v^w$ is empty.

Lemma 4. If $2k+2 \leq n \leq 3k+1$ and if v and w are two adjacent vertices of a graph Γ which is locally C_n^k, then

(i) $|A_w^v| = n - 2k - 1$ and the subgraph induced by Γ on A_w^v is isomorphic to K_{n-2k-1}.

(ii) $|M_w^v| = 2(3k+2-n)$ and $M_w^v = M_v^w$.

Proof. By Lemma 1, $|N_w^v| = 2k$. Since $|\Gamma(v)| = n$, it follows that $|A_w^v| = n - 2k - 1$, and so $1 \leq |A_w^v| \leq k$ because $2k+2 \leq n \leq 3k+1$. Therefore $\Gamma(v) \simeq C_n^k$ induces on A_w^v a subgraph isomorphic to K_{n-2k-1}; moreover $|M_w^v| = 2(k+1-|A_w^v|) = 2(3k+2-n)$. By applying similar arguments to $\Gamma(w) \simeq C_n^k$, we get $|M_v^w| = 2(3k+2-n)$. Thus M_w^v and M_v^w have the same cardinality and, in order to prove that $M_w^v = M_v^w$, it suffices to show that $M_w^v \subset M_v^w$.

Let x be any vertex of M_w^v. In the subgraph $\Gamma(v)$, x is adjacent to w and to the $n - 2k - 1$ vertices of A_w^v. Therefore, by Lemma 1, x must be adjacent to exactly $2k - 1 - |A_w^v|$ vertices of N_w^v.

On the other hand, since $N_w^v = N_v^w$, x is also a vertex of N_v^w. Suppose that $x \notin M_v^w$. Then x is not adjacent to all the vertices of A_v^w. Therefore, by Lemma 1, the number of neighbours of x in N_v^w is less than $2k - 1 - |A_v^w|$. Since $|A_v^w| = |A_w^v|$

and $N_v^w = N_w^v$, this contradicts the conclusion of the preceding paragraph. It follows that $x \in M_v^w$, and so $M_w^v \subset M_v^w$.

3 Proof of the theorem

Let v be any vertex of a graph Γ which is locally C_n^k. Since the case $k = 2$ has already been examined in [15], we may assume $3 \leq k < n$. It is no restriction of generality to denote by $v_0, \ldots, v_{n-1}$ the vertices of $\Gamma(v)$, the edges of $\Gamma(v)$ being those of a graph C_n^k constructed over the basic cycle $v_0 \sim v_1 \sim \ldots \sim v_{n-1} \sim v_0$.

1) If $k+1 \leq n \leq 2k+1$, then $C_n^k \simeq K_n$, and so obviously $\Gamma \simeq K_{n+1}$.

2) If $n = 2k+2$, then C_n^k is isomorphic to the complete $(k+1)$-partite graph $K_{2,\ldots,2}$ and it is very easy to conclude that Γ is necessarily the complete $(k+2)$-partite graph $K_{2,\ldots,2,2}$ (see for example Brouwer, Cohen and Neumaier [4], Proposition 1.1.5).

3) If $n \geq 2k+3$, $\Gamma(v_0)$ contains $v, v_1, \ldots, v_k, v_{n-k}, \ldots, v_{n-1}$ and no other vertex of $\Gamma(v)$, and so vo must be adjacent to $n-2k-1 \geq 2$ new vertices $v_n, v_{n+1}, \ldots, v_{2n-2k-2}$ which form the set $A_v^{v_0}$. It is no restriction of generality to assume that the path $v_n \sim v_{n+1} \sim \ldots \sim v_{2n-2k-2}$ is a subgraph of a basic cycle $B(v_0)$ of $\Gamma(v_0) \simeq C_n^k$, and that v_{n+j} and $v_{2n-2k-2-j}$ are at distance $k+1+j$ from v in $B(v_0)$ $(0 \leq j < \frac{1}{2}(n-2k-1))$.

Let $n = 2k+1+i$, where $i \geq 2$.

Case I : $i \leq k-1$.

Note first that $2k+3 \leq n \leq 3k$, so that Lemma 4 can be applied.

Each of the sets $A_{v_n}^{v_0}$ and $A_{v_{n+1}}^{v_0}$ is of cardinality $n-2k-1 = i$. Since v_n and v_{n+1} are adjacent on the basic cycle $B(v_0)$, the set $A_{v_n}^{v_0} \cup A_{v_{n+1}}^{v_0}$ consists of $i+1$ consecutive vertices of $B(v_0)$. But $i+1 \leq k$, and so there is at least one vertex $w \in N_{v_n}^{v_0} \cap N_{v_{n+1}}^{v_0}$ which is adjacent to the $i+1$ vertices of $A_{v_n}^{v_0} \cup A_{v_{n+1}}^{v_0}$. In other words, $w \in M_{v_n}^{v_0} \cap M_{v_{n+1}}^{v_0}$. By Lemma 4 (ii), it follows that $w \in M_{v_0}^{v_n} \cap M_{v_0}^{v_{n+1}}$, which means that w is adjacent to the i vertices of $A_{v_0}^{v_n}$ and to the i vertices of $A_{v_0}^{v_{n+1}}$. On the other hand, the only vertices of Γ adjacent to w are v_0, the $2k$ vertices of $N_{v_0}^w$ and the i vertices of $A_{v_0}^w$ (by definition of $N_{v_0}^w$ and $A_{v_0}^w$). Since the vertices of $A_{v_0}^{v_n}$ and $A_{v_0}^{v_{n+1}}$ are all non adjacent to v_0 and since $|A_{v_0}^w| = |A_{v_0}^{v_n}| = |A_{v_0}^{v_{n+1}}| = i$, we deduce that $A_{v_0}^w = A_{v_0}^{v_n} = A_{v_0}^{v_{n+1}}$.

The vertices of $N_{v_{n+1}}^{v_n}$ are either adjacent to v_0 (there are exactly $2k-2$ such vertices in $\Gamma(v_0)$) or non adjacent to v_0 (there are exactly i such vertices because $A_{v_0}^{v_n} \cap A_{v_0}^{v_{n+1}} = A_{v_0}^w$ has cardinality i). Therefore $|N_{v_{n+1}}^{v_n}| = 2k-2+i$. If $i > 2$, this contradicts Lemma 1.

If $i = 2$, then $n = 2k+3$, $A_{v_0}^v = \{v_{k+1}, v_{k+2}\}$, $A_v^{v_0} = \{v_n, v_{n+1}\}$ and $M_{v_0}^v = \{v_2, \ldots, v_k, v_{k+3}, \ldots, v_{2k+1}\}$. Note that $w \sim v$ since w is adjacent to v_0, v_n and v_{n+1}. Thus the $2k+3$ vertices of $\Gamma(w)$ are the $2k$ vertices of N_w^v together with v, v_n and v_{n+1}. On the other hand, as we have seen before, w is adjacent to the two vertices of $A_{v_0}^{v_n}$ and to the two vertices of $A_{v_0}^{v_{n+1}}$. Therefore $A_{v_0}^{v_n} \cup A_{v_0}^{v_{n+1}} \subset N_w^v - \Gamma(v_0)$, and so necessarily $A_{v_0}^{v_n} = A_{v_0}^{v_{n+1}} = A_{v_0}^v = \{v_{k+1}, v_{k+2}\}$. It follows that $\Gamma(v_n)$ contains the vertices $v_0, v_{n+1}, v_{k+1}, v_{k+2}$, the vertices $v_2, \ldots, v_k, v_{k+3}, \ldots, v_{2k+1}$ of $M_v^{v_0} = M_{v_0}^v$ and one vertex of $\{v_1, v_{2k+2}\}$, which means that $M_{v_0}^{v_n}$ must contain the $2k-2$ vertices of $M_v^{v_0}$ and v_{n+1}, i.e. at least $2k-1$ vertices, contradicting Lemma 4 (ii) since $2k-i > 2k-2$.

Case II: $i \geq k$

Since $n \geq 3k+1$, Lemma 2 shows that the subgraph induced by Γ on the set $N_v^{v_0} = \{v_1, \ldots, v_k, v_{n-k}, \ldots, v_{n-1}\}$ is isomorphic to P_{2k}^{k-1}. Thus, using Lemma 3, it is no restriction of generality to assume that $v \sim v_1 \sim \ldots \sim v_k \sim v_n \sim \ldots \sim v_{2n-2k-2} \sim v_{n-k} \sim \ldots \sim v_{n-1} \sim v$ is a basic cycle of $\Gamma(v_0) \simeq C_n^k$. Therefore $N_{v_{k-1}}^{v_k}$ contains the vertex v, $2k-2$ vertices of $\Gamma(v) \simeq C_n^k$ and $k-1$ vertices of $A_v^{v_0}$ (namely $v_n, \ldots, v_{n+k-2}$). It follows that $|N_{v_{k-1}}^{v_k}| \geq 3k-2 > 2k$ (because $k \geq 3$), which contradicts Lemma 1 and finishes the proof of our theorem.

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