On the embedding of a finite group as Frattini subgroup

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Introduction

First of all, the groups featuring in this paper are finite, no one excepted. We remind that $\Phi(G)$ stands for the *Frattini subgroup* of the group G, i.e. $\Phi(G)$ is the intersection of all the maximal subgroups of G.

Consider the following problem.

Problem 1.*Given a group G. Does there exist a group H containing a normal subgroup $\bar{G}$ isomorphic to G with $\bar{G} \leq \Phi(H)$?*

A remarkable step forward concerning Problem 1 was made by R.B.J.T. Allenby ([**1**]) who showed that if Problem 1 admits an affirmative answer for G, then there exists also a group K with $G \cong \Phi(K)$. Therefore it is maybe of an advantage that one focusses the attention to the eventually equivalent Problem 1'.

Problem 1'. *Given a group G. Does there exist a group H with $G \cong \Phi(H)$?*

Due to Allenby's result the following notation-definition seems appropriate.

$G \in \Phi \Leftrightarrow$ there exists a group U for which $G \cong \Phi(U)$;

$G \notin \Phi \Leftrightarrow$ there does not exist a group U satisfying $G \cong \Phi(U)$.

Problem 1 has a long history; among others the papers [**4**] and [**8**] deal with it. In the last paper it was finally established which of the groups P of order p^4, p any prime, satisfy $P \in \Phi$ and which of them do satisfy $P \notin \Phi$.

This paper is written in the same spirit as in [**8**]. Up to the groups numbered 16 and 40 in Hall's list of all the groups of order 32 (see [**2**] or equivalently [**7**]), we

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give a conclusive answer to the question as posed in Problem 1, for each one of the groups of order 32. In order to do so some construction principles around 2-groups had to be invented; see for instance Theorem 1.4 and Corollary 1.5.

Each of the exceptional groups $32/16$ and $32/40$ mentioned above satisfies the following two statements:

(1) it does not occur as isomorphy type of a Frattini subgroup of some 2-group; and

(2) its inner automorphism group is contained in the Frattini subgroup of its full automorphism group.

Due to this result it is quite clear that at least one of the following folklore conjectures must be *false* in general.

Conjecture 2. *A p-group P occurs as Frattini subgroup of some group if and only if P occurs as Frattini subgroup of some p-group.*

Conjecture 3. *The converse to Gaschütz's theorem* ([**5**],III.3.13) *holds, i.e. if* $\mathrm{Inn}(N) \leq \Phi(\mathrm{Aut}(N))$ *for some group N, then there exists a group G with* $N \cong \Phi(G)$.

A word on notations *et all.* is in order. It will be standard as in [**5**] or otherwise self-explanatory. A symbol like 32/17 stands for the group with number 17 in Hall's list of all the groups of order 32; see [**2**] and [**7**]. Results extracted from the lists [**2**] and [**7**] will be used or quoted freely. The symbol $N \trianglelefteq G$ means that N is a normal subgroup of the group G.

1 Groups of order 32 as Frattini subgroups.

There exist precisely fifty-one isomorphism types of groups of order 32, see [**2**] and [**7**]. For each representative of them, it will be studied whether it will occur as Frattini subgroup of some group or not. We start with the following easy observation.

Theorem 1.1 *Each abelian group A of prime power order satisfies $A \in \Phi$.*

Proof Suppose A is abelian of order p^a, p prime, $a \geq 1$. Then
$A \cong C_{p^{i_1}} \times \ldots \times C_{p^{i_n}}$ for suitable $i_j \geq 1$. By ([**5**],III.3.14.a)) it holds that $\Phi(A) = \{g^p \mid g \in A\}$. Thus from
$\Phi(C_{p^{i_1+1}} \times \ldots \times C_{p^{i_n+1}}) \cong C_{p^{i_1}} \times \ldots \times C_{p^{i_n}}$ the Theorem immediately follows. □

By Theorem 1.1 each of the abelian groups of order 32 occurs as Frattini subgroup of some group. The abelian groups are numbered $32/1, \ldots, 32/7$; see [**2**].

Our next theorem has to do with direct products of groups.

Theorem 1.2 *Suppose $G \in \Phi$. Then $G \times C_{p^a} \in \Phi$ for any prime p and nonnegative integer a.*

Proof. Suppose $G \cong \Phi(K)$ for some group K. Then
$\Phi(K \times C_{p^{a+1}}) \cong \Phi(K) \times \Phi(C_{p^{a+1}}) \cong G \times C_{p^a}$ proves the assertion. □

Corollary 1.3 $32/8 \in \Phi$ *and* $32/9 \in \Phi$.

Proof It is mentioned in [**3**] and repeated in [**8**] that $D_4 \times C_2 \in \Phi$ and that $Q_4 \times C_2 \in \Phi$; here D_4 and Q_4 stand for the dihedral group and quaternion group respectively, each of order eight. Since $32/8 \cong D_4 \times C_2 \times C_2$ and $32/9 \cong Q_4 \times C_2 \times C_2$, the corollary follows from Theorem 1.2. $\square$

It may happen that $G \notin \Phi$ whereas $G \times C_{2^n} \in \Phi$ for a suitable $n \geq 1$ ($G = D_4$ for instance, with $n = 1$). This phenomenon can be elucidated by means of Theorem 1.4 and its Corollary 1.5.

Theorem 1.4 *Let* $K = G \wr C_t$ *be the regular wreath product of a 2-group* G *with a cyclic group* $C_t = \langle \delta \rangle$ *of order* $t \geq 2$. *Put* $B = GG^{\delta}G^{\delta^2} \ldots G^{\delta^{t-1}} < K$, *where* $[G^{\delta^i}, G^{\delta^j}] = 1$ *whenever* $0 \leq i < j \leq t-1$. *Now assume* t *is a power of 2. Put* $X = \langle (gg^{\delta} \ldots g^{\delta^{t-2}} g^{\delta^{t-1}}) \mid g \in G \rangle$. *Then* $XB' \in \Phi$.

Proof Let $\alpha \in K$. Then $(gg^{\delta} \ldots g^{\delta^{t-2}} g^{\delta^{t-1}})^{\alpha} \in (gg^{\delta} \ldots g^{\delta^{t-2}} g^{\delta^{t-1}})B' \leq XB'$. So XB' is a normal subgroup of K; note that XB' is already a normal subgroup of B. Furthermore, as t is a power of 2 at least equal to 2, K turns out to be a 2-group satisfying $gg^{\delta} \ldots g^{\delta^{t-2}} g^{\delta^{t-1}} \in K' \langle k^2 \mid k \in K \rangle = \Phi(K)$ for $g \in G$. Hence $XB' \in \Phi$. $\square$

Corollary 1.5 *Adopt the hypotheses of Theorem 1.4. Let* $t = 2$. *Then* $X \cong G$ *and* $XB' = XG'$ *with* $X \cap G' = \{1\}$. *In particular, if in addition* G *is nilpotent of class 2, it follows that* $XG' \cong G \times G'$ *whence that* $G \times G' \in \Phi$. $\square$

As a consequence each of

$$\begin{aligned}
&32/10 : \langle a, b, c \mid a^4 = b^2 = c^2 = 1,\ a^b = a^c = a,\ b^c = a^2 b \rangle \times C_2 \\
&32/11 : \langle a, b, c \mid a^4 = b^2 = c^2 = 1,\ b^a = b^c = b,\ a^c = a^3 b \rangle \times C_2 \\
&32/12 : \langle a, b \mid a^4 = b^4 = 1,\ a^b = a^{-1} \rangle \times C_2 \\
&32/13 : \langle a, b \mid a^8 = b^2 = 1,\ a^b = a^5 \rangle \times C_2
\end{aligned}$$

is embeddable as Frattini subgroup in a group, since

$$\begin{aligned}
&32/10 \cong (16/8) \times C_2,\ (16/8)' \cong C_2, \\
&32/11 \cong (16/9) \times C_2,\ (16/9)' \cong C_2, \\
&32/12 \cong (16/10) \times C_2,\ (16/10)' \cong C_2, \\
&32/13 \cong (16/11) \times C_2,\ (16/11)' \cong C_2,
\end{aligned}$$

We proceed with the following"positive" result.

Theorem 1.6 $32/i \in \Phi$ *for* $i \in \{19, 34, 35, 39\}$.

Proof The proof to Theorem 1.6 is indirect. We were told that each of the groups

$$\begin{aligned}
&32/19 : \langle a, b \mid a^8 = b^4 = 1,\ a^b = a^5 \rangle \\
&32/34 : \langle a, b, c \mid a^4 = b^4 = c^2 = 1,\ a^b = a,\ a^c = a^{-1},\ b^c = b^{-1} \rangle \\
&32/35 : \langle a, b, c \mid a^4 = b^4 = 1,\ a^2 = c^2,\ a^b = a,\ a^c = a^{-1},\ b^c = b^{-1} \rangle \\
&32/39 : \langle a, b, c \mid a^4 = b^4 = c^2 = 1,\ a^b = a,\ a^c = a^{-1},\ b^c = a^2 b^{-1} \rangle
\end{aligned}$$

does occur as Frattini subgroup of a 2-group; see the Acknowledgement at the end of this paper. $\square$

Now consider the following general observation.

Theorem 1.7 *Let H be a characteristic subgroup of G. Suppose $G \in \Phi$. Then $H \in \Phi$ and $G/H \in \Phi$.*

Proof Suppose there exists a group T with $G = \Phi(T)$. Then, as H is characteristic in G, and as $G \trianglelefteq T$, it holds that $H \trianglelefteq T$ while $H \leq \Phi(T)$. By [**1**] it thus follows that $H \in \Phi$. Also, as $H \leq \Phi(T)$, we see by applying ([**5**],III.3.4.b)) that $\Phi(T/H) = \Phi(T)/H = G/H$. So $G/H \in \Phi$. □

As a consequence of the part "$G \in \Phi \Rightarrow H \in \Phi$" of Theorem 1.7 we have that none of the groups $32/14, 32/17, 32/36, 32/37, 32/38$ belongs to Φ. Namely, focus the attention on

$H = \langle a, bc\rangle \cong 16/10$
for $32/14 : \langle a, b \mid a^4 = b^2 = 1,\ a^b = a^{-1}\rangle \times \langle c \mid c^4 = 1\rangle$,

$H = \langle c, bd\rangle \cong 16/18$
for $32/17 : \langle a, b, c \mid a^8 = b^2 = c^2 = 1,\ a^b = a^c = a,\ b^c = a^4 b\rangle$,

$H = \langle c, bd\rangle \cong 16/10$
for $32/36 : \langle < a > \times < b > \times < c >, d \mid a^2 = b^2 = c^4 = d^2 = 1,$
$b^d = ab,\ c^d = c^{-1},\ a^d = a\rangle$,

$H = \langle c, bd\rangle \cong 16/10$
for $32/37 : \langle < a > \times < b > \times < c >, d \mid a^2 = b^2 = c^4 = 1,$
$c^2 = d^2,\ b^d = ab,\ c^d = c^{-1}, a^d = a\rangle$,

$H = \langle c, ad\rangle \cong 16/9$
for $32/38 : \langle < a > \times < b > \times < c >, d \mid a^4 = b^2 = c^2 = d^2 = 1,$
$a^d = ab,\ c^d = a^2 c,\ b^d = b\rangle$.

By the part "$G \in \Phi \Rightarrow G/H \in \Phi$" of Theorem 1.7 we have that none of the groups $32/15, 32/18, 32/20, 32/21$ belongs to Φ. Namely, apply

$H = \langle a^2 c^2\rangle$ for $32/15 : \langle a, b \mid a^4 = 1,\ a^2 = b^2,\ a^b = a^{-1}\rangle \times \langle c \mid c^4 = 1\rangle$ with $(32/15)/H \cong 16/8$;
$H = \langle ab^2, ac^2\rangle$ for $32/18 : \langle a, b, c \mid a^2 = b^4 = c^4,\ a^b = a^c = a,\ b^c = ab\rangle$ with $(32/18)/H \cong 8/5$;
$H = \langle b^4\rangle$ for $32/20 : \langle a, b, c \mid a^2 = b^8 = c^2 = 1,\ a^b = a^c = a,\ b^c = ab\rangle$ with $(32/20)/H \cong 16/9$; and finally
$H = \langle b^4\rangle$ for $32/21 : \langle a, b \mid a^4 = b^8 = 1,\ a^b = a^{-1}\rangle$ with $(32/21)/H \cong 16/10$.

In order to obtain more results of a "negative" nature, we mention

Theorem 1.8 *Let P be a p-group of order p^n of nilpotency class bigger than $\frac{1}{2}n$. Then $P \notin \Phi$.*

Proof By [**1**] the statement of the theorem is equivalent to a result obtained by Hill and Parker; see ([**3**], Theorem 1). □

The nilpotency class of the eighteen groups $32/23, 32/24, \ldots, 32/32,$ $32/44, 32/45, \ldots, 32/51$ is at least three; see [**2**]. So Theorem 1.8 reveals that none of these groups can occur as Frattini subgroup of a group.

It was shown in [**6**] that the following is true.

Theorem 1.9 (paraphrased; see ([**6**], Theorem 1.1)). *A non-abelian p-group with cyclic center can be embedded in a group as its Frattini subgroup if and only if it is either an extraspecial 2-group of order at least 128 or the central product of a cyclic group C of order at least 4 and an extraspecial group E of order at least 32 with $\Omega_1(C)$ and $Z(E)$ amalgamated.* □

Corollary 1.10 $32/22 \notin \Phi$, $32/42 \notin \Phi$, $32/43 \notin \Phi$.

Proof Each of the groups $32/22, 32/42$ and $32/43$ has a cyclic center; now apply Theorem 1.9. □

We now have almost reached the end of our journey, but some tough obstacles are on our way yet. First, we recall Gaschütz's theorem.

Theorem 1.11 ([**5**],III.3.13). *Let G be a group and $N \trianglelefteq G$. Suppose $N \leq \Phi(G)$. Then the inner automorphism group of N is contained in the Frattini subgroup of the full automorphism group of N.* □

We will omit the proof of fact that $32/33 \notin \Phi$. Namely, $\mathrm{Inn}\,(32/33) \not\leq \Phi(\mathrm{Aut}(32/33))$ holds, so that Theorem 1.11 yields indeed $32/33 \notin \Phi$. We leave the (rather technical) proof of $\mathrm{Inn}\,(32/33) \not\leq \Phi(\mathrm{Aut}(32/33))$ to the reader.

The following theorem will help us to provide a comparatively easy proof of the fact that $32/41 \notin \Phi$.

Theorem 1.12 *Suppose P is a normal Sylow p-subgroup of G. Then $\Phi(P) = \Phi(G) \cap P$; or otherwise said, $\Phi(P)$ is the (unique and normal) Sylow p-subgroup of $\Phi(G)$.*

Proof See ([**8**], Lemma 5). □

Corollary 1.13 $32/41 \notin \Phi$

Proof Put $G = 32/41$. It holds by inspection that $\#\mathrm{Aut}(G) = 192$, that $\mathrm{Aut}(G)$ contains an elementary abelian normal subgroup T of order 64 where each $t \in T$ operates on G in such a way that $g^t\Phi(\mathrm{Aut}(G)) = g\Phi(\mathrm{Aut}(G))$ whenever $g \in G$. Hence by Theorem 1.12, $\Phi(T) = \Phi(\mathrm{Aut}(G)) \cap T$. Now $\Phi(T) = \langle t^2 \mid t \in T \rangle$; by ([**5**],II.3.14.b)). Thus $\Phi(T) = \{1\}$. Therefore $\#\Phi(\mathrm{Aut}(G)) \leq \frac{192}{64} = 3$. However, $\#\mathrm{Inn}(G) = \#G/Z(G) = 8$. Hence $\mathrm{Inn}(G)$ is not contained in $\Phi(\mathrm{Aut}(G))$. Therefore, $32/41 \notin \Phi$ by Theorem 1.11 and [**1**]. □

Thus "only" the groups $32/16$ and $32/40$ are left to investigate. For $P \in \{32/16,\ 32/40\}$ it can be verified by inspection that $\mathrm{Inn}(P)$ is contained in $\Phi(\mathrm{Aut}(P))$.

On the other hand we will next prove the Theorems 1.14 and 1.15. Despite several strenuous efforts over long a period of time to improve those results, we were not able to show that $P \notin \Phi$, nor to disprove that.

Theorem 1.14 *Let P be isomorphic to the group $32/16$. Then there does not exist a 2-group G satisfying $P = \Phi(G)$.*

Proof Define $P = \langle a, b, c \mid a^4 = b^4 = c^2 = 1,\ a^b = a,\ a^c = a,\ b^c = a^2 b\rangle$. We have $P' = \langle a^2\rangle$, $\Phi(P) = \langle a^2, b^2\rangle$ and $Z(P) = \langle a, b^2\rangle$. Suppose on the contrary that a finite 2-group G exists satisfying $P = \Phi(G)$. Consider the fact that $G/C_G(P)$ can be regarded as subgroup of $\mathrm{Aut}(P)$. From [**2**] we borrow that $|J| = 2^6$, where $J := \{\alpha \in \mathrm{Aut}(P) \mid g^\alpha \Phi(P) = g\Phi(P) \text{ for all } g \in P\}$. It is easily seen that J consists precisely of the 2^6 maps t

$$\begin{cases} a \mapsto a(b^2)^\delta (a^2)^\varepsilon \\ b \mapsto b(b^2)^\rho (a^2)^\psi \\ c \mapsto c(b^2)^\sigma (a^2)^\tau \end{cases} \quad \text{with} \quad \delta, \varepsilon, \rho, \psi, \sigma, \tau \in \{0, 1\},$$

whence that J is an elementary abelian normal subgroup of $\mathrm{Aut}(P)$. From [**2**] we also extract that $|\mathrm{Aut}(P)/J| = 4$. Observe that the set $\{1, \xi_1, \xi_2, \xi_3\}$ provides a full set of representatives of the cosets of J in $\mathrm{Aut}(P)$, where

$$\xi_1 : \begin{cases} a \to a \\ b \to ba \\ c \to c \end{cases}; \qquad \xi_2 : \begin{cases} a \to a \\ b \to bc \\ c \to cb^2 \end{cases}; \qquad \xi_3 : \begin{cases} a \to a \\ b \to bac \\ c \to cb^2 \end{cases}.$$

Consider $(\xi_i t)^2$ with $i \in \{1, 2, 3\}$. Now $P = \Phi(G) = \langle g^2 \mid g \in G\rangle$. Therefore, under the assumption that $\xi_i t$ corresponds to an element of $G \setminus C_G(P)$ for $i \in \{1, 2, 3\}$ it follows that

$$(\xi_1 t)^2 : \begin{cases} a \to a \\ b \to ba^2(a^2)^{\varepsilon+\rho} \\ c \to c(a^2)^\sigma \end{cases}; \qquad (\xi_2 t)^2 : \begin{cases} a \to a \\ b \to b(a^2)^{\rho+\tau} \\ c \to c \end{cases};$$

$$(\xi_3 t)^2 : \begin{cases} a \to a \\ b \to ba^2(a^2)^{\varepsilon+\tau} \\ c \to c \end{cases}.$$

As to the action of $(\xi_2 t)^2$ and $(\xi_3 t)^2$ it is used that $\{b, ba^2\}$ constitute a full class of conjugates of P, whereas for $(\xi_1 t)^2$ and $(\xi_2 t)^2$ it is used that $a \in Z(P)$. Thus we observe that $c^{(\xi_i t)^2} = c$ when $i \in \{2, 3\}$. Regard $\xi_1 t$ as an element of $G \setminus C_G(P)$. Thus $(\xi_1 t)^2 = b^i c^j z$ for suitable $i, j \in \mathbb{Z}$ and $z \in Z(P)$. Hence, as $a \in Z(P)$, $b^i c^j z = (\xi_1 t)^2 = ((\xi_1 t)^2)^{(\xi_1 t)} = (b^i c^j z)^{(\xi_1 t)} = (b^i a^i c^j z^{\xi_1})^t \in b^i a^i c^j z^{\xi_1 t} \langle a^2, b^2\rangle$. Since $z^{\xi_1 t} \in z\langle a^2, b^2\rangle$ (remember $Z(P) = \langle a, b^2\rangle$ en $\Phi(P) = \langle a^2, b^2\rangle$ are characteristic subgroups of P), we conclude that $a^i \in \langle a^2, b^2\rangle$, that is, i is even. Thus $(\xi_1 t)^2 = c^r u$ for suitable $r \in \mathbb{Z}$ and $u \in Z(P)$, i.e. $c^{(\xi_1 t)^2} = c$. Therefore, $c^{g^2} = c$ for all $g \in G$. As $c^b = ca^2$, we have found a contradiction to the assumption $P = \Phi(G) = \langle g^2 \mid g \in G\rangle$.

The proof of the theorem is complete. $\square$