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Some recurrence relations for the generalized basic hypergeometric functions *

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Abstract

In the present paper, we express the generalized basic hypergeometric function ${}_r\Phi_s(\cdot)$ (for $r = s+1$) in terms of an iterated q-integrals involving the basic analogue of the Gauss's hypergeometric function. Further, using the relations between q-contiguous hypergeometric series, we obtain some recurrence relations for the generalized basic hypergeometric functions of one variable.

1 Introduction

The generalized basic hypergeometric series cf. Gasper and Rahman [4] is given by

$$
{}_r\Phi_s\left(a_1,\cdots,a_r;b_1,\cdots,b_s;\ q,x\right) = {}_r\Phi_s\left[\begin{array}{ll} a_1,\cdots,a_r & ; \\ & q,x \\ b_1,\cdots,b_s & ; \end{array}\right]
$$

$$
= {}_r\Phi_s\left[(a_r);(b_s);\ q,x\right] = \sum_{n=0}^{\infty}\frac{(a_1,\cdots,a_r;q)_n}{(q,b_1,\cdots,b_s;q)_n}\,x^n\,\left\{(-1)^n q^{n(n-1)/2}\right\}^{(1+s-r)}, \tag{1.1}
$$

where for real or complex a

$$
(a;q)_n = \begin{cases} 1 & ;\text{if}\quad n=0 \\ (1-a)(1-aq)\cdots(1-aq^{n-1}) & ;\text{if}\quad n\in N, \end{cases} \tag{1.2}
$$

is the q-shifted factorial, r and s are positive integers, and variable x, the numerator parameters $a_1,\cdots,a_r$, and the denominator parameters $b_1,\cdots,b_s$ being any complex quantities provided that

$$
b_j \neq q^{-m}, m = 0,1,\cdots; j = 1,2,\cdots,s.
$$

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If $|q| < 1$, the series (1.1) converges absolutely for all x if $r \leq s$ and for $|x| < 1$ if $r = s+1$. This series also converges absolutely if $|q| > 1$ and $|x| < |b_1 b_2 \cdots b_s|/|a_1 a_2 \cdots a_r|$.

Further, in terms of the q-gamma function, (1.2) can be expressed as

$$(a;q)_n = \frac{\Gamma_q(a+n)(1-q)^n}{\Gamma_q(a)}, \qquad n > 0, \tag{1.3}$$

where the q-gamma function (cf. Gasper and Rahman [4]) is given by

$$\Gamma_q(a) = \frac{(q;q)_\infty}{(q^a;q)_\infty (1-q)^{a-1}} = \frac{(q;q)_{a-1}}{(1-q)^{a-1}}, \tag{1.4}$$

where $a \neq 0, -1, -2, \cdots$.

The theory of basic hypergeometric functions of one and more variables has a wide range of applications in various fields of Mathematical, Physical and Engineering Sciences, namely-Number theory, Partition theory, Combinatorial analysis, Lie theory, Fractional calculus, Integral transforms, Quantum theory etc. (see [1,2,4,5])

In the present work, we express the generalized basic hypergeometric function ${}_{s+1}\Phi_s(\cdot)$ in terms of an iterated q-integrals involving the q-Gauss hypergeometric function ${}_2\Phi_1(\cdot)$. Using q-contiguous relations for ${}_2\Phi_1(\cdot)$, we obtain some recurrence relations for the generalized basic hypergeometric functions of one variable. The above mentioned technique is a q-version of the technique used by Galué and Kalla [3].

2 Integral representation

In this section, we express the generalized basic hypergeometric function ${}_r\Phi_s(\cdot)$ (for $r = s+1$) in terms of an iterated integral involving the basic analogue of Gauss hypergeometric function.

Theorem: *Let $Re(b_{s-i}) > 0$, for all $i = 0, 1, \cdots, s-2$ and $|q| < 1$, then the iterated q-integral representaion of ${}_{s+1}\Phi_s(\cdot)$ is given by*

$${}_{s+1}\Phi_s\left[\begin{array}{ll} q^\alpha, q^\beta, a_3, a_4, \cdots, a_s, a_{s+1} & ; \\ & q, x \\ q^\gamma, b_2, b_3, \cdots, b_s & ; \end{array}\right] = \prod_{i=0}^{s-2} \Gamma_q \left[\begin{array}{c} b_{s-i} \\ a_{s+1-i}, b_{s-i} - a_{s+1-i} \end{array}\right]$$

$$\times \underbrace{\int_0^1 \cdots \int_0^1}_{(s-1) times} \prod_{i=0}^{s-2} t_{i+1}^{a_{s+1-i}-1} (t_{i+1}q;q)_{b_{s-i}-a_{s+1-i}-1}$$

$$\times {}_2\Phi_1\left[\begin{matrix} q^{\alpha}, q^{\beta} & ; & \\ & & q, t_{s-1}\cdots t_2 t_1 x \\ q^{\gamma} & ; & \end{matrix}\right] d_q t_{s-1}\cdots d_q t_2 d_q t_1, \quad (2.1)$$

where $|x| < 1$ *and* $|t_{s-1}\cdots t_2 t_1 x| < 1$.

Proof: To prove the theorem, we consider the well-known q-integral representation of ${}_r\Phi_s(\cdot)$, namely

$${}_r\Phi_s\left[\begin{matrix} a_1,\cdots,a_r & ; & \\ & & q, x \\ b_1,\cdots,b_s & ; & \end{matrix}\right] = \Gamma_q\left[\begin{matrix} b_s \\ a_{s+1}, b_s - a_{s+1}\end{matrix}\right]$$

$$\times \int_0^1 t^{a_r-1}(tq;q)_{b_s-a_r-1}\ {}_{r-1}\Phi_{s-1}\left[\begin{matrix} a_1,\cdots,a_{r-1} & ; & \\ & & q, tx \\ b_1,\cdots,b_{s-1} & ; & \end{matrix}\right] d_q t, \quad (2.2)$$

which is the generalization of q-analogue of Euler's integral representation, namely (cf. Gasper and Rahman [4, eqn.(1.11.9), p.19])

$${}_2\Phi_1\left[\begin{matrix} q^{a}, q^{b} & ; & \\ & & q, x \\ q^{c} & ; & \end{matrix}\right] = \frac{\Gamma_q(c)}{\Gamma_q(b)\Gamma_q(c-b)}\int_0^1 t^{b-1}(tq;q)_{c-b-1}$$

$$\times\ {}_1\Phi_0\left[\begin{matrix} q^{a} & ; & \\ & & q, tx \\ - & ; & \end{matrix}\right] d_q t. \quad (2.3)$$

Therefore, relation (2.2) can also be written as

$${}_{s+1}\Phi_s\left[\begin{matrix} q^{\alpha}, q^{\beta}, a_3, a_4,\cdots,a_s, a_{s+1} & ; & \\ & & q, x \\ q^{\gamma}, b_2, b_3,\cdots,b_s & ; & \end{matrix}\right] = \Gamma_q\left[\begin{matrix} b_s \\ a_{s+1}, b_s - a_{s+1}\end{matrix}\right]$$

$$\times\int_0^1 t_1^{a_{s+1}-1}(t_1 q;q)_{b_s-a_{s+1}-1}\ {}_s\Phi_{s-1}\left[\begin{matrix} q^{\alpha}, q^{\beta}, a_3, a_4,\cdots,a_s & ; & \\ & & q, t_1 x \\ q^{\gamma}, b_2, b_3,\cdots,b_{s-1} & ; & \end{matrix}\right] d_q t_1. \quad (2.4)$$

Repeating the process in the right-hand side of (2.4), we get

$${}_{s+1}\Phi_s\left[\begin{matrix} q^{\alpha}, q^{\beta}, a_3, a_4,\cdots,a_s, a_{s+1} & ; & \\ & & q, x \\ q^{\gamma}, b_2, b_3,\cdots,b_s & ; & \end{matrix}\right] = \Gamma_q\left[\begin{matrix} b_s \\ a_{s+1}, b_s - a_{s+1}\end{matrix}\right]$$

$$\times\Gamma_q\left[\begin{matrix} b_{s-1} \\ a_s, b_{s-1} - a_s\end{matrix}\right]\int_0^1\int_0^1 t_2^{a_s-1}(t_2 q;q)_{b_{s-1}-a_s-1}\ t_1^{a_{s+1}-1}(t_1 q;q)_{b_s-a_{s+1}-1}$$

$$\times {}_{s-1}\Phi_{s-2}\left[\begin{matrix} q^{\alpha}, q^{\beta}, a_3, a_4,\cdots,a_{s-1} & ; & \\ & & q, t_2 t_1 x \\ q^{\gamma}, b_2, b_3,\cdots,b_{s-2} & ; & \end{matrix}\right] d_q t_2 d_q t_1. \quad (2.5)$$

Successive operations $(s-3)$ times in the right-hand side of (2.5) leads to the desired result (2.1)

3 Recurrence relations

In this section, as an application of the integral representation for ${}_{s+1}\Phi_s(.)$, given by (2.1), we shall derive certain recurrence relation for the generalized basic hypergeometric series.

Using the relation between q-contiguous basic hypergeometric functions [4, p.22]

$$
{}_2\Phi_1\left[\begin{matrix} q^{\alpha}, q^{\beta} ; \\ q, x \\ q^{\gamma-1} ; \end{matrix}\right] - {}_2\Phi_1\left[\begin{matrix} q^{\alpha}, q^{\beta} ; \\ q, x \\ q^{\gamma} ; \end{matrix}\right] = q^{\gamma}\, x\, \frac{(1-q^{\alpha})(1-q^{\beta})}{(q-q^{\gamma})(1-q^{\gamma})} \times {}_2\Phi_1\left[\begin{matrix} q^{\alpha+1}, q^{\beta+1} ; \\ q, x \\ q^{\gamma+1} ; \end{matrix}\right], \tag{3.1}
$$

we get

$$
{}_2\Phi_1\left[\begin{matrix} q^{\alpha}, q^{\beta} ; \\ q, t_{s-1}\cdots t_2 t_1 x \\ q^{\gamma} ; \end{matrix}\right] = {}_2\Phi_1\left[\begin{matrix} q^{\alpha}, q^{\beta} ; \\ q, t_{s-1}\cdots t_2 t_1 x \\ q^{\gamma-1} ; \end{matrix}\right] - q^{\gamma}\, t_{s-1}\cdots t_2 t_1 x\, \frac{(1-q^{\alpha})(1-q^{\beta})}{(q-q^{\gamma})(1-q^{\gamma})}\, {}_2\Phi_1\left[\begin{matrix} q^{\alpha+1}, q^{\beta+1} ; \\ q, t_{s-1}\cdots t_2 t_1 x \\ q^{\gamma+1} ; \end{matrix}\right]. \tag{3.2}
$$

On substituting value from relation (3.2) in the right-hand side of the result (2.1), we have

$$
{}_{s+1}\Phi_s\left[\begin{matrix} q^{\alpha}, q^{\beta}, a_3, a_4, \cdots, a_s, a_{s+1} ; \\ q, x \\ q^{\gamma}, b_2, b_3, \cdots, b_s ; \end{matrix}\right] = \prod_{i=0}^{s-2} \Gamma_q\left[\begin{matrix} b_{s-i} \\ a_{s+1-i}, b_{s-i}-a_{s+1-i} \end{matrix}\right]
\times \underbrace{\int_0^1 \cdots \int_0^1}_{(s-1)times} \prod_{i=0}^{s-2} t_{i+1}^{a_{s+1-i}-1} (t_{i+1}q;q)_{b_{s-i}-a_{s+1-i}-1}
\times {}_2\Phi_1\left[\begin{matrix} q^{\alpha}, q^{\beta} ; \\ q, t_{s-1}\cdots t_2 t_1 x \\ q^{\gamma-1} ; \end{matrix}\right] d_q t_{s-1}\cdots d_q t_2 d_q t_1
$$

$$
-q^{\gamma}\, x\, \frac{(1-q^{\alpha})(1-q^{\beta})}{(q-q^{\gamma})(1-q^{\gamma})} \prod_{i=0}^{s-2} \Gamma_q\left[\begin{matrix} b_{s-i} \\ a_{s+1-i}, b_{s-i}-a_{s+1-i} \end{matrix}\right]
\times \underbrace{\int_0^1 \cdots \int_0^1}_{(s-1)times} \prod_{i=0}^{s-2} t_{i+1}^{1+a_{s+1-i}-1}\, (t_{i+1}q;q)_{b_{s-i}-a_{s+1-i}-1}
$$

$$\times\, {}_2\Phi_1\left[\begin{array}{ll} q^{\alpha+1}, q^{\beta+1} & ; \\ & q, t_{s-1}\cdots t_2 t_1 x \\ q^{\gamma+1} & ; \end{array}\right] d_q t_{s-1}\cdots d_q t_2 d_q t_1. \quad (3.3)$$

Again, on making use of the result (2.1), the above result (3.3) leads to the following recurrence relation:

$$\begin{aligned} &{}_{s+1}\Phi_s\left[\begin{array}{ll} q^{\alpha}, q^{\beta}, a_3, a_4, \cdots, a_s, a_{s+1} & ; \\ & q, x \\ q^{\gamma}, b_2, b_3, \cdots, b_s & ; \end{array}\right] \\ &= {}_{s+1}\Phi_s\left[\begin{array}{ll} q^{\alpha}, q^{\beta}, a_3, a_4, \cdots, a_s, a_{s+1} & ; \\ & q, x \\ q^{\gamma-1}, b_2, b_3, \cdots, b_s & ; \end{array}\right] - q^{\gamma}\ x\ \frac{(1-q^{\alpha})(1-q^{\beta})}{(q-q^{\gamma})(1-q^{\gamma})} \\ &\times \prod_{i=0}^{s-2}\left[\frac{(1-q^{a_{s+1-i}})}{(1-q^{b_{s-i}})}\right] {}_{s+1}\Phi_s\left[\begin{array}{ll} q^{\alpha+1}, q^{\beta+1}, a_3 q, a_4 q, \cdots, a_s q, a_{s+1} q & ; \\ & q, x \\ q^{\gamma+1}, b_2 q, b_3 q, \cdots, b_s q & ; \end{array}\right], \end{aligned} \quad (3.4)$$

where $Re(b_{s-i}) > 0$, for all $i = 0, 1, \cdots, s-2$ and $|x| < 1$.

Similarly, if we consider the following q-contiguous relations (cf. Gasper and Rahman [4, p.22])

$$\begin{aligned} &{}_2\Phi_1\left[\begin{array}{ll} q^{\alpha+1}, q^{\beta} & ; \\ & q, x \\ q^{\gamma} & ; \end{array}\right] - {}_2\Phi_1\left[\begin{array}{ll} q^{\alpha}, q^{\beta} & ; \\ & q, x \\ q^{\gamma} & ; \end{array}\right] = q^{\alpha}\ x\ \frac{(1-q^{\beta})}{(1-q^{\gamma})} \\ &\qquad \times\, {}_2\Phi_1\left[\begin{array}{ll} q^{\alpha+1}, q^{\beta+1} & ; \\ & q, x \\ q^{\gamma+1} & ; \end{array}\right], \end{aligned} \quad (3.5)$$

$$\begin{aligned} &{}_2\Phi_1\left[\begin{array}{ll} q^{\alpha+1}, q^{\beta} & ; \\ & q, x \\ q^{\gamma+1} & ; \end{array}\right] - {}_2\Phi_1\left[\begin{array}{ll} q^{\alpha}, q^{\beta} & ; \\ & q, x \\ q^{\gamma} & ; \end{array}\right] = q^{\alpha}\ x\ \frac{(1-q^{\gamma-\alpha})(1-q^{\beta})}{(1-q^{\gamma+1})(1-q^{\gamma})} \\ &\qquad \times\, {}_2\Phi_1\left[\begin{array}{ll} q^{\alpha+1}, q^{\beta+1} & ; \\ & q, x \\ q^{\gamma+2} & ; \end{array}\right], \end{aligned} \quad (3.6)$$

and

$$\begin{aligned} &{}_2\Phi_1\left[\begin{array}{ll} q^{\alpha+1}, q^{\beta-1} & ; \\ & q, x \\ q^{\gamma} & ; \end{array}\right] - {}_2\Phi_1\left[\begin{array}{ll} q^{\alpha}, q^{\beta} & ; \\ & q, x \\ q^{\gamma} & ; \end{array}\right] = q^{\alpha}\ x\ \frac{(1-q^{\beta-\alpha+1})}{(1-q^{\gamma})} \\ &\qquad \times\, {}_2\Phi_1\left[\begin{array}{ll} q^{\alpha+1}, q^{\beta} & ; \\ & q, x \\ q^{\gamma+1} & ; \end{array}\right]. \end{aligned} \quad (3.7)$$

and make use of the result (2.1), we obtain the following respective recurrence relations for generalized basic hypergeometric functions, namely

$$
{}_{s+1}\Phi_s\left[\begin{array}{ll} q^{\alpha}, q^{\beta}, a_3, a_4, \cdots, a_s, a_{s+1} & ; \\ & q, x \\ q^{\gamma}, b_2, b_3, \cdots, b_s & ; \end{array}\right]
$$

$$
= {}_{s+1}\Phi_s\left[\begin{array}{ll} q^{\alpha+1}, q^{\beta}, a_3, a_4, \cdots, a_s, a_{s+1} & ; \\ & q, x \\ q^{\gamma}, b_2, b_3, \cdots, b_s & ; \end{array}\right] - q^{\alpha} x \frac{(1-q^{\beta})}{(1-q^{\gamma})}
$$

$$
\times \prod_{i=0}^{s-2}\left[\frac{(1-q^{a_{s+1-i}})}{(1-q^{b_{s-i}})}\right] {}_{s+1}\Phi_s\left[\begin{array}{ll} q^{\alpha+1}, q^{\beta+1}, a_3 q, a_4 q, \cdots, a_s q, a_{s+1} q & ; \\ & q, x \\ q^{\gamma+1}, b_2 q, b_3 q, \cdots, b_s q & ; \end{array}\right], \tag{3.8}
$$

where $Re(b_{s-i}) > 0$, for all $i = 0, 1, \cdots, s-2$ and $|x| < 1$.

$$
{}_{s+1}\Phi_s\left[\begin{array}{ll} q^{\alpha}, q^{\beta}, a_3, a_4, \cdots, a_s, a_{s+1} & ; \\ & q, x \\ q^{\gamma}, b_2, b_3, \cdots, b_s & ; \end{array}\right]
$$

$$
= {}_{s+1}\Phi_s\left[\begin{array}{ll} q^{\alpha+1}, q^{\beta}, a_3, a_4, \cdots, a_s, a_{s+1} & ; \\ & q, x \\ q^{\gamma+1}, b_2, b_3, \cdots, b_s & ; \end{array}\right] - q^{\alpha}\ x\ \frac{(1-q^{\gamma-\alpha})(1-q^{\beta})}{(1-q^{\gamma+1})(1-q^{\gamma})}
$$

$$
\times \prod_{i=0}^{s-2}\left[\frac{(1-q^{a_{s+1-i}})}{(1-q^{b_{s-i}})}\right] {}_{s+1}\Phi_s\left[\begin{array}{ll} q^{\alpha+1}, q^{\beta+1}, a_3 q, a_4 q, \cdots, a_s q, a_{s+1} q & ; \\ & q, x \\ q^{\gamma+2}, b_2 q, b_3 q, \cdots, b_s q & ; \end{array}\right], \tag{3.9}
$$

where $Re(b_{s-i}) > 0$, for all $i = 0, 1, \cdots, s-2$ and $|x| < 1$.

$$
{}_{s+1}\Phi_s\left[\begin{array}{ll} q^{\alpha}, q^{\beta}, a_3, a_4, \cdots, a_s, a_{s+1} & ; \\ & q, x \\ q^{\gamma}, b_2, b_3, \cdots, b_s & ; \end{array}\right]
$$

$$
= {}_{s+1}\Phi_s\left[\begin{array}{ll} q^{\alpha+1}, q^{\beta-1}, a_3, a_4, \cdots, a_s, a_{s+1} & ; \\ & q, x \\ q^{\gamma}, b_2, b_3, \cdots, b_s & ; \end{array}\right] - q^{\alpha} x \frac{(1-q^{\beta-\alpha+1})}{(1-q^{\gamma})}
$$

$$
\times \prod_{i=0}^{s-2}\left[\frac{(1-q^{a_{s+1-i}})}{(1-q^{b_{s-i}})}\right] {}_{s+1}\Phi_s\left[\begin{array}{ll} q^{\alpha+1}, q^{\beta}, a_3 q, a_4 q, \cdots, a_s q, a_{s+1} q & ; \\ & q, x \\ q^{\gamma+1}, b_2 q, b_3 q, \cdots, b_s q & ; \end{array}\right], \tag{3.10}
$$

where $Re(b_{s-i}) > 0$, for all $i = 0, 1, \cdots, s-2$ and $|x| < 1$.

4 Special cases

In view of the limit formulae

$$\lim_{q\to 1^-} \Gamma_q(a) \;=\; \Gamma(a) \quad \textit{and} \quad \lim_{q\to 1^-} \frac{(q^a;q)_n}{(1-q)^n} \;=\; (a)_n \;, \tag{4.1}$$

where

$$(a)_n = a(a+1)\cdots(a+n-1), \tag{4.2}$$

one can note that the result (2.1) is the q-extension of the known result due to Galu*é* and Kalla [3, eqn.(4), p.52], namely

$$ {}_{s+1}F_s \left[\begin{array}{lr} \alpha,\beta,a_3,a_4,\cdots,a_s,a_{s+1} & ; \\ & x \\ \gamma,b_2,b_3,\cdots,b_s & ; \end{array} \right] = \prod_{i=0}^{s-2} \Gamma \left[\begin{array}{c} b_{s-i} \\ a_{s+1-i}, b_{s-i}-a_{s+1-i} \end{array} \right]$$

$$\times \underbrace{\int_0^1 \cdots \int_0^1}_{(s-1)times} \prod_{i=0}^{s-2} t_{i+1}^{a_{s+1-i}-1}(1-t_{i+1})^{b_{s-i}-a_{s+1-i}-1}$$

$$\times\, {}_2F_1 \left[\begin{array}{lr} \alpha,\beta & ; \\ & t_{s-1}\cdots t_2 t_1 x \\ \gamma & ; \end{array} \right] dt_{s-1}\cdots dt_2 dt_1, \tag{4.3}$$

where $Re(b_{s-i}) > 0$, for all $i = 0,1,\cdots,s-2$, $|x| < 1$ and $|t_{s-1}\cdots t_2 t_1 x| < 1$.

Similarly, if we let $q \to 1^-$ and use the formula (1.4), then the results (3.4) and (3.8)-(3.10) correspond to the recurrence relations for generalized hypergeometric functions.

We conclude with the remark that the results deduced in the present article appears to be a new contribution to the theory of basic hypergeometric series. Secondly, one can easily obtain number of recurrence relations for the basic hypergeometric functions by the applications of iterated q-integral representation for ${}_r\Phi_s(.)$.

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