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Common Fixed Point Results and its Applications to Best Approximation in Ordered Semi-Convex Structure *

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Abstract

In this paper, we prove some results concerning the existence of invariant best approximation in Banach spaces. Our main result improves the corresponding results of Jungck and Hussain, Hussain et.al. In the sequel, we discuss some results on best simultaneous approximation in ordered semi-convex structure.

1 Introduction

Fixed point theorems have been used in many instances in best approximation theory. It is pertinent to say that in Best Approximation Theory, it is viable, meaningful and potentially productive to know whether some useful properties of the function being approximated is inherited by the approximating function. In this perspective, Meinardus [25] observed the general principle that could be applied, while doing so the author has employed a fixed point theorem as a tool to establish it. Further, Brosowski [4] obtained a celebrated result and generalized the Meinardus's result. The result of Brosowski was further generalized by Habiniak [13], Smoluk [40] and Subrahmanyam [41]. Sahab, Khan and Sessa [33] extended the result of Hicks and Humpheries [14] and Singh [37] by considering one linear and the other nonexpansive mappings.

On the other hand, Ai-Thagafi and Shahzad [2], Singh [37, 38], Hussain, O'Regan and Agarwal [15], Hussain and Rhoades [17], O'Regan and Hussain [27], Pathak, Cho and Kang [30] and many others have used fixed point theorems in approximation theory, to prove existence of best approximation. Various types of applications of fixed point theorems may be seen in Klee [24], Meinardus [25] and Pathak and Shahzad [31]. Some applications of the fixed point theorems to best simultaneous approximation is given by Sahney and Singh [34]. For the

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detail survey of the subject we refer the reader to Cheney [6] and Singh, Watson and Srivastava [39].

2 Preliminaries and Definitions

Let $X, \|\cdot\|$ be a normed space, M a subset of of X. We shall use $\mathbb{N}$ to denote the set of positive integers, $cl(M)$ to denote the closure of a set M, $\mathcal{D}(M)$ to denote the derived set of M and $wcl(M)$ to denote the weak closure of a set M. Let $I : M \to M$ be a mapping. A mapping $T : M \to M$ is called

(1) an I-contraction if there exists $0 \le k < 1$ such that $\|Tx - Ty\| \le k\|Ix - Iy\|$ for any $x, y \in M$. If $k = 1$, then T is called I-nonexpansive.

(2) asymptotically I-nonexpansive if there exists a sequence $\{k_n\}$ of real numbers with $k_n \ge 1$ and $lim_n k_n = 1$ such that $\|T^n x - T^n y\| \le k_n \|Ix - Iy\|$ for all $x, y \in M$ and $n = 1, 2, 3, ...$

(3) uniformly asymptotically regular on M [3, 10], if for each $\eta > 0$, there exists $N(\eta) = N$ such that $\|T^n x - T^{n+1} x\| < \eta$ for all $\eta \ge N$ and all $x \in M$.

The set of fixed points of $T($ resp. $I)$ is denoted by $F(T)$(resp. $F(I)$). A point $x \in M$ is a coincidence point (common fixed point) of I and T if $Ix = Tx$ $(x = Ix = Tx)$. The set of coincidence points of I and T is denoted by $C(I, T)$. A point $x \in M$ is called an m-th order coincidence point of the pair (I, T) if $I^m(x) = T^m(x)$ and $I^m(x)(orT^m(x)$ is called a point of m-th order coincidence of the pair (I, T). 1-st order coincidence point of the pair (I, T) is simply called coincidence point of (I, T). The set of all m-th order coincidence points of the pair (I, T) in M is denoted by $C_M^m(I, T)$; i.e., $C_M^m(I, T) = \{u \in M : u = I^m(x) = T^m(x),$ for some $x \in M\}$. It is conventional to define $C_M^0(I, T) = M$.

It may be remarked that the set M need not always have a coincidence point. To see this we observe the following example.

Example 2.1. Let $X = l^2$ be endowed with usual norm and $M = \{(x_1, x_2, 0, 0, \cdots) : x_1, x_2 \neq 0\}$. Define $T, I : M \to M$ by $T(x) = (-x_1, -x_2, 0, 0, \cdots)$ and $I(x) = (x_2, x_1, 0, 0, \cdots)$ for all $x = (x_1, x_2, 0, 0, \cdots)$ in M. Then $C_M^2(I, T) = M$, but $C_M^1(I, T) = \emptyset$.

Let $T,\ I : M \to M$ be mappings. Then the pair $\{I, T\}$ is called

(1°) commuting if $TIx = ITx$ for all $x \in M$,

(2°) R-weakly commuting if for all $x \in M$, there exists $R > 0$ such that $\|ITx - TIx\| \le R\|Ix - Tx\|$. If $R = 1$, then the maps are called weakly commuting;

(3°) compatible if $\lim_n \|TIx_n - ITx_n\| = 0$ whenever $\{x_n\}$ is a sequence such that $\lim_n Tx_n = \lim_n Ix_n = t$ for some t in M;

(4°) weakly compatible if they commute at their coincidence points, i.e.,if $ITx = TIx$ whenever $Ix = Tx$.

The set M is called q-starshaped with $q \in M$, if the segment $[q, x] = \{(1-k)q + kx : 0 \leq k \leq 1\}$ joining q to x is contained in M for all $x \in M$. Suppose that M is q-starshaped with $q \in F(I)$ and is both T- and I-invariant. Then T and I are called

(5°) C_q-commuting [2, 17] if $ITx = TIx$ for all $x \in C_q(I, T)$, where $C_q(I, T) = \cup\{C(I, T_k) : 0 \leq k \leq 1\}$ where $T_k = (1-k)q + kT$;

(6°) R-subweakly commuting on M if for all $x \in M$, there exists a real number $R > 0$ such that $\|ITx - TIx\| \leq R dist(Ix, [q, Tx])$;

(7°) uniformly R-subweakly commuting on $M \setminus \{q\}$ (see [3]) if there exists a real number $R > 0$ such that $\|IT^n x - T^n Ix\| \leq R dist(Ix, [q, T^n x])$, for all $x \in M \setminus \{q\}$ and $n \in \mathbb{N}$.

The ordered pair (T, I) of two self maps of a metric space (X, d) is called a Banach operator pair, if the set $F(I)$ is T-invariant, namely $T(F(I)) \subseteq F(I)$. Obviously commuting pair (T, I) is Banach operator pair but not conversely in general, see [5]. If (T, I) is Banach operator pair then (I, T) need not be Banach operator pair (cf. Example 1 [5]). If the self-maps T and I of X satisfy

$$d(ITx, Tx) \leq kd(Ix, x),$$

for all $x \in X$ and $k \geq 0$, then (T, I) is Banach operator pair. In particular , when $I = T$ and X is a normed space, the above inequality can be rewritten as

$$\|T^2 x - Tx\| \leq k\|Tx - x\|$$

for all $x \in X$. Such T is called Banach operator of type k in [41] and [13].

Now we introduce the following definition which encompasses the class of C_q-commuting mappings.

Definition 2.2. Let $T, I : M \to M$ be mappings. Suppose that M is q-starshaped with $q \in F(I)$ and is both T- and I-invariant. Then T and I are called C_q^{m-1}-commuting for some $m \in \mathbb{N}$ if $ITx = TIx$ for all $x \in C_q^{m-1}(I, T)$, where $C_q^{m-1}(I, T) = \cup\{C^{m-1}(I, T_k) : 0 \leq k \leq 1\}$ where $T_k = (1-k)q + kT$.

Definition 2.3. Let $T, I : M \to M$ be mappings. Suppose that M is q-starshaped with $q \in F(I)$ and is both T- and I-invariant. Then T and I are called uniformly C_q^{m-1}-commuting for some $m \in \mathbb{N}$ if $IT^n x = T^n Ix$ for all $x \in C_q^{m-1}(I,T)$ and $n \in \mathbb{N}$, where $C_q^{m-1}(I,T) = \cup\{C^{m-1}(I,T_k) : 0 \leq k \leq 1\}$ where $T_k = (1-k)q + kT$.

Now we give the notion of convex structure introduced by Gudder [12](see also, Petrusel [32]).

Definition 2.4. Let X be a set and $F : [0,1] \times X \times X \to X$ a mapping. Then the pair (X,F) forms a *convex prestructure*. Let (X,F) be a convex prestructure. If F satisfies the following conditions:

(i) $F(\lambda, x, F(\mu, y, z)) = F(\lambda + (1-\lambda)\mu,\ F(\lambda(\lambda + (1-\lambda)\mu)^{-1},\ x,\ y), z)$ for every $\lambda, \mu \in (0,1)$ with $\lambda + (1-\lambda)\mu \neq 0$ and $x, y, z \in X$.
(ii) $F(\lambda, x, x) = x$ for any $x \in X$ and $\lambda \in (0,1)$,

then (X,F) forms a *semi-convex structure*. If (X,F) is a semi-convex structure, then

$(SC1)$ $F(1, x, y) = x$ for any $x, y \in X$.

A semi-convex structure is said to be *regular* if

$(SC2)$ $\lambda \leq \mu \Rightarrow F(\lambda, x, y) \leq F(\mu, x, y)$ where $\lambda, \mu \in (0,1)$.

A semi-convex structure (X,F) is said to form a *convex structure* if F also satisfies the conditions

(iii) $F(\lambda, x, y) = F(1-\lambda, y, x)$ for every $\lambda \in (0,1)$ and $x, y \in X$.
(iv) if $F(\lambda, x, y) = F(\lambda, x, z)$ for some $\lambda \neq 1, x \in X$ then $y = z$.

Let (X,F) be a convex structure. A subset Y of X is called

(a) *F-starshaped* if there exist $p \in Y$ so that for any $x \in Y$ and $\lambda \in (0,1), F(\lambda, x, p) \in Y$.

(b) *F-convex* if for any x, y in Y and $\lambda \in (0,1), F(\lambda, x, y) \in Y$.

For $F(\lambda, x, y) = \lambda x + (1-\lambda)y$, we obtain the known notion of starshaped convexity from linear spaces. Petrusel [32] noted with an example that a set can be a F-semi convex structure without being a convex structure. Let (X,F) be a semi-convex structure. A subset Y of X is called F *semi-starshaped* if there exists $p \in Y$ so that for any $x \in Y$ and $\lambda \in (0,1)$, $F(\lambda, x, p) \in Y$. A Banach space X with semi-convex structure F is said to satisfy condition (P_1) at $p \in K$ (where K is semi-starshaped and p is star centre) if F is continuous relative to the following argument : for any $x, y \in X, \lambda \in (0,1)$

$$\parallel (F(\lambda, x, p) - F(\lambda, y, p) \leq \lambda \parallel x - y \parallel .$$

In this paper, we prove some results in approximation theory using the general type of starshaped condition on Banach space with semi-convex structure, based upon the general theory of convexity given by Gudder [12].

3 Common Fixed Point results

Theorem 3.1. Let M be a subset of metric space (X, d), and I and T be weakly compatible self-maps of M. Assume that $cl\Big(T(M)\Big) \subset I(M)$, $clT(M)$ is complete, and for some $m \in \mathbb{N}$, $\mathcal{D}(clT(M)) \subset I(C_M^{m-1}(I,T))$ and suppose that T and I satisfy for all $x, y \in M$ and $0 < h < 1$,

$$d(Tx,Ty) \leq h\, max\{d(Ix,Iy), d(Ix,Tx), d(Iy,Ty), d(Ix,Ty), d(Iy,Tx)\} \tag{3.1}$$

Then $I(C_M^{m-1}(I,T)) \cap F(I) \cap F(T)$ is a singleton.

Proof. It follows from our assumption that $T(M) \subset I(M)$. So, we can choose $x_n \in M$, for $n \in \mathbb{N}$, such that $Tx_n = Ix_{n+1}$. Set $y_n = Tx_n$ and let $O(y_k; n) = \{y_k, y_{k+1}, \cdots y_{k+n}\}$. Then following the arguments of [28, Lemma 2.1], we infer that $\{y_n\} = \{Tx_n\}$ is a Cauchy sequence. It follows from the completeness of $clT(M)$ that $Tx_n \to w$ for some $w \in \mathcal{D}(clT(M))$ and hence $Ix_n \to w$ as $n \to \infty$. As a consequence we have

$$lim_n Ix_n = lim_n Tx_n = w \in \mathcal{D}(clT(M)) \subset I(C_M^{m-1}(I,T))$$

for some $m \in \mathbb{N}$. Thus $w = Iy$ for some $y \in C_M^{m-1}(I,T)$. For $n \geq 1$, we notice that

$$\begin{aligned} d(w,Ty) &\leq d(w,Tx_n) + d(Tx_n,Ty) \\ &\leq d(w,Tx_n) + h\, max\{d(Ix_n,Iy), d(Ix_n,Tx_n), \\ &\quad\ d(Iy,Ty), d(Ix_n,Ty), d(Iy,Tx_n)\}. \end{aligned}$$

Letting $n \to \infty$, we obtain $Ty = w = Iy$. Since $y \in C_M^{m-1}(I,T)$, it follows that $T^m x = w = I^m x$ for some $y = T^{m-1}x = I^{m-1}x$ in $C_M^{m-1}(I,T)$. We now show that the point of m-th order coincidence $T^m x$ is unique. So, we suppose that for some $z \in M, T^m z = w = I^m z$. Then, from inequality (2.1), we obtain

$$\begin{aligned} d(I^m x, I^m z) &= d(T^m x, T^m z) \\ &\leq h\, max\{d(I^m x, I^m z), d(I^m x, T^m x), d(I^m z, T^m z), \\ &\quad\ d(I^m x, T^m z), d(I^m z, T^m x)\} \\ &\leq h\, d(I^m x, I^m z), \end{aligned}$$

a contradiction. Hence $I^m z = I^m x = T^m x$. Thus the point $T^m x = w = I^m x$ of m-th order coincidence is unique.
Since I and T are weakly compatible and $II^{m-1}x = TT^{m-1}x$; i.e., $y = T^{m-1}x =$

$I^{m-1}x$ is a coincidence point of T and I, it follows that $Tw = TI^m x = TII^{m-1}x = ITT^{m-1}x = IT^m x = Iw$. Now using (2.1) we obtain

$$\begin{aligned} d(w, Tw) &= d(TI^{m-1}x, TT^m x) \\ &\leq h\, max\{d(I^m x, IT^m x), d(I^m x, TI^{m-1}x), d(IT^m x, TT^m z), \\ &\quad d(I^m x, TT^m x), d(IT^m x, T^m x)\} \\ &\leq h\, d(w, Tw). \end{aligned}$$

Hence $Tw = w$ as $h \in (0, 1)$. Thus $w = Tu$ is a common fixed point of T and I. But $w = T^m x = I^m x = Iu$, a common fixed point of T and I, is also a point of m-th order coincidence of T and I, and is therefore unique.

It may be observe that Theorem 3.1 is more sharper than the following result due to Jungck and Hussain ([21], Theorem 2.1) from geometrical point of view in the sense that geometrically we can identify the location of common fixed point of T and I in a restricted region of M. Indeed, when $m = 1$, we have $C_M^{m-1}(I, T) = M$ and so in view of the hypothesis $clT(M) \subset I(M)$ the condition $\mathcal{D}(clT(M)) \subset I(C_M^{m-1}(I, T))$ is trivially true. Thus we have the following result as corollary of Theorem 3.1.

Corollary 3.2.([21]) Let M be a subset of metric space (X, d), and I and T be weakly compatible self-maps of M. Assume that $clT(M) \subset I(M), clT(M)$ is complete, and T and I satisfy for all $x, y \in M$ and $0 < h < 1$,

$$d(Tx, Ty) \leq h\, max\{d(Ix, Iy), d(Ix, Tx), d(Iy, Ty), d(Ix, Ty), d(Iy, Tx)\}$$

Then $M \cap F(I) \cap F(T)$ is a singleton.

Example 3.3. Let $X = \mathbb{R}$ be endowed with usual metric and let $M = \{0, 1, 2, 3\}$. Define maps $T, I : M \to M$ by $T0 = T1 = T2 = 0,\ I0 = I1 = 0,\ I2 = 1,\ T3 = 1$ and $I3 = 2$. Then $C_M^1(I, T) = \{0 \in M : 0 = I0 = T0,\ 0 = I1 = T1\},\ C_M^2(I, T) = \{0 \in M : 0 = I^2 2 = T^2 2\},\ C_M^3(I, T) = \{0 \in M : 0 = I^3 3 = T^3 3\}$ and $C_M^m(I, T) = \emptyset$ for all $m > 3$. Clearly $T0 = 0$ is the unique point of 1-th, 2-nd and 3-rd order coincidence of the pair (I, T), whereas 0 and 1 are 1-st order concidence points of the pair (I, T) in M, 2 is a 2-nd order coincidence point of the pair (I, T) in M and 3 is a 3-rd order coincidence point of the pair (I, T) in M. Moreover, for any $h \in [\frac{1}{2}, 1)$, the hypothesis of Theorem 2.5 is satisfied. Clearly, $I(C_M^{m-1}(I, T)) \cap F(I) \cap F(T) = \{0\}$ for $m = 1, 2, 3, 4$.

We can extend these concepts on F-starshaped set in the convex structure (X, F)(see [15, 16]). We define

$$Y_p^{T^n x} = \{F(\lambda, T^n x, p) : 0 \leq \lambda \leq 1\}.$$

Let $(X, F, \leq)$ be an ordered semi-convex structure and M a nonempty subset of X. Call M to be weakly closed if the weak limit of every weakly convergent sequence from M belongs to M. Notice that every weakly closed subspace of a normed linear space is closed.

The following result improves and extends Lemma 3.3 [4].

Lemma 3.4. Let $(X, F, \leq)$ be an ordered semi-convex structure and, I and T be self-maps on a nonempty subset M of X. Suppose that M is F-starshaped with respect to an element p in $F(I)$, I satisfies $F(\lambda, Ix, p) = I(F(\lambda, x, p))$ and $I(M) = M$. Assume that T and I are uniformly C_p^{m-1}-commuting and satisfy for each $n \geq 1$

$$\|T^n x - T^n y\| \leq k_n max \left\{ \begin{array}{c} \|Ix - Iy\|, dist(Ix, Y_p^{T^n x}), dist(Iy, Y_p^{T^n y}), \\ dist(Ix, Y_p^{T^n y}), dist(Iy, Y_p^{T^n x}) \end{array} \right\} \quad (3.2)$$

for all $x, y \in M$, where $\{k_n\}$ is a sequence of real numbers with $k_n \geq 1$ and $lim_n k_n = 1$. For each $n \geq 1$, define a mapping T_n on M by

$$T_n x = F(\mu_n, T^n x, p)$$

where $\mu_n = \frac{\lambda_n}{k_n}$ and $\{\lambda_n\}$ is a sequence of numbers in $(0, 1)$ such that $lim_n \lambda_n = 1$. Then for each $n \geq 1$, T_n and I have exactly one common fixed point x_n in $C_M^{m-1}(I, T_n)$ such that

$$Ix_n = x_n = F(\mu_n, T^n x_n, p)$$

provided one of the following conditions hold:
(i) M is closed and for each n, $clT_n(M)$ is complete and $\mathcal{D}(clT(M)) \subset I(C_M^{m-1}(I, T))$,
(ii) M is weakly closed and for each n, $wclT_n(M)$ is complete and $\mathcal{D}(clT(M)) \subset I(C_M^{m-1}(I, T))$.

Proof. By definition,

$$T_n x = F(\mu_n, T^n x, p).$$

As I and T are uniformly C_p^m-commuting and $F(\lambda, Ix, p) = I(F(\lambda, x, p))$, then for each $y \in C_M^{m-1}(I, T_n) \subseteq C_p^{m-1}(I, T^n)$ for which $Iy = T_n y$,

$$\begin{aligned} T_n Iy &= F(\mu_n, T^n Iy, p) \\ &= F(\mu_n, IT^n y, p) \\ &= I(F(\mu_n, T^n y, p)) \\ &= IT_n y. \end{aligned}$$

Hence I and T_n are weakly compatible for all n. Also by (3.2),

$$\begin{aligned} \|T_n x - T_n y\| &= \mu_n \|T^n x - T^n y\| \\ &\leq \lambda_n \max\{\|Ix - Iy\|, dist(Ix, Y_p^{T^n x}), dist(Iy, Y_p^{T^n y}), \\ &\quad dist(Ix, Y_p^{T^n y}), dist(Iy, Y_p^{T^n x})\} \\ &\leq \lambda_n \max\{\|Ix - Iy\|, \|Ix - T_n x\|, \|Iy - T_n y\|, \\ &\quad \|Ix - T_n y\|, \|Iy - T_n x\|\}, \end{aligned}$$

for each $x, y \in M$.
(i) As M is closed, therefore, for each n, $clT_n(M) \subset M = I(M)$ and $\mathcal{D}(clT(M)) \subset I(C_M^{m-1}(I,T))$. By Theorem 3.1, for each $n \geq 1$, there exists $x_n \in I(C_M^{m-1}(I,T))$ such that $x_n = Ix_n = T_n x_n$. Thus for each $n \geq 1$, $I(C_M^{m-1}(I,T)) \cap F(T_n) \cap F(I) \neq \emptyset$.
(ii) As $wclT_n(M) \subset M = I(M)$ and $\mathcal{D}(clT(M)) \subset I(C_M^{m-1}(I,T))$, for each n, by Theorem 3.1, the conclusion follows.

The following result extends the recent results due to Al-Thagafi and Shahzad [2], Theorems 2.2-2.4) to asymptotically I-nonexpansive maps defined on F-starshaped domain.

Theorem 3.5. Let $(X, F, \leq)$ be an ordered semi-convex structure with F regular and, I and T be self-maps on a nonempty subset M of X. Suppose that M is F-starshaped with respect to an element p in $F(I)$, I satisfies $F(\lambda, Ix, p) = I(F(\lambda, x, p))$ and $I(M) = M$. Assume that T and I are uniformly C_p^{m-1}-commuting maps, T is uniformly asymptotically regular and asymptotically I-nonexpansive map on $I(C_M^{m-1}(I,T))$. Then $F(T) \cap F(I) \neq \emptyset$, provided one of the following conditions holds;
(i) M is closed and $clT(M)$ is compact and $\mathcal{D}(clT(M)) \subset I(C_M^{m-1}(I,T))$,
(ii) X is complete, M is weakly closed, I is weakly continuous, $wclT(M)$ is weakly compact, $\mathcal{D}(clT(M)) \subset I(C_M^{m-1}(I,T))$ and either $I_d - T$ is demiclosed at 0 or X satisfies Opial's condition.

Proof. (i) Notice that compactness of $clT(M)$ implies that $clT_n(M)$ is compact and hence complete. From Theorem 3.1, for each $n \geq 1$, there exists $x_n \in I(C_M^{m-1}(I,T_n))$ such that $x_n = Ix_n = T_n x_n = F(\mu_n, T^n x_n, p)$. Hence $x_n \in C_p^{m-1}(I,T^n)$.

$$
\begin{aligned}
\text{Therefore } x_n - T^{n+1}x_n &= T_n x_n - T^{n+1}x_n \\
&= F(\mu_n, T^n x_n, p) - T^{n+1}x_n \\
&\leq F(\limsup_{n\to\infty} \mu_n, T^n x_n, p) - T^{n+1}x_n \\
&\leq F(1, T^n x_n, p) - T^{n+1}x_n \\
&\leq T^n x_n - T^{n+1}x_n.
\end{aligned}
$$

Applying the same argument as above, we also have

$$x_n - T^n x_n \leq 0.$$

Since T is uniformly asymptotically regular on $I(C_M^{m-1}(I,T))$ it follows that

$$T^n x_n - T^{n+1}x_n \to 0 \text{ as } n \to \infty.$$

Therefore $x_n - T^{n+1}x_n \to 0$ as $n \to \infty$.

Now $\| x_n - Tx_n \| \leq \| x_n - T^{n+1}x_n \| + \| T^{n+1}x_n - Tx_n \|$

$$\leq \| x_n - T^{n+1} \| + k_1 \| S(T^n x_n) - Sx_n \| \quad \text{for some } k_1 \geq 1$$

$$= \| x_n - T^{n+1} x_n \| + k_1 \| T^n x_n - x_n \|$$

Since I commutes with T^n on $C_p^{m-1}(I, T^n)$ and $x_n \in C_p^{m-1}(I, T^n)$, $x_n = Ix_n$, therefore $(I_d - T)x_n \to 0$ as $n \to \infty$
Since $clT(M)$ is compact, there exists a subsequence $\{Tx_m\}$ of $\{Tx_n\}$ such that $Tx_m \to z$ as $m \to \infty$. By the continuity of I and T and the fact $\|x_m - Tx_m\| \to 0$, we have $z \in F(T) \cap F(I)$. Thus $F(T) \cap F(I) \neq \emptyset$.

(ii) The weak compactness of $wclT(M)$ implies that $wclT_n(M)$ is weakly compact and hence complete due to completeness of X (see [2, 18]). From Theorem 3.1, for each $n \geq 1$, there exists $x_n \in M$ such that $x_n = Ix_n = F(\mu_n, T^n x_n, p)$. The analysis in (i), implies that $\|x_n - Tx_n\| \to 0$ as $n \to \infty$. The weak compactness of $wclT(M)$ implies that there is a subsequence $\{x_m\}$ of $\{x_n\}$ converging weakly to $z \in M$ as $m \to \infty$. As I is weakly continuous, so $Iz = z$. Also we have, $Ix_m - Tx_m = x_m - Tx_m \to 0$ as $m \to \infty$. If $I - T$ is demiclosed at 0, then $Iz = Tz$. Thus $F(T) \cap F(I) \neq \emptyset$.

If X satisfies Opial's condition and $z \neq Tz$, then

$$\begin{aligned}
\liminf_{m\to\infty} \|x_m - z\| &< \liminf_{m\to\infty} \|x_m - Tz\| \\
&\leq \liminf_{m\to\infty} \|x_m - Tx_m\| + \liminf_{m\to\infty} \|Tx_m - Tz\| \\
&= \liminf_{m\to\infty} \|Tx_m - Tz\| \leq \liminf_{m\to\infty} k_m \|Ix_m - Iz\| \\
&= \liminf_{m\to\infty} \|x_m - z\|,
\end{aligned}$$

which is a contradiction. Thus $Iz = Tz = z$ and hence $F(T) \cap F(I) \neq \emptyset$.
This completes the proof.

Corollary 3.6. (see, [4], Theorem 3.4) Let I and T be continuous self-maps on a q-starshaped subset M of a normed space X. Assume that $clT(M) \subset I(M)$, $q \in F(I)$, I is linear, T is uniformly asymptotically regular and asymptotically I-nonexpansive . If $clT(M)$ is compact, T and I are uniformly R-subweakly commuting on M, then $F(T) \cap F(I) \neq \emptyset$.

Remark 3.7 . Notice that the conditions of the continuity and linearity of S are not needed in Theorem 3.4 of Beg et al. [3]. The result is also true for affine mapping S.

Now we introduce the concept of lower semi-convex structure in a Banach space as follows:

Definition 3.7. Let $(X, \|\cdot\|)$ be a Banach space with semi-convex structure F. A continuous map $F : [0, \frac{1}{2}] \times X \times X \to X$ is said to be a *lower semi-convex* *structure* on X if for all x, y in X, λ in $[0, \frac{1}{2}]$,

$$\|u - F(\lambda, x, F(\lambda, y, y))\| \leq \lambda\|u - x\| + (1 - \lambda)\|u - y\|$$

for all u in X.

Definition 3.8. Let $(X, \| \, . \, \|)$ be a Banach space with lower semi-convex structure F. Then the triplet $(X, F, \| \cdot \|)$ is called a *lower semi-convex Banach space* (or, in brief, *LSCBS*).

Definition 3.9. Let $(X, F, \| \cdot \|)$ be a lower semi-convex Banach space, K a subset of X and let '$\leq$' be an order relation defined on K by

$$x \leq y \text{ iff } y - x \in K.$$

Then the triplet $(X, F, \| \cdot \|)$ is said to be an *ordered* $LSCBS$ induced by $(K, \leq)$.

The following result extends main theorems in [7, 8, 9, 20].

Lemma 3.10. Let M be a nonempty, closed subset of an ordered $LSCBS$ $(X, F, \| \cdot \|)$ induced by $(M, \leq)$, and $T, I : M \to M$ be weakly compatible pair satisfying the following condition:

$$\|Tx - Ty\|^p \leq a\|Ix - Iy\|^p + (1 - a)\, max\{\|Tx - Ix\|^p, \|Ty - Iy\|^p\} \tag{3.3}$$

for all $x, y \in M$, where $0 < a < 1$ and $0 < p \leq 1$. If $C_q^{m-1}(T, I)$ is nonempty and $cl(T(M)) \cup F\Big([0, \frac{1}{2}] \times T(M) \times T(M)\Big) \subseteq I(M)$, $\mathcal{D}(clT(M)) \subset I(C_M^{m-1}(I, T))$, where F is a lower semi-convex structure on M, then T and I have a unique common fixed point in $I(C_M^{m-1}(I, T))$; i.e., $I(C_M^{m-1}(I, T)) \cap F(T) \cap F(I)$ is singleton.

Proof. Let x be an arbitrary point of M. Choose points x_1, x_2, x_3 in M and some $\lambda \in [0, \frac{1}{2}]$ such that

$$Ix_1 = Tx, Ix_2 = Tx_1, Ix_3 = F(\lambda, Tx_1, Tx_2).$$

This choice is possible because $Tx, Tx_1, Tx_2, F(\lambda, Tx_1, Tx_2)$ are in $I(M)$.
By (3.1), we have

$$\begin{aligned} \|Ix_1 - Ix_2\|^p &= \|Tx - Tx_1\|^p \\ &\leq a\,\|Ix - Ix_1\|^p + (1 - a)\, \max\{\|Ix - Tx\|^p, \|Ix_1 - Tx_1\|^p\} \\ &= a\,\|Ix - Ix_1\|^2 + (1 - a)\, \max\{\|Ix - Ix_1\|^2, \|Ix_1 - Ix_2\|^2\}. \end{aligned}$$

Hence we have

$$\|Ix_1 - Ix_2\| \leq \|Ix - Ix_1\|. \tag{3.4}$$

Form (3.3) and (3.4),

$$\begin{aligned} \|Ix_2 - Tx_2\|^p &= \|Tx_1 - Tx_2\|^p \\ &\leq a\,\|Ix_1 - Ix_2\|^p + (1 - a)\, \max\{\|Ix_1 - Tx_1\|^p, \|Ix_2 - Tx_2\|^p\} \\ &\leq a\,\|Ix - Ix_1\|^p + (1 - a)\, \max\{\|Ix - Ix_1\|^p, \|Ix_2 - Tx_2\|^p\} \end{aligned}$$

which implies

$$\|Ix_2 - Tx_2\| \leq \|Ix - Ix_1\| \tag{3.5}$$

As $f(x) = x^p$ is increasing for $x \geq 0$, we have from (3.3),

$$\|Ix_1 - Tx_2\|^p = \|Tx - Tx_2\|^p$$

$$\leq a\,\|Ix - Ix_2\|^p + (1-a)\,\max\{\|Ix - Tx\|^p, \|Ix_2 - Tx_2\|^p\}$$

$$\leq a\,[\|Ix - Ix_1\| + \|Ix_1 - Ix_2\|]^p + (1-a)\,\max\{\|Ix - Ix_1\|^p, \|Ix_2 - Tx_2\|^p\}.$$

Hence, using (3.4) and (3.5), we have

$$\|Ix_1 - Tx_2\|^p \leq (2^p a + 1 - a)\,\|Ix - Ix_1\|^p. \tag{3.6}$$

Now using Definition 3.9 and convexity of $f(x) = x^p (p \geq 1)$, we have

$$\|Ix_1 - Ix_3\|^p = \|Ix_1 - F(\lambda, Tx_1, Tx_2)\|^p$$

$$= \|Ix_1 - F(\lambda, Tx_1, F(\lambda, Tx_2, Tx_2))\|^p$$

$$\leq [\lambda\,\|Ix_1 - Tx_1\| + (1-\lambda)\,\|Ix_1 - Tx_2\|]^p$$

$$\leq \lambda^p\,\|Ix_1 - Ix_2\|^p + (1-\lambda)^p\,\|Ix_1 - Tx_2\|^p.$$

Hence, from (3.4) and (3.6), we obtain

$$\|Ix_1 - Ix_3\|^p \leq [\lambda^p + (1-\lambda)^p\{2^p a + (1-a)\}]\|Ix - Ix_1\|^p. \tag{3.7}$$

Further,

$$\|Ix_2 - Ix_3\|^p = \|Ix_2 - F(\lambda, Tx_1, Tx_2)\|^p$$

$$= \|Ix_2 - F(\lambda, Tx_1, F(\lambda, Tx_2, Tx_2))\|^p$$

$$\leq [\lambda\,\|Ix_2 - Ix_2\| + (1-\lambda)\,\|Ix_2 - Tx_2\|]^p$$

hence by (3.5) we get

$$\|Ix_2 - Ix_3\| \leq (1-\lambda)\,\|Ix - Ix_1\|. \tag{3.8}$$

Now we choose $x_4 \in M$ such that $Ix_4 = Tx_3$. Then from (3.3), (3.4) and (3.5) we have

$$\|Ix_3 - Ix_4\|^p = \|Tx_3 - F(\lambda, Tx_1, Tx_2)\|^p$$

$$= \|Tx_3 - F(\lambda, Tx_1, F(\lambda, Tx_2, Tx_2))\|^p$$

$$\leq [\lambda\,\|Tx_1 - Tx_3\| + (1-\lambda)\,\|Tx_2 - Tx_3\|]^p$$

$$\leq\ \lambda^p\,[a\,[\|Ix_1 - Ix_3\|^p + (1-a)\,max\{\|Ix_1 - Ix_2\|^p, \|Ix_3 - Ix_4\|^p\}]$$

$$+(1-\lambda)^p\,[a\,[\|Ix_2 - Ix_3\|^p + (1-a)\,max\{\|Ix_2 - Tx_2\|^p, \|Ix_3 - Ix_4\|^p\}]$$

$$\leq a\,[\lambda^p\,\|Ix_1 - Ix_3\|^p + (1-\lambda)^p\,\|Ix_2 - Ix_3\|^p]$$

$$+(1-a)[\lambda^p + (1-\lambda)^p]\,max\{\|Ix - Ix_1\|^p, \|Ix_3 - Ix_4\|^p\}.$$

Hence, using (3.7) and (3.8), we have

$$\|Ix_3 - Ix_4\|^p \le \mu^p\, max\{\|Ix - Ix_1\|^p, \|Ix_3 - Ix_4\|^p\},$$

where $\mu^p = \Big(a\,\lambda^p[\lambda^p+(1-\lambda)^p\{2^p a+(1-a)\}+(1-\lambda)^p]+(1-a)[\lambda^p+(1-\lambda)^p]\Big)$. Since $p \ge 1$, $0 < a < 1$ and $\lambda \in [0, \frac{1}{2}]$, we obtain $\mu^p < 1$. To see this, we observe that

$$\begin{aligned}
\mu^p &= \Big(a\,\lambda^p[\lambda^p+(1-\lambda)^p\{2^p a+(1-a)\}+(1-\lambda)^p]+(1-a)[\lambda^p+(1-\lambda)^p]\Big)\\
&= \Big(a\,\lambda^p[\lambda^p+(1-\lambda)^p+a(1-\lambda)^p\{2^p-1\}+(1-\lambda)^p]+(1-a)[\lambda^p+(1-\lambda)^p]\Big)\\
&\le \Big(a\,2^{-p}[2^{1-p} + a2^{-p}\{2^p - 1\} + 2^{-p}] + (1-a)2^{1-p}\Big)\\
&= \Big(3a\,2^{-2p} + a^2(2^{-p} - 2^{-2p}) + (1-a)2^{1-p}\Big)\\
&< \Big(3a\,2^{-2p} + a(2^{-p} - 2^{-2p}) + (1-a)2^{1-p}\Big), \text{ as } 0 < a^2 < a < 1\\
&= \Big(2a\,2^{-2p} + a2^{-p} + 2^{1-p} - 2a2^{-p}\Big)\\
&= \Big(2a\,2^{-2p} + 2^{1-p} - a2^{-p}\Big)\\
&= \Big(2^{1-p} - a(2^{-p} - 2^{1-2p})\Big)\\
&< 1, \text{ as } 0 < a < 1 \text{ and } p \ge 1.
\end{aligned}$$

Therefore,

$$\|Ix_3 - Ix_4\| \le \mu\, \|Ix - Ix_1\| \qquad (0 < k < 1). \tag{3.9}$$

Now we shall consider the sequence $\{Sx_n\}_{n=0}^{\infty}$ which possess the properties (3.4), (3.5), (3.8) and (3.9); i.e., the sequence $\{Ix_n\}_{n=0}^{\infty}$ is defined as follows:

$$Ix_{3k+1} = Tx_{3k}; Ix_{3k+2} = Tx_{3k+1}; Ix_{3(k+1)} = F(\lambda, Tx_{3k+1}, Tx_{3k+2}),\ k = 0, 1, 2, \cdots$$

By induction it can easily be shown that from (3.9), (3.4) and (3.8) we have

$$\|Ix_{3k} - Ix_{3k+1}\| \le \mu\, \|Ix_{3(k-1)} - Ix_{3(k-1)+1}\| \le \cdots \le \mu^k\, \|Ix - Ix_1\|,$$

$$\|Ix_{3k+1} - Ix_{3k+2}\| \le \|Ix_{3k} - Ix_{3k+1}\| \le \mu^k\, \|Ix - Ix_1\|,$$

$$\|Ix_{3k+2} - Ix_{3(k+1)}\| \le (1-\lambda)\, \|Ix_{3k} - Ix_{3k+1}\| \le (1-\lambda)\, \mu^k\, \|Ix - Ix_1\|. \tag{3.10}$$

Hence for $m > n > N$, we have

$$\|Ix_m - Ix_n\| \le \sum_{i=N}^{\infty} \|Ix_i - Ix_{i+1}\| \le \Big((3-\lambda)\mu^{[N/3]}/(1-\mu)\Big)\|Ix - Ix_1\|,$$

where $[N/3]$ means the greatest integer not exceeding $N/3$. Take $x_0 = x$, then it follows from the above inequality that the sequence $\{Ix_n\}_{n=0}^{\infty}$ is a Cauchy

sequence in M, hence convergent. So, let $\lim_{n\to\infty} Ix_n = u$.
As $Tx_{3k} = Ix_{3k+1}, Tx_{3k+1} = Ix_{3k+2}$, from (3.5) and (3.10) we have

$$\|Tx_{3k+2} - Ix_{3k+2}\| \leq \|Ix_{3k} - Ix_{3k+1}\| \leq \mu^p \|Ix - Ix_1\|.$$

Therefore,

$$\lim_{n\to\infty} Tx_n = \lim_{n\to\infty} Ix_n = u.$$

Let $z \in C_q^{m-1}(T, I)$. Then from (3.3) we have

$$\| Tx_n - Tz \|^p \leq a \| Ix_n - Iz \|^p + (1-a) \max\{\| Tx_n - Ix_n \|^p, \| Tz - Iz \|^p\}.$$

Letting $n \to \infty$, we obtain

$$\| u - Tz \|^p \leq a \| u - Tz \|^p,$$

a contradiction. Hence, $Tz = u = Iz$. Since, T and I commutes at each $z \in C_q^{m-1}(T, I)$ we have $TIz = ITz = Iu = IIz$. Again, from (3.3) we have

$$\| TIz - Tz \|^p \leq a \| IIz - Iz \|^p + (1-a) \max\{\| TIz - IIz \|^p, \| Tz - Iz \|^p\}$$

$$\| Iu - u \|^p \leq a \| Iu - u \|^p,$$

which gives $Iu = u$. Hence, $Tu = Iu = u$; i.e., u is a common fixed point of T and I. Condition (3.3) ensures that u is the unique common fixed point of T and I; i.e., $I(C_q^{m-1}(T, I)) \cap F(T) \cap F(I)$ is singleton.

Theorem 3.11. Let $(X, F, \|\cdot\|)$ be an ordered $LSCBS$ induced by $(M, \leq)$, where F is a lower semi-convex structure on M and let $T, I : M \to M$ be C_p^{m-1}-commuting pair of continuous mappings. Let M be closed F-starshaped with respect to an element $p \in F(I)$ and I satisfies $F(\lambda, Ix, p) = I(F(\lambda, x, p))$ for each $x \in M$. If $M = I(M)$, $cl(T(M))$ is compact, $\mathcal{D}(clT(M)) \subset I(C_M^{m-1}(I, T))$, and satisfies, for all $x, y \in M$, and all $k \in (0, 1)$,

$$\| Tx - Ty \| \leq \| Ix - Iy \| + \frac{1-k}{k} \max\{dist(Ix, Y_p^{Tx}),\, dist(Iy, Y_p^{Ty})\}, \quad (3.11)$$

then $I(C_p^{m-1}(I, T)) \bigcap F(I) \bigcap F(T) \neq \emptyset$.

Proof. Define $T_n : M \to M$ by

$$T_n x = F(k_n, Tx, p)$$

for some $p \in F(I)$ and all $x \in M$ and a fixed sequence of real numbers $k_n (0 < k_n < 1)$ converging to 1. As I and T are C_p^{m-1}-commuting and $F(\lambda, Ix, p) = I(F(\lambda, x, p))$ with $Ip = p$, then for each $u \in C_p^{m-1}(I, T_n)$ for which $Iu = T_n u$,

$$\begin{aligned} T_n Iu &= F(k_n, TIu, p) \\ &= F(k_n, ITu, p) \\ &= I(F(k_n, Tu, p)) \\ &= IT_n u. \end{aligned}$$

Thus $IT_nu = T_nIu$ for each $u \in C_p^{m-1}(I, T_n) \subset C_p^{m-1}(I, T)$. Hence I and T_n are weakly compatible for all n. Also

$$\begin{aligned}
\| T_nx - T_ny \| &= k_n \| Tx - Ty \| \\
&\leq k_n\{\| Ix - Iy \| + \frac{1-k_n}{k_n} \max\{\| Ix - T_nx \|, \| Iy - T_ny \|\}\} \\
&= k_n \| Ix - Iy \| + (1-k_n) \max\{\| Ix - T_nx \|, \| Iy - T_ny \|\}
\end{aligned}$$

for each $x, y \in M$ and $0 < k_n < 1$. By Lemma 3.10, for each $n \geq 1$, there exist an $x_n \in I(C_p^{m-1}(I, T_n))$ such that $x_n = Ix_n = T_nx_n$. The compactness of $cl(T(M))$ implies that there exists a subsequence x_{n_i} such that $x_{n_i} \to z$ as $i \to \infty$. Since T is continuous, $T(x_{n_i}) \to T(z)$ as $i \to \infty$. Again

$$z = \lim x_{n_i} = \lim T_{n_i}(x_{n_i}) = \lim F(k_{n_i}, T(x_{n_i}), p) = F(1, T(z), p) = T(z).$$

By continuity of I, we have $Iz = z$. This shows that $I(C_p^{m-1}(I, T)) \cap F(I) \cap F(T) \neq \emptyset$.

Theorem 3.11 extends Theorem 2.2 [1] and Theorem 2.2 [2].

Lemma 3.12. Let M be a nonempty, closed subset of an ordered $LSCBS$ $(X, F, \|\cdot\|)$ induced by $(M, \leq)$, and $T, I : M \to M$ be a pair of maps satisfying inequality (3.3), where F is a lower semi-convex structure on M and $F(I)$. Suppose that $cl(T(M))$ is complete, $\mathcal{D}(clT(M)) \subset I(C_M^{m-1}(I, T))$, (T, I) is Banach operator pair, I is continuous and $F(I)$ is nonempty, then T and I have a unique common fixed point in $I(C_M^{m-1}(I, T))$.

Proof. By our assumptions, $T(F(I)) \subseteq F(I)$ and $F(I)$ is nonempty closed and has a lower semi-convex structure. Further for all $x,\ y \in F(I)$, we have by inequality (3.3),

$$\begin{aligned}
\|Tx - Ty\| &\leq a \| Ix - Iy \|^p + (1-a) \max\{\| Tx - Ix \|^p, \| Ty - Iy \|^p\} \\
&= a \| x - y \|^p + (1-a) \max\{\| Tx - x \|^p, \| Ty - y \|^p\}
\end{aligned}$$

By Lemma 3.10, T has a unique fixed point y in $F(I)$ and consequently $I(C_q^{m-1}(T, I)) \cap F(T) \cap F(I)$ is singleton.

The following result extends and improves Theorem 3.3 of [5].

Theorem 3.13. Let $(X, F, \|\cdot\|)$ be an ordered $LSCBS$ induced by $(M, \leq)$ and let $T, I : M \to M$ be pair of continuous mappings. Let M be closed F-starshaped with respect to an element p in $F(I)$. Assume that (T, I) is Banach operator pair on M, $F(I)$ is F-starshaped with respect to an element $p \in F(I)$, where F is a lower semi-convex structure on M and $F(I)$. If $cl(T(M))$ is compact, $\mathcal{D}(clT(M)) \subset I(C_M^{m-1}(I, T))$ and (T, I) satisfies (3.11), for all $x, y \in M$, and all $k \in (0, 1)$, then $I(C_M^{m-1}(I, T)) \cap F(I) \cap F(T) \neq \emptyset$.

Proof. Define $T_n : M \to M$ as in Theorem 3.11. As $F(I)$ is F-starshaped with respect to an element p in $F(I)$, for each $x \in F(I)$, $T_n x = F(k_n, Tx, q) \in F(I)$, since $Tx \in F(I)$. Thus (T_n, I) is Banach operator pair for each n. Also

$$\| T_n x - T_n y \| = k_n \| Tx - Ty \|$$

$$\leq k_n\{\| Ix - Iy \| + \tfrac{1-k_n}{k_n} \max\{\| Ix - T_n x \|, \| Iy - T_n y \|\}\}$$

$$= k_n \| Ix - Iy \| + (1 - k_n) \max\{\| Ix - T_n x \|, \| Iy - T_n y \|\}$$

for each $x, y \in M$ and $0 < k_n < 1$. By Lemma 3.12, for each $n \geq 1$, there exist an $x_n \in M$ such that $x_n = Ix_n = T_n x_n$. The compactness of $cl(T(M))$ implies that there exist a subsequence x_{n_i} such that $x_{n_i} \to z$ as $i \to \infty$. Since T is continuous, $T(x_{n_i}) \to T(z)$ as $i \to \infty$. Again

$$z = \lim x_{n_i} = \lim T_{n_i}(x_{n_i}) = \lim F(k_{n_i}, T(x_{n_i}), p) = F(1, T(z), p) = T(z).$$

By continuity of I, we also have $Iz = z$. This shows that $I(C_q^{m-1}(T, I)) \cap F(I) \cap F(T) \neq \emptyset$.

4 Some Invariant Approximation Results

Let M be a subset of a Banach space $(X, \| \, . \, \|)$. The set $P_M(u) = \{x \in M : \| x - u \| = dist(u, M)\}$ is called the set of best approximants to $u \in X$ out of M, where $dist(u, M) = inf\{\| y - u \| : y \in M\}$. Suppose A , G, are bounded subsets of X, then we write

$$r_G(A) = inf_{g \in G} sup_{a \in A} \| a - g \|$$

$$cent_G(A) = \{g_0 \in G : sup_{a \in A} \| a - g_0 \| = r_G(A)\}.$$

The number $r_G(A)$ is called the *Chebyshev radius* of A w.r.t G and an element $y_0 \in cent_G(A)$ is called a *best simultaneous approximation* of A w.r.t G. If $A = \{u\}$, then $r_G(A) = d(u, G)$ and $cent_G(A)$ is the set of all best approximations, $P_G(u)$, of u out of G. We also refer the reader to Cheney [6], Klee [24] and Milman [26] for further details.

Sahab et al. [33], Jungck and Sessa [22] and Al-Thagafi [1] generalized main result of Singh [38] to nonexpansive mapping T with respect to continuous mapping S in the context of best approximation in normed linear space. In this section, as an application of our common fixed point results, we prove the corresponding results in semi-convex structure in the context of best simultaneous approximation for more general pair of mappings.

In the following result we extend Theorem 3.1-3.4 due to Al-Thagafi and Shahzad [2] to asymptotically I-nonexpansive maps defined on F-starshaped domain.

Theorem 4.1. Let $(X, F, \leq)$ be an ordered semi-convex structure with F regular and, G and A are nonempty subset of X such that $cent_G(A)$, set of best simultaneous approximation of elements in A by G, is nonempty. Let T and I are self mapping on $cent_G(A)$. Suppose that $cent_G(A)$ is F-starshaped with respect to an element p in $F(I)$, $F(\lambda, Ix, p) = I(F(\lambda, x, p))$ for all $x \in cent_G(A)$ and $I(cent_G(A)) = cent_G(A)$. Assume that T and I are uniformly $C_p^{m-1}-$commuting, T is uniformly asymptotically regular and asymptotically $I-$nonexpansive. Then $F(T) \cap F(I) \cap cent_G(A) \neq \emptyset$, provided one of the following conditions holds:
(i) $cent_G(A)$ is closed and $clT(cent_G(A))$ is compact.
(ii) X is complete, $cent_G(A)$ is weakly closed, I is weakly continuous, $wclT(cen t_G(A))$ is weakly compact and either $I_d - T$ is demiclosed at 0 or X satisfies Opial's condition.

Proof. In both of the cases (i) -(ii), Lemma 3.10 implies that, for each $n \geq 1$, there exists $x_n \in cent_G(A)$ such that $x_n = Ix_n = F(\mu_n, T^n x_n, p)$. The result now follows from Theorem 3.5.

Corollary 4.2.([42], Theorem 2.3) Let K be a nonempty subset of a normed space X and $y_1, y_2 \in X$. Suppose that T and S are self-mappings of K such that T is asymptotically $I-$nonexpansive. Suppose that the set $F(S)$, fixed point of I, is nonempty. Let the set D, of best simultaneous K-approximates to y_1 and y_2, is nonempty compact and starshaped with respect to an element p in $F(I)$ and D is invariant under T. Assume further that T and I are commuting, T is uniformly asymptotically regular on D, I is affine with $I(D) = D$. Then D contains a $T-$ and $I-$invariant point.

Remark 4.3. As an application of Theorems 3.11 and 3.13, invariant best simultaneous approximation results similar to Theorem 4.1 can be established for C_p-commuting and Banach operator pair (T, I) which extend the recent results of Al-Thagafi [1], Al-Thagafi and Shahzad [2], Chen and Li [5], Habiniak [13], Hussain, O'Regan and Agarwal [15], Hussain and Rhoades [17], Jungck and Sessa [22], Khan et al. [23], Sahab, Khan and Sessa [33], Sahney and Singh [34], Singh [37, 38], Smoluk [40], Subrahmanyam [41] and Vijayraju [42] to ordered semi-convex structure $(X, F, \leq)$.

References

[1] M. A. Al-Thagafi, Common fixed point and best approximation, *J. Approx. Theory* **85**, (1996), 318-323.

[2] M. A. Al-Thagafi and N. Shahzad, Noncommuting selfmaps and invarient approximations, *Nonlinear Anal.* **64**, (2006), 2778-2786.

[3] I. Beg, D. R. Sahu and S. D. Diwan, Approximation of fixed points of uniformly R-subweakly commuting mappings, *J. Math. Anal. Appl.* **324**, (2006), 1105-1114.

[4] B. Brosowski, Fixpunktsatze in der Approximationstheorie, *Mathematica (Cluj)* **11**, (1969), 195-220.

[5] J. Chen and Z. Li, Common fixed points for Banach operator pairs in best approximation, *J. Math. Anal. Appl.* **336**, (2007), 1466-1475.

[6] E. W. Cheney, Application of fixed point theorems to approximation theory, *Theory of Approximations, Academic Press* (1976), 1-8.

[7] Lj. B. Ćirić, On a common fixed point theorem of a Gregus type, *Publ. Inst. Math.* **49**, (1991), 174-178.

[8] Lj. B. Ćirić, On Diviccaro, Fisher and Sessa open questions, *Arch. Math. (BRNO)* **29**, (1993), 145-152.

[9] M. L. Diviccaro, B. Fisher and S. Sessa, A common fixed point theorem of Gregus type, *Publ. Math. Debrecean* **34**, (1987), 83-89.

[10] M. Edelstein and R.C.O'Brien, Nonexpansive mappings, asymptotic regularity and successive approximations, *J. London Math Soc.* **17**, (1978), 547-554.

[11] K. Goeble and W. A. Kirk, A fixed point theorem for asymptotically nonexpansive mappings, *Proc. Amer. Math. Soc.* **35**, (1972), 171-174.

[12] S. P. Gudder, A general theory of convexity, *Rend. Sem. Mat. Milano* **49**, (1979), 89-96.

[13] L. Habiniak, Fixed point theorems and invarient approximations, *J. Approximation Theory* **56**, (1989), 241-244.

[14] T. L. Hicks and M. D. Humpheries, A note on fixed point theorems, *J. Approximation Theory* **34**, (1982), 221-225.

[15] N. Hussain, D. O'Regan and R. P. Agarwal, Common fixed point and invarient approximation results on non-starshaped domain, *Georgian Math. J.* **12**, (2005), 659-669.

[16] N. Hussain, H. K. Pathak and Satyaj Tiwari, Common fixed point theorems and best simultaneous approximation in ordered semi-convex structure, (2007) submitted.

[17] N. Hussain and B. E. Rhoades, C_q-commuting maps and invarient approximations, *Fixed point Theory and Appl.* **2006**, (2006), 1-9.

[18] G. Jungck, Commuting mapping and fixed points, *Amer. Math. Monthly* **83**, (1976), 261-263.

[19] G. Jungck, Compatible mappings and common fixed points, *Internat. J. Math. Math. Sci.* **9**, (1986), 771-779.

[20] G. Jungck, On a fixed point theorem of Fisher and Sessa, *Internat. J. Math. Math. Sci.* **13**, (1990), 497-500.

[21] G. Jungck and N. Hussain, Compatible maps and invarient approximations, *J. Math. Anal.* **325**, (2007), 1003-1012.

[22] G. Jungck and S. Sessa, Fixed point theorems in best approximation theory, *Math. Japon.* **42**, (1995), 249-252.

[23] A. R. Khan, N. Hussain and A. B. Thaheem, Application of fixed point theorems to invarient approximation, *Approx. Theory and Appl.* **16**, (2000), 48-55.

[24] V. Klee, Convexity of Chebyshev sets, *Math. Ann.* **142**, (1961), 292-304.

[25] G. Meinardus, Invaiauz Bei Lineaeu Approximation, *Arch.Rational Mech. Anal.* **14**, (1963), 301-303.

[26] P. D. Milman, On best simultaneous approximation in normed linear spaces, *J. Approximation Theory* **20**, (1977), 223-238.

[27] D. O'Regan and N. Hussain, Generalized I-contractions and pointwise R-subweakly commuting maps, *Acta Math. Sinica* **23(8)**, (2007), 1505-1508.

[28] D. O'Regan and N. Shahzad, Invariant approximations for generalized I-contractions, *Numerical Functional Analysis and Optimization* **26**, no. 4-5, (2005), 565575.

[29] R. P. Pant, Common fixed points of non-commuting mappings, *J. Math. Anal. Appl.* **88**, (1994), 436-440.

[30] H. K. Pathak, Y.J. Cho and S.M. Kang, An application of fixed point theorems in best approximation theory, *Internat. J. Math. Math. Sci.* **21**, (1998), 467-470.

[31] H. K. Pathak and N. Shahzad, Fixed points for generalized contractions and applications to control theory, *Nonlinear Anal.* **68**, (2007), 2181-2193.

[32] A. Petrusel, Starshaped and fixed points, Seminar on fixed point theory (Cluj-Napoca), *Stud.Univ."Babes-Bolyai"* **3**, (1987), 19-24.

[33] S. A. Sahab, M. S. Khan and S. Sessa, A result in best approximation theory, *J. Approx. Theory* **55**, (1988), 349-351.

[34] B. N. Sahney and S. P. Singh, On best simultaneous approximation, *Approximation Theory III, Academic Press* (1980), 783-789.

[35] S. Sessa, On a weak commutative condition in fixed point consideration, *Publ. Inst. Math. (Beograd)* **32**, (1982), 146-153.

[36] N. Shahzad, Invariant approximations and R-subweakly commuting maps, *J. Math. Anal. Appl.* **257**, (2001), 39-45.

[37] S. P. Singh, Application of fixed point theorems in approximation theory, *Applied Nonlinear Analysis, Academic Press* (1979), 389-394.

[38] S. P. Singh, Application of a fixed point theorem to approximation theory, *J. Approx. Theory* **25** (1979), 89-90.

[39] S. P. Singh, B. Watson and P. Srivastava, *Fixed Point Theory and Best Approximation: The KKM-map Principle*, Kluwer Academic Publishers 1997.

[40] A. Smoluk, Invarient approximations, *Mathematyka Stosowana* **17**, (1981), 17-22 (in Polish).

[41] P. V. Subrahmanyam, An application of a fixed point theorem to best approximations, *J. Approx. Theory* **20**, (1977), 165-172.

[42] P. Vijayraju, Applications of fixed point theorem to best simultaneous approximations, *Indian J. Pure Appl. Math.* **24(1)**, (1993), 21-26.

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