Bulletin of Mathematical analysis and Applications
ISSN: 1821-1291, URL: http://www.bmathaa.org
Volume 1, Issue 2, (2009), Pages 17-22

An ordinary differentail operator and its applications to certain classes of multivalently meromorphic functions *

H. Irmak, G. Tınaztepe, N. Tuneski & M. Şan

Abstract

In the present work, using an ordinary differential operator of order q ($q \in \mathbb{N}_0 := \{0, 1, 2, \cdots\}$), a general class of meromorphic functions which are analytic and multivalent in the punctured unit disk is firstly introduced. Sufficient condition for a function in the related class is then obtained. Several useful consequences of the main results are also pointed out.

1 Introduction and Definitions

Let $\mathcal{M}(p)$ denote the class of functions f of the following form

$$f(z) = z^{-p} + \sum_{k=p+1}^{\infty} a_k z^k \quad (p \in \mathbb{N} := \mathbb{N}_0 \setminus \{0\}; a_k \in \mathbb{C}), \tag{1}$$

which are *analytic* and *multivalently meromorphic* in the *punctured unit disk*

$$\mathbb{D} := \{z \ : \ z \in \mathbb{C} \text{ and } 0 < |z| < 1\} = \mathbb{U} - \{0\},$$

where $\mathbb{C}$ denotes the set of complex numbers.

Also let $\mathcal{MS}(p;\alpha)$ and $\mathcal{MC}(p;\alpha)$ be the well-known subclasses of the class $\mathcal{M}(p)$ consisting, respectively, of functions which are *multivalently meromorphic starlike of order* α and *multivalently meromorphic convex of order* α in $\mathbb{D}$, *where* $0 \leq \alpha < p$ ($p \in \mathbb{N}$). (See [2], [3] and [7] for further details).

Upon differentiating both sides of (1), q-times with respect to the complex variable z, one easily obtains the following (ordinary) differential operator

$$f^{(q)}(z) = \frac{(p+q-1)!}{(p-1)!}(-1)^q z^{-p-q} + \sum_{k=p+1}^{\infty} \frac{k!}{(k-q)!} a_k z^{k-q}, \tag{2}$$

** Mathematics Subject Classifications:* 30C45.

Key words: Multivalently meromrophic function, multivalently meromrophic starlikeness, multivalently meromrophic convexity, ordinary differential operator, differential inequalities, Jack's lemma.

Submitted May, 2009. Published June, 2009.

where $f \in \mathcal{M}(p)$, $p > q$, $p \in \mathbb{N}$, and $q \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$.

The operator defined by (2) has been studied earlier by several researchers (see, for example, [1], [4] and [5]).

Using the ordinary differential operator defined in (2), we now introduce a general subclass $\Omega_q^\lambda(p;\alpha)$ of the class of multivalently meromorphic functions $\mathcal{M}(p)$, which consists of functions f satisfying the following inequality:

$$\Re e \left(\frac{z\mathcal{F}'(z)}{\mathcal{F}(z)} \right) < -\alpha \quad (z \in \mathbb{D}; 0 \le \alpha < p+q; p \in \mathbb{N}; q \in \mathbb{N}_0),$$

where, here and throughout this paper, the above function $\mathcal{F}$ is defined by

$$\mathcal{F}(z) = (1-\lambda) f^{(q)}(z) + \lambda z f^{(1+q)}(z) \quad (0 \le \lambda \le 1; p > q; p \in \mathbb{N}; q \in \mathbb{N}_0). \qquad (3)$$

One can note that by choosing specific values of p, q and/or λ we receive some well known classes of multivalently meromorphic functions. Namely,

$$\begin{aligned}
&\Omega_0^\lambda(p;\alpha) =: \mathcal{V}_\lambda(p;\alpha) \quad (0 \le \lambda \le 1; 0 \le \alpha < p; p \in \mathbb{N}),\\
&\Omega_0^\lambda(1;\alpha) =: \mathcal{W}_\lambda(\alpha) \quad (0 \le \lambda \le 1; 0 \le \alpha < 1),\\
&\Omega_q^0(p;\alpha) =: \mathcal{A}_q(p;\alpha) \quad (0 \le \alpha < p+q; p \in \mathbb{N}; q \in \mathbb{N}_0),\\
&\Omega_q^1(p;\alpha) =: \mathcal{B}_q(p;\alpha) \quad (0 \le \alpha < p+q; p \in \mathbb{N}; q \in \mathbb{N}_0),\\
&\Omega_0^0(p;\alpha) \equiv \mathcal{S}(p;\alpha) \ (0 \le \alpha < 1),\\
&\Omega_0^1(p;\alpha) \equiv \mathcal{K}(p;\alpha) \ (0 \le \alpha < 1),\\
&\Omega_0^0(1;\alpha) \equiv \mathcal{S}(\alpha) \quad (0 \le \alpha < 1),\\
&\Omega_0^1(1;\alpha) \equiv \mathcal{K}(\alpha) \quad (0 \le \alpha < 1).
\end{aligned}$$

In this investigation we obtain sufficient condition for a function $f \in \mathcal{M}(p)$ to be in $\Omega_q^\lambda(p;\alpha)$. In addition we give several corollaries of the main result. For that purpose we will use the method of the differential inequalities and the well-known assertion of Jack [6].

Lemma 1.1 *Let the function w be non-constant and analytic in $\mathbb{U}$ with $w(0) = 0$. If $|w(z)|$ attains its maximum value on the circle $|z| = r < 1$ at a point z_0, then $z_0 w'(z_0) = c w(z_0)$, where c is real number and $c \ge 1$.*

2 The Main result

Theorem 2.1 *Let the functions f and $\mathcal{F}$ be defined by (1) and (3), respectively. Also let the function $\mathcal{H}$ be defined by*

$$\mathcal{H}(z) := \left(1 + q + \frac{z\mathcal{F}''(z)}{\mathcal{F}'(z)} + p\right) \cdot \left(q + \frac{z\mathcal{F}'(z)}{\mathcal{F}(z)} + p\right)^{-1} \quad (z \in \mathbb{D}). \qquad (4)$$

If $\mathcal{H}$ satisfies

$$\Re e \{\mathcal{H}(z)\} > 1 - \beta \qquad (5)$$

for all $z \in \mathbb{D}$, *then* $f \in \Omega_q^{\lambda}(p;\alpha)$ *and*

$$\Re e\left\{[\mathcal{H}(z)]^{-1}\right\} < (1-\beta)^{-1} \quad (z \in \mathbb{D}), \tag{6}$$

where $\beta := [2(p+q)-\alpha]^{-1}$ *and* $0 \leq \alpha < p+q$.

Proof.

Let f be of the form (1) Then, in view of (3), one easily obtains that

$$\frac{z\mathcal{F}'(z)}{\mathcal{F}(z)} = \frac{(p+q)\phi(q,\lambda;p)(-1)^{1+q} + \sum_{k=p+1}^{\infty}(k-q)\psi(q,\lambda;k)a_k z^{k+p}}{\phi(q,\lambda;p)(-1)^{q} + \sum_{k=p+1}^{\infty}\psi(q,\lambda;k)a_k z^{k+p}},$$

where

$$\phi(q,\lambda;p) := \frac{(p+q-1)![1-\lambda(p+q+1)]}{(p-1)!}$$

and

$$\psi(q,\lambda;k) := \frac{k![1+\lambda(k-q-1)]}{(k-q)!},$$

$$(k \geq p+1; p > q; p \in \mathbb{N}; q \in \mathbb{N}_0).$$

Now let us define a function $w(z)$ with

$$-q - \frac{z\mathcal{F}'(z)}{\mathcal{F}(z)} - p = (p+q-\alpha)w(z) \quad (w(z) \neq 0). \tag{7}$$

It is clear that the function w is both analytic in $\mathbb{U}$ with $w(0) = 0$ and meromorphic in $\mathbb{D}$. We also find from (7) that

$$\begin{aligned} &-1-q-\frac{z\mathcal{F}''(z)}{\mathcal{F}'(z)} - p \\ &\quad = (p+q-\alpha)w(z)\left[1 - \frac{zw'(z)}{w(z)} \cdot \frac{1}{p+q+(p+q-\alpha)w(z)}\right]. \end{aligned} \tag{8}$$

By using (7) and (8), we easily arrive at

$$\mathcal{G}(z) := 1 - \mathcal{H}(z) = \frac{zw'(z)}{w(z)} \cdot \frac{1}{p+q+(p+q-\alpha)w(z)}, \tag{9}$$

where $\mathcal{H}$ is defined by (4).

Now let suppose that there exists a point $z_0 \in \mathbb{U}$ such that $max\{|w(z)| : |z| \leq |z_0|\} = |w(z_0)| = 1$ and $w(z_0) = e^{i\theta}$ $(0 \leq \theta < 2\pi)$. By applying Jack lemma we then have $z_0 w'(z_0) = c w(z_0)$ $(c \geq 1)$. Thus, in view of (9), we obtain

$$\Re e\left\{\mathcal{G}(z_0)\right\} = c\ \Re e[(p+q+(p+q-\alpha)e^{i\theta})^{-1}] \geq [2(p+q)-\alpha]^{-1} = \beta$$

or, equivalently,

$$\Re e\left\{\mathcal{H}(z_0)\right\} \leq 1-\beta,$$

which is a contradiction to (5), *where* β is given by in the statement of the Theorem 2.1. Hence, we conclude that $|w(z)| < 1$ for all z in $\mathbb{U}$, and the definition (7) immediately yields the inequality

$$\left| q + \frac{z\mathcal{F}'(z)}{\mathcal{F}(z)} + p \right| < p + q - \alpha,$$

which implies

$$\Re e \left(-\frac{z\mathcal{F}'(z)}{\mathcal{F}(z)} \right) > \alpha$$

$$(z \in \mathbb{D}; 0 \leq \alpha < p + q; p \in \mathbb{N}; q \in \mathbb{N}_0),$$

that is $f \in \Omega_q^\lambda(p; \alpha)$.

At the end (5) implies (6) since $1 - \beta > 0$ because of $2(p+q) - \alpha > 1$. This completes the proof of the Theorem 1.2.

3 Certain consequences of the main result

As we indicated in the Section 1, i.e., by fixing some specific admissible values of parameters p, q and/or λ, from Theorem 2.1 we easily receive many interesting results concerning the functions f in the classes $\mathcal{V}_\lambda(p;\alpha)$, $\mathcal{W}_\lambda(\alpha)$, $\mathcal{A}_q(p;\alpha)$, $\mathcal{B}_q(p;\alpha)$, $\mathcal{S}(p;\alpha)$, $\mathcal{K}(p;\alpha)$, $\mathcal{S}(\alpha)$ and also $\mathcal{K}(\alpha)$. Here we only state some of them as corollaries.

By taking $q = 0$ in Theorem 2.1, we first obtain the following corollary.

Corollary 3.1 *Let* $f \in \mathcal{M}(p)$, $p \in \mathbb{N}$, $0 \leq \alpha < p$, $0 \leq \lambda \leq 1$, *and also let* $\mathcal{F}(z) = (1-\lambda)f(z) + \lambda z f'(z)$. *If*

$$\Re e \left(\frac{1 + \frac{z\mathcal{F}''(z)}{\mathcal{F}'(z)} + p}{\frac{z\mathcal{F}'(z)}{\mathcal{F}(z)} + p} \right) > 1 - \beta$$

for all $z \in \mathbb{D}$, *then* $f \in \mathcal{V}_\lambda(p;\alpha)$ *and*

$$\Re e \left(\frac{\frac{z\mathcal{F}'(z)}{\mathcal{F}(z)} + p}{1 + \frac{z\mathcal{F}''(z)}{\mathcal{F}'(z)} + p} \right) < \frac{1}{1-\beta} \quad (z \in \mathbb{D}),$$

where $\beta := 1/(2p - \alpha)$.

By setting $q = 0$ and $\lambda = 0$ in Theorem 2.1, we receive

Corollary 3.2 *Let* $f \in \mathcal{M}(p)$, $p \in \mathbb{N}$, *and* $0 \leq \alpha < p$. *If*

$$\Re e \left(\frac{1 + \frac{zf''(z)}{f'(z)} + p}{\frac{zf'(z)}{f(z)} + p} \right) > 1 - \beta$$

for all $z \in \mathbb{D}$, *then* $f \in \mathcal{S}(p;\alpha)$ *and*

$$\Re e\left(\frac{\frac{zf'(z)}{f(z)}+p}{1+\frac{zf''(z)}{f'(z)}+p}\right)<\frac{1}{1-\beta}\quad (z\in\mathbb{D}),$$

where $\beta := 1/(2p-\alpha)$.

By letting $q = 0$ and $\lambda = 1$ in Theorem 2.1, we have

Corollary 3.3 *Let* $f \in \mathcal{M}(p)$, $p \in \mathbb{N}$, *and* $0 \leq \alpha < p$. *If*

$$\Re e\left(\frac{1+\frac{z(zf'(z))''}{(zf'(z))'}+p}{\frac{(zf'(z))'}{f'(z)}+p}\right)>1-\beta$$

for all $z \in \mathbb{D}$, *then* $f(z) \in \mathcal{K}(p;\alpha)$ *and*

$$\Re e\left(\frac{\frac{(zf'(z))'}{f'(z)}+p}{1+\frac{z(zf'(z))''}{(zf'(z))'}+p}\right)<\frac{1}{1-\beta}\quad (z\in\mathbb{D}),$$

where $\beta := 1/(2p-\alpha)$.

Acknowledgements The work on this paper was supported by the Joint Research Project financed by The Ministry of Education and Science of the Republic of Macedonia (MESRM) (Project No.17-1383/1) and The Scientific and Technical Research Council of Turkey (TUBITAK) (Project No. TBGA-U-105T056).

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H. Irmak & M. Şan
Department of Mathematics
Faculty of Science and Letters
Çankırı Karatekin University
Tr-18100, Çankırı, Turkey
e-mails: hirmak70@gmail.com, mufitsan@hotmail.com

N. Tuneski
Faculty of Mechanical Engineering
Karpoš II b.b., 1000 Skopje, Republic of Macedonia
e-mail: nikolat@mf.ukim.edu.mk

G. Tinaztepe
Vocational School of Technical Sciences
Akdeniz University
Tr-07058, Antalya, Turkey
e-mail: gtinaztepe@akdeniz.edu.tr