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Some properties of paranormal and hyponormal operators *

N. L. Braha, M.Lohaj, F.H. Marevci, Sh.Lohaj

Abstract

In this article we will give some properties of paranormal and hyponormal operators. Exactly we will give some conditions which are generalization of concepts of paranormal, hyponormal, N-paranormal, N-hyponormal operators.

1 Introduction

Let us denote by H the complex Hilbert space and with $B(H)$ the space of all bounded linear operators defined in Hilbert space H. In the following we will mention some known classes of operators defined in Hilbert space H. Let T be an operator in $B(H)$. The operator T is called normal if it satisfies the following condition: $T^*T = TT^*$. The operator T is called quasi-normal if: $T(T^*T) = (T^*T)T$, it is hyponormal if: $T^*T \geq TT^*$, which is equivalent to the condition: $||T^*(x)|| \leq ||T(x)||$, for all x in H. We say that an operator T is quasi-hyponormal if the following condition: ${T^*}^2T^2 \geq (T^*T)^2$ holds, and the last one is equivalent with: $||T^*T(x)|| \leq ||T^2(x)||$, for all x in H. In paper [1], some properties of *-paranormal operators are given. One of that is necessary and sufficient condition under which the operator T is *-paranormal. For an operator T we say that it belongs to the class of *-paranormal operators if : $||T^*(x)||^2 \leq ||T^2(x)||$, for every unit vector x in H. This condition is equivalent to the following one: An operator T belongs to the class of *-paranormal operators if and only if: ${T^*}^2T^2 - 2\lambda TT^* + \lambda^2 \geq 0$, for all $\lambda \in \mathbb{R}$. In paper [2] is defined the class of M^*-paranormal operators. T is M^*-paranormal, if $||T^*(x)||^2 \leq M||T^2(x)||$ for every unit vector x in H. Also in [2] is given the necessary and sufficient condition under which an operator T is from the class of M^*-paranormal operators and that condition is $M^2{T^*}^2T^2 - 2\lambda TT^* + \lambda^2 \geq 0$, for all $\lambda \in \mathbb{R}$. In this paper we will study some properties of some new classes which are generalization of paranormal, hyponormal, quasi-hyponormal, N- hyponormal, N- paranormal and N-*paranormal operators . We say that an operator T

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is of $(M,k)_N$ class if $NT^{*k}T^k \geq (T^*T)^k$ for $k \geq 2$ and a fixed constant $N > 0$ and we say that T is of $(M,k)_N^*$ class if $NT^{*k}T^k \geq (TT^*)^k$ for $k \geq 1$ and a fixed constant $N > 0$.

For an operator $T \in B(H)$ we will say that is N- quasi-hyponormal, if $||T^*T(x)|| \leq N||T^2(x)||$, T is N- hyponormal, if $||T^*(x)|| \leq N||T(x)||$. T is N-paranormal, if $N||T^2(x)|| \geq ||T(x)||^2$ and T is N-*paranormal, if $N||T^2(x)|| \geq ||T^*(x)||^2$ for all unit vectors x in H. All notations which are not mentioned here are same like in [5].

2 Shift Operators

Let T be unilateral shift operator then it is easy to proof the following results.

Lemma 2.1 *If T is unilateral shift operator, then it is quasi-normal operator.*

Lemma 2.2 *If T is bilateral shift operator, then it is quasi-normal operator.*

The above facts also are valid for unilateral weighted and bilateral weighted shift.

Lemma 2.3 *If T is unilateral weighted shift, with weighted sequence (α_n), $\alpha_n \neq 0, n \in \mathbb{N}$, then it is quasi-normal if and only if $|\alpha_n| = |\alpha_{n+1}|$.*

Lemma 2.4 *If T is bilateral weighted shift, with weighted sequence (α_n), $\alpha_n \neq 0, n \in \mathbb{N}$, then it is quasi-normal if and only if $|\alpha_n| = |\alpha_{n+1}|$.*

In the sequel we give one example, where it is shown that there exists operators from the class of *paranormal operators which are not hyponormal operators. Firstly, we give the following lemma the proof of which is trivial and we omit it.

Lemma 2.5 *Let T be a bilateral weighted shift with weighted sequence (α_n). T is *paranormal if and only if $|\alpha_{n-1}|^2 \leq |\alpha_n||\alpha_{n+1}|$.*

Example 2.6 *Let $T \in B(H)$ be a bilateral weighted shift with weighted sequence (α_n) given as follows:*

$$\alpha_n = \begin{cases} \frac{1}{2} & \text{for } n \leq -1 \\ 1 & \text{for } n = 0 \\ \frac{1}{2} & \text{for } n = 1 \\ 2 & \text{for } n = 2 \\ \frac{1}{4} & \text{for } n = 3 \\ 64 & \text{for } n \geq 4 \end{cases}$$

After some calculations it follows that T is * paranormal operator and it is not hyponormal operator.

3 Paranormal and Hyponormal Operators

In this section we will show some properties of paranormal and hyponormal operators.

Proposition 3.1 *Let $T \in B(H)$, then T is hyponormal operator if and only if $T^*T + 2\lambda TT^* + \lambda^2 T^*T \geq 0$, for all $\lambda \in \mathbb{R}$.*

Proof. Let $\lambda \in \mathbb{R}$ and $x \in H$ be given. T is hyponormal operator if and only if

$$||T^*(x)|| \leq ||T(x)|| \Leftrightarrow 4||T^*(x)||^4 - 4||T(x)||^2 \cdot ||T(x)||^2 \leq 0 \Leftrightarrow$$

$$||T(x)||^2 + 2\lambda||T^*(x)||^2 + \lambda^2||T(x)||^2 \geq 0 \Leftrightarrow$$

$$(T(x), T(x)) + 2\lambda(T^*(x), T^*(x)) + \lambda^2(T(x), T(x)) \geq 0$$

$$(T^*T(x), x) + 2\lambda(TT^*(x), x) + \lambda^2(T^*T(x), x) \geq 0 \Leftrightarrow$$

$$((T^*T + 2\lambda TT^* + \lambda^2 T^*T)(x), x) \geq 0 \Leftrightarrow T^*T + 2\lambda TT^* + \lambda^2 T^*T \geq 0.$$

In the next proposition we show some generalized conditions under which an operator is quasi-*paranormal.

Proposition 3.2 *Let $T \in B(H)$. Then:*

$$||T^*T(x)||^2 \leq ||T^n(x)|| \cdot ||T(x)|| \Leftrightarrow {T^*}^n T^n + 2\lambda(T^*T)^2 + \lambda^2 T^*T \geq 0,$$

$n \in \mathbb{N}$, $\lambda \in \mathbb{R}$ and $x \in H$.

Proof. Let $n \in \mathbb{N}$, $\lambda \in \mathbb{R}$ and $x \in H$. Then:

$${T^*}^n T^n + 2\lambda(T^*T)^2 + \lambda^2 T^*T \geq 0 \Leftrightarrow ({T^*}^n T^n + 2\lambda(T^*T)^2 + \lambda^2 T^*T)(x), x) \geq 0 \Leftrightarrow$$

$$(({T^*}^n T^n)(x), x) + 2\lambda((T^*T)^2(x), x) + \lambda^2((T^*T)(x), x) \geq 0 \Leftrightarrow$$

$$(T^n(x), T^n(x)) + 2\lambda((T^*T)(x), (T^*T)(x)) + \lambda^2(T(x), T(x)) \geq 0 \Leftrightarrow$$

$$||T(x)||^2\lambda^2 + 2\lambda||T^*T(x)||^2 + ||T^n(x)||^2 \geq 0 \Leftrightarrow 4||T^*T(x)||^4 \leq 4||T^n(x)||^2||T(x)||^2$$

$$\Leftrightarrow ||T^*T(x)||^2 \leq ||T^n(x)||||T(x)||.$$

Corollary 3.3 *If $n = 3$, we get the following relation $||T^*T(x)||^2 \leq ||T^3(x)|| \cdot ||T(x)|| \Leftrightarrow {T^*}^3 T^3 + 2\lambda(T^*T)^2 + \lambda^2 T^*T \geq 0$, which is the definition of the quasi *-paranormal operator.*

Proposition 3.4 *If T is unilateral weighted shift operator with weighted sequence (α_n), then it is quasi-hyponormal if and only if $|\alpha_n| \leq |\alpha_{n+1}|, \alpha_n \neq 0, n \in \mathbb{N}$.*

Proof. Let us denote by (e_n) the orthonormal basis in H. Then we get the following:

$$||T^*T(e_n)|| = ||T^*(\alpha_n e_{n+1})|| = ||\alpha_n T^*(e_{n+1})|| = ||\alpha_n \overline{\alpha_n} e_n|| = |\alpha_n|^2. \qquad (1)$$

On the other hand we have:

$$||T^2(e_n)|| = ||T(\alpha_n e_{n+1})|| = |\alpha_n| \cdot |\alpha_{n+1}|. \qquad (2)$$

Now, since T is quasi-hyponormal we have

$$T^{*2}T^2 \geq (T^*T)^2 \Leftrightarrow ||T^*T(x)|| \leq ||T^2(x)||, \qquad (3)$$

for every $x \in H$. Now the proof of the proposition follows from relations (1), (2) and (3).

Proposition 3.5 *Let T be an operator in $B(H)$. The following relation*

$$T^{*k}T^k \geq (T^*T)^k \Leftrightarrow \left\|(T^*T)^{\frac{k}{2}}(x)\right\| \leq ||T^k(x)||,$$

holds for every $x \in H$ and $k \geq 2$.

Proof.

$$T^{*k}T^k \geq (T^*T)^k \Leftrightarrow T^{*k}T^k - (T^*T)^k \geq 0$$

$$\Leftrightarrow ((T^{*k}T^k - (T^*T)^k)(x), x) \geq 0, \text{for all} \quad x \in H$$

$$\Leftrightarrow (T^{*k}T^k(x), x) - ((T^*T)^k(x), x) \geq 0, \text{for all} \quad x \in H$$

$$\Leftrightarrow (T^k(x), T^k(x)) - ((T^*T)^{\frac{k}{2}}(x), (T^*T)^{\frac{k}{2}}(x)) \geq 0, \text{for all} \quad x \in H$$

$$\Leftrightarrow \left\|(T^*T)^{\frac{k}{2}}(x)\right\| \leq ||T^k(x)||, \text{for all} \quad x \in H.$$

Corollary 3.6 *By Proposition 3.5, for $k = 2$ we have that T is quasi-hyponormal operator.*

Remark 3.7 *In [4] was defined the class of (M, k)- operators, by the relation $T^{*k}T^k \geq (T^*T)^k$, for $k \geq 2$. From Proposition 3.5, it follows that the operator T belongs to the class (M, k) if and only if*

$$\left\|(T^*T)^{\frac{k}{2}}(x)\right\| \leq ||T^k(x)||,$$

holds for every $x \in H$ and $k \geq 2$.

In the following proposition we will give generalized necessary and sufficient conditions under which an operator $T \in B(H)$ is hyponormal.

Proposition 3.8 *Let T be operator from $B(H)$. The following relation*

$$T^{*k}T^k \geq (TT^*)^k \Leftrightarrow \left\|(TT^*)^{\frac{k}{2}}(x)\right\| \leq ||T^k(x)||,$$

holds for every $x \in H$ and $k \geq 1$.

Proof.

$$T^{*k}T^k \geq (TT^*)^k \Leftrightarrow T^{*k}T^k - (TT^*)^k \geq 0$$
$$\Leftrightarrow ((T^{*k}T^k - (TT^*)^k)(x), x) \geq 0, \text{for all} \quad x \in H$$
$$\Leftrightarrow (T^{*k}T^k(x), x) - ((TT^*)^k(x), x) \geq 0, \text{for all} \quad x \in H$$
$$\Leftrightarrow (T^k(x), T^k(x)) - ((TT^*)^{\frac{k}{2}}(x), (TT^*)^{\frac{k}{2}}(x)) \geq 0, \text{for all} \quad x \in H$$
$$\Leftrightarrow \left\|(TT^*)^{\frac{k}{2}}(x)\right\| \leq ||T^k(x)||, \text{for all} \quad x \in H.$$

Remark 3.9 *An operator $T \in B(H)$ belongs to class $(M,k)^*$ (see [4]) if and only if $T^{*k}T^k \geq (TT^*)^k$. From Proposition 3.8 it follows that an operator $T \in B(H)$ belongs to class $(M,k)^*$ if and only if $\left\|(TT^*)^{\frac{k}{2}}(x)\right\| \leq ||T^k(x)||$, holds for every $x \in H$ and $k \geq 1$.*

From Proposition 3.8 it follows that:

Corollary 3.10 *Let $T \in B(H)$ and $k = 1$, then it follows that T is a hyponormal operator.*

In the next proposition we give some necessary and sufficient condition under which an operator is a quasi-hyponormal.

Proposition 3.11 *Let T be operator from $B(H)$. Then the following relation*

$$||(T^*T)^n(x)|| \leq ||T^{2n}(x)|| \Leftrightarrow T^{*2n}T^{2n} + 2\lambda(T^*T)^{2n} + \lambda^2 T^{*2n}T^{2n} \geq 0,$$

holds, for every $n \in \mathbb{N}$, $\lambda \in \mathbb{R}$ and $x \in H$.

Proof. Let $n \in \mathbb{N}$, $\lambda \in \mathbb{R}$ and $x \in H$. Then we get:

$$||(T^*T)^n(x)|| \leq ||T^{2n}(x)|| \Leftrightarrow 4||(T^*T)^n(x)||^4 \leq 4||T^{2n}(x)||^2 \cdot ||T^{2n}(x)||^2 \Leftrightarrow$$
$$||T^{2n}(x)||^2 + 2\lambda||(T^*T)^n(x)||^2 + \lambda^2||T^{2n}(x)||^2 \geq 0 \Leftrightarrow$$
$$(T^{2n}(x), T^{2n}(x)) + 2\lambda((T^*T)^n(x), (T^*T)^n(x)) + \lambda^2(T^{2n}(x), T^{2n}(x)) \geq 0$$
$$\Leftrightarrow T^{*2n}T^{2n} + 2\lambda(T^*T)^{2n} + \lambda^2 T^{*2n}T^{2n} \geq 0.$$

Corollary 3.12 *Let $T \in B(H)$. The following relation*

$$T^{*2n}T^{2n} \geq (T^*T)^{2n} \Leftrightarrow T^{*2n}T^{2n} + 2\lambda(T^*T)^{2n} + \lambda^2 T^{*2n}T^{2n} \geq 0,$$

holds for every $n \in \mathbb{N}$, $\lambda \in \mathbb{R}$ and $x \in H$.

Remark 3.13 *If we take $n = 1$ in the above corollary we have that T is a quasi-hyponormal operator.*

Proposition 3.14 *Let $T \in (M, 2)$, then T is a paranormal operator.*

Proof. Let $T \in (M, 2)$, then we get

$$T^{*2}T^2 \geq (T^*T)^2 \Rightarrow (T^{*2}T^2 - (T^*T)^2(x), x) \geq 0,$$

for all $x \in H$.

$$\Rightarrow (T^{*2}T^2(x), x) - ((T^*T)^2(x), x) \geq 0 \Rightarrow (T^2(x), T^2(x)) - ((T^*T)(x), (T^*T)(x)) \geq 0$$

$$\Rightarrow ||(T^*T)(x)|| \leq ||T^2(x)||, \text{for all} \quad x \in H. \tag{4}$$

On the other hand:

$$||T(x)||^2 = |(Tx, Tx)| = |(T^*Tx, x)| \leq ||T^*Tx|| \cdot ||x||. \tag{5}$$

Now from relations (4) and (5) follows that T is a paranormal operator.

Corollary 3.15 *Every quasi-hyponormal operator is a paranormal operator.*

The following lemma will be very useful to demonstrate the necessary and sufficient conditions under which an operator $T \in B(H)$ is a quasi-hyponormal.

Lemma 3.16 *Let $A, B, T \in B(H)$, such that $\overline{R(T)} = H$. Then the following relation*

$$A \geq B \Leftrightarrow T^*AT \geq T^*BT,$$

holds.

Proof. Let $A \geq B$. Then we get the following relation:

$$(T^*AT(x), x) = (AT(x), T(x)).$$

For $y = T(x) \in H$, it follows that:

$$(A(y), y) \geq (B(y), y) = (BT(x), T(x)) = (T^*BT(x), x).$$

On the other hand let $T^*AT \geq T^*BT$. Because T has a dense range in H, we get the following relation:

$$(T^*AT(x), x) \geq (T^*BT(x), x) \Rightarrow (AT(x), T(x)) \geq (BT(x), T(x)),$$

for $y = T(x)$,

$$\Rightarrow (A(y), y) \geq (B(y), y) \Rightarrow A \geq B.$$

Remark 3.17 *Above Lemma is an especial case of the known result given in [6] (see Lemma 8).*

Proposition 3.18 *Let $T \in B(H)$, be a quasi-normal operator in H. Then it is quasi-hyponormal if and only if $T \in (M,k)$, for $k \geq 2$.*

Proof. Let $T \in (M,k)$, for $k \geq 2$, it means that $T \in (M,2)$. One side of proposition follows directly from definition of classes (M,k). Let us prove the other side of the proposition. Consider that operator T is quasi-hyponormal. We will construct our proof following mathematical induction. For $k = 2$, it follows from definition of quasi-hyponormal operators. Let us consider that our claim is valid for $k = n$ and we will prove it for $k = n+1$. We have:

$$T^{*n+1}T^{n+1} = T^*(T^{*n}T^n)T \geq T^*(T^*T)^nT = T^*\left[\underbrace{(T^*T)\cdot(T^*T)\cdots(T^*T)}_{n-\text{times}}\right]T$$

$$= T^*T\left[\underbrace{(T^*T)\cdot(T^*T)\cdots(T^*T)}_{n-\text{times}}\right] = (T^*T)^{n+1},$$

by which we have proved the proposition.

4 N-paranormal and N-hyponormal operators

In this section we will introduce some other classes of operators which are related to the previous classes.

We say that an operator T is of $(M,k)_N$ class if $NT^{*k}T^k \geq (T^*T)^k$ for $k \geq 2$ and a fixed constant $N > 0$ and we say that T is of $(M,k)^*_N$ class if $NT^{*k}T^k \geq (TT^*)^k$ for $k \geq 1$ and a fixed constant $N > 0$. For an operator $T \in B(H)$ we will say that is N- quasi-hyponormal, if $N||T^2(x)|| \geq ||T^*T(x)||$, T is N- hyponormal, if $||T^*(x)|| \leq N||T(x)||$. T is N- paranormal, if $N||T^2(x)|| \geq ||T(x)||^2$ and T is N-*paranormal, if $N||T^2(x)|| \geq ||T^*(x)||^2$ for all unit vectors x in H. T is quasi-normal, if $T(T^*T) = (T^*T)T$.

Proposition 4.1 *An operator $T \in B(H)$ belongs to the class $(M,k)_N$ if and only if*

$$\left\|(T^*T)^{\frac{k}{2}}(x)\right\| \leq \sqrt{N}||T^k(x)||,$$

holds for every $x \in H$ and $k \geq 2$.

Proof. The proof of the Proposition is similar to that of Proposition 3.5.

Proposition 4.2 *An operator $T \in B(H)$ belongs to the class $(M,k)^*_N$ if and only if*

$$\left\|(TT^*)^{\frac{k}{2}}(x)\right\| \leq \sqrt{N}||T^k(x)||,$$

holds for every $x \in H$ and $k \geq 1$.

Proof. The proof of the Proposition is similar to that of Proposition 3.8.

Proposition 4.3 *Let $T \in B(H)$. Then $T \in (M,2)_N$ if and only if T is $\sqrt{N}$-quasi-hyponormal operator.*

Proof. Let $T \in (M,2)_N$, then $N{T^*}^2T^2 \geq (T^*T)^2$, holds. Respectively

$$(N{T^*}^2T^2 - (T^*T)^2(x), x) \geq 0 \Leftrightarrow (N{T^*}^2T^2(x), x) - ((T^*T)^2(x), x) \geq 0$$

for all $x \in H$.

$$\Leftrightarrow \sqrt{N}||T^2(x)|| \geq ||T^*T(x)||,$$

from which follows that T is $\sqrt{N}$-quasi-hyponormal operator.

Corollary 4.4 *If $T \in (M,2)_N$ then T is $\sqrt{N}$-paranormal operator.*

Proof. From Proposition 4.3 it follows that the following relation holds:

$$\sqrt{N}||T^2(x)|| \geq ||T^*T(x)||,$$

for all $x \in H$. And in what follows without lose of generality we can take that $||x|| = 1$. From this we have

$$\sqrt{N}||T^2(x)|| \geq ||T^*T(x)|| \geq ||T(x)||^2,$$

hence T is $\sqrt{N}$-paranormal operator.

Proposition 4.5 *If $T \in (M,2)^*_N$ then T is $\sqrt{N}$-*paranormal operator.*

Proof. Suppose $T \in (M,2)^*_N$, then we have: $N{T^*}^2T^2 \geq (TT^*)^2$, respectively

$$(N{T^*}^2T^2 - (TT^*)^2(x), x) \geq 0 \Rightarrow N({T^*}^2T^2(x), x) - ((TT^*)^2(x), x) \geq 0$$

for all $x \in H$,

$$\sqrt{N}||T^2(x)|| \geq ||TT^*(x)||.$$

In what follows without lose of generality we can take that $||x|| = 1$. From this we have

$$\sqrt{N}||T^2(x)|| \geq ||TT^*(x)|| \geq ||T^*(x)||^2.$$

Therefore T is $\sqrt{N}$-*paranormal operator.

Proposition 4.6 *Let $T \in (M,k)_N$ and let T^k be a compact operator for some $k \in \mathbb{N}$, then it follows that T is compact too.*

Proof. From the fact that $T \in (M,k)_N$ for $k \geq 2$, following proposition 4.1, we have:

$$\left\|(T^*T)^{\frac{k}{2}}(x)\right\| \leq \sqrt{N}||T^k(x)||. \tag{6}$$

Let $(x_n) \in H$ be weakly convergent sequence with limit 0 in H. From compactness of T^k and relation (6) we get the following relation:

$$\left\|(T^*T)^{\frac{k}{2}}(x_n)\right\| \to 0, n \to \infty.$$

From the last relation it follows that T^*T is compact operator, respectively T is compact(see [3]).

Proposition 4.7 *Let $T \in (M,k)_N^*$ and let T^k be compact operator for some $k \in \mathbb{N}$, then it follows that T is compact too.*

Proof. The proof is similar with the proof of the proposition 4.6 .

Example 4.8 *An operator from $(M,k)_{\frac{1}{N}}^*$ which is not in $(M,k)_{\frac{1}{N}}$. We will construct our example following ideas given in example 3.5 in [4]. Let us consider that $N > 3$ and let us suppose that T is weighted shift operator in l_2, given by relation: $T : l_2 \to l_2$, such that $T(x_1, x_2, \cdots,) = (0, 0, \alpha_1 x_1, \alpha_2 x_2, \cdots,)$, $T^*(x_1, x_2, \cdots,) = (\alpha_1 x_3, \alpha_2 x_4, \cdots,)$, with weights $\alpha_1 = \alpha_2 = \frac{3\sqrt{N}}{2}$, $\alpha_3 = \alpha_4 = N$, $\alpha_5 = \alpha_6 = N \cdot \sqrt{N}$, $\alpha_7 = \alpha_8 = N^2$, $\alpha_9 = \alpha_{10} = N^2 \cdot \sqrt{N}, \cdots$.*

Let us denote by $e_n = (0, 0, \cdots, \underbrace{1}_{n\text{-position}}, 0, \cdots)$ the orthogonal basis. Then we have

$$\left(\frac{1}{N}T^{*2}T^2 - (TT^*)^2\right)(e_n) = \left(\frac{1}{N}\alpha_n^2\alpha_{n+2}^2 - \alpha_{n-2}^4\right)(e_n) \geq 0, \tag{7}$$

for $n > 2$. Further, for $n = 1, 2$ we have:

$$\left(\frac{1}{N}T^{*2}T^2 - (TT^*)^2\right)(e_1) = \left(\frac{1}{N}\alpha_1^2\alpha_3^2\right)(e_1) \geq 0,$$

and

$$\left(\frac{1}{N}T^{*2}T^2 - (TT^*)^2\right)(e_2) = \left(\frac{1}{N}\alpha_4^2\alpha_2^2\right)(e_2) \geq 0.$$

Finally we get the following estimation:

$$\left(\left(\frac{1}{N}T^{*2}T^2 - (TT^*)^2\right)(e_n), e_n\right) \geq 0,$$

for every $n \in \mathbb{N}$, therefore $T \in (M,2)_{\frac{1}{N}}^*$. On the other hand, since

$$\frac{1}{N}T^{*2}T^2(e_1) - (T^*T)^2(e_1) = \left(\frac{1}{N}{\alpha_1}^2{\alpha_3}^2 - {\alpha_1}^4\right)(e_1) = \left(\frac{1}{N}\left(\frac{3}{2}\right)^2 N^3 - \left(\frac{3}{2}\right)^4 N^2\right)(e_1)$$

$$= -\frac{45}{16}N^2(e_1) < 0,$$

we conclude that $T \notin (M,2)_{\frac{1}{N}}$.

Example 4.9 *An operator from* $(M,k)_{\frac{1}{N}}$ *which is not in* $(M,k)^*_{\frac{1}{N}}$. *Suppose* H *is a direct sum of denumerable copies of two dimensional Hilbert space* $\mathbb{R}\times\mathbb{R}$. *Let* A *and* B *be any two positive operators on* $\mathbb{R}\times\mathbb{R}$. *For any fixed* $n \in \mathbb{N}$ *define operator* $T = T_{A,B,n}$ *on* H *as follows:*

$$T(x_1, x_2, \cdots,) = (0, Ax_1, Ax_2, \cdots, Ax_n, Bx_{n+1}, \cdots),$$

and from this we have the adjoint operator:

$$T^*(x_1, x_2, \cdots,) = (Ax_2, Ax_3, \cdots, Ax_{n+1}, Bx_{n+2}, \cdots).$$

In what follows we will consider that $N > \frac{1}{2}$. *The operator* T *belong to* $(M,2)_{\frac{1}{N}}$ *class if and only if* $AB^2A - N\cdot A^4 \geq 0$. *Let us denote by* $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ *and* $B = \begin{pmatrix} N & N \\ N & N \end{pmatrix}$ *matrices which satisfies the condition* $AB^2A - N\cdot A^4 \geq 0$, *respectively condition under which operator* $T \in (M,2)_{\frac{1}{N}}$. *Let*

$$x = \left(0, 0, \cdots, 0, \underbrace{\left(-\frac{1}{8N^4 - N}, \frac{1}{8N^4}\right)}_{(n+1)-position}, 0, \cdots\right).$$

Based in the definition of operators A, B *and* T *we will get the following relation:*

$$\big(T^{*2}T^2 - N(TT^*)^2(x), x\big) = ||B^2(x_{n+1})||^2 - N||A^2(x_{n+1})||^2 =$$

$$\left(\begin{pmatrix} 8N^4 - N & 8N^4 \\ 8N^4 & 8N^4 \end{pmatrix}\left(-\frac{1}{8N^4 - N}, \frac{1}{8N^4}\right), \left(-\frac{1}{8N^4 - N}, \frac{1}{8N^4}\right)\right) =$$

$$\left(\left(0, -\frac{N}{8N^4 - N}\right), \left(-\frac{1}{8N^4 - N}, \frac{1}{8N^4}\right)\right) = -\frac{N}{(8N^4 - N)8N^4} < 0,$$

from which it follows that $T \notin (M,2)^*_{\frac{1}{N}}$.

Proposition 4.10 *Let* T *be an operator from* $(M,k)^*_N$ *class. Then*

$$(M,k)^*_N \subset (M,k+1)_N,$$

holds true. The conversely is also true in case T *has dense range in* H.

Proof. Let us suppose that $T \in (M,k)_N^*$. Then for $k \geq 2$, it follows that

$$NT^{*k}T^k \geq (TT^*)^k.$$

This is equivalent with:

$$((NT^{*k}T^k - (TT^*)^k)(x), x) \geq 0,$$

for every $x \in H$. Further:

$$((NT^{*k+1}T^{k+1} - (T^*T)^{k+1})(x), x) = (NT^*(T^{*k}T^k - (TT^*)^k)T(x), x) =$$

$$((NT^{*k}T^k - (TT^*)^k)T(x), T(x)) \geq 0.$$

From this it follows that $NT^{*k+1}T^{k+1} \geq (T^*T)^{k+1} \Rightarrow T \in (M,k+1)_N$. Now let us suppose that T is an operator from $(M,k+1)_N$ with dense range, we will show that $T \in (M,k)_N^*$. Let $x \in H$. Since $\overline{R(T)} = H$, it follows that there exists a sequence $(x_n) \in H$ such that $T(x_n) \to x$. On the other hand, because $T \in (M,k+1)_N$, it follows that

$$((NT^{*k}T^k - (TT^*)^k)T(x_n), T(x_n)) = ((NT^{*k+1}T^{k+1} - (T^*T)^{k+1})(x_n), x_n) \geq 0. \tag{8}$$

Taking the limit as $n \to \infty$, and taking into the consideration that the inner product is continuous function we will get the following relation:

$$((NT^{*k}T^k - (TT^*)^k)T(x_n), T(x_n)) \to ((NT^{*k}T^k - (TT^*)^k)x, x). \tag{9}$$

From relations (8) and (9) it follows that

$$((NT^{*k}T^k - (TT^*)^k)x, x) \geq 0,$$

for every $x \in H$, respectively

$$NT^{*k}T^k \geq (TT^*)^k,$$

and therefore $T \in (M,k)_N^*$.

Corollary 4.11 *If T is N quasi-hyponormal operator with dense range then T is N-hyponormal too.*

Proposition 4.12 *If $T \in B(H)$ is a quasi-normal operator, then it is a $\sqrt{N}$-hyponormal operator if and only if $T \in (M,k)_N^*$, for all $k \geq 1$.*

Proof. The proof is similar to that in Theorem 3.11 [4].

Now we will show that if T_i are operators from $(M,k)_N$ then it follows that their direct sum is from $(M,k)_N$, also.

Proposition 4.13 *Let $H = \oplus_{i\in\mathbb{N}} H_i, H_i \cong H_j$ and $T = \oplus_{i\in\mathbb{N}} T_i$. Where $T_i : H_i \to H_i$ are operators from $(M,k)_N$, $T \in B(H)$, then also $T \in (M,k)_N$ and $N||T^{*k}T^k|| \geq ||T^*T||^k$.*

Proof. From $T_i \in (M,k)_N$ we have:

$$N{T_i}^{*k}{T_i}^k \geq (T_i^*T_i)^k.$$

Hence:

$$NT^{*k}T^k = N(\oplus_{i\in\mathbb{N}} T_i)^{*k}(\oplus_{i\in\mathbb{N}} T_i)^k = N(\oplus_{i\in\mathbb{N}} {T_i}^{*k})(\oplus_{i\in\mathbb{N}} {T_i}^k) = N\left(\oplus_{i\in\mathbb{N}} T_i^{*k}T_i^k\right)$$

$$(*) \qquad \geq \oplus_{i\in\mathbb{N}}(T_i^*T_i)^k = (\oplus_{i\in\mathbb{N}} T_i^*T_i)^k = (T^*T)^k$$

respectively $T \in (M,k)_N$. Now for every $x \in H$, $x = \sum_{i\in\mathbb{N}} x_i$, $x_i \in H_i$,

$$T(x) = \oplus_{i=1}^{\infty} T_i x_i \Rightarrow ||Tx||^2 = \sum_{i=1}^{\infty} ||T_i x_i||^2 \leq \sum_{i=1}^{\infty} ||T_i||^2 \cdot ||x_i||^2,$$

therefore

$$||Tx||^2 \leq \sup_{i\in\mathbb{N}} ||T_i||^2 \sum_{i=1}^{\infty} ||x_i||^2 = \sup_{i\in\mathbb{N}} ||T_i||^2 ||x||^2.$$

On the other hand:

$$||T||^2 = \sup_{||x||=1} ||Tx||^2 \leq \sup_{||x||=1} \{\sup_{i\in\mathbb{N}} ||T_i||^2 ||x||^2\} = \sup_{i\in\mathbb{N}} ||T_i||^2. \qquad (10)$$

Further, for $x = x_i \in H$, it follows that $Tx = T_i x_i$ and $||Tx|| = ||T_i x_i||$. Further more

$$||T|| = \sup_{||x||=1} ||Tx|| \geq \sup_{||x_i||=1} ||T_i x_i|| = ||T_i||,$$

for all $i \in \mathbb{N}$ which implies

$$||T|| \geq \sup_{i\in\mathbb{N}} ||T_i||. \qquad (11)$$

From (10) and (11) it follows:

$$||T|| = \sup_{i\in\mathbb{N}} ||T_i||. \qquad (12)$$

Finally, from relations (*) and (12) it is obvious that $N||T^{*k}T^k|| \geq ||T^*T||^k$.

Proposition 4.14 *Let $H = \oplus_{i\in\mathbb{N}} H_i, H_i \cong H_j$ and $T = \oplus_{i\in\mathbb{N}} T_i$. Where $T_i : H_i \to H_i$ are operators from $(M,k)_N^*$, $T \in B(H)$, then also $T \in (M,k)_N^*$ and $N||T^{*k}T^k|| \geq ||TT^*||^k$.*

Proof. The proof of proposition is similar to the previous proposition.

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N. L. Braha and M. Lohaj
Department of Mathematics and Computer Sciences,
Avenue Mother Teresa Nr=5,
University of Prishtina, Prishtinë 10000, Kosovë
e-mail: nbraha@yahoo.com, muhib_lohaj@yahoo.com

F. H. Marevci
Faculty of Education
University of Prishtina, Prishtinë 10000, Kosovë
e-mail: fahrimarevci@hotmail.com

Sh. Lohaj
Faculty of Electrical Engineering
University of Prishtina, Prishtinë 10000, Kosovë
e-mail: shqipe_ lohaj@hotmail.com