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EXISTENCE RESULTS FOR SEMILINEAR FRACTIONAL DIFFERENTIAL INCLUSIONS WITH STATE-DEPENDENT DELAY

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Abstract. Our goal In this paper is to establish sufficient conditions for the existence of mild solution of some class of semilinear fractional differential inclusions of order $0 < \alpha \leq 1$ with state dependent delay in separable Banach space. The existence result is established when the multivalued function has convex values. The result is obtained via the nonlinear alternative of Leray-Schauder type.

1. Introduction

Our aim in this paper is to study the existence of mild solutions defined on a compact interval for fractional semilinear differential inclusions with state dependent delay in a separable Banach space E of the form:

$$D_t^{\alpha} y(t) \in Ay(t) + F(t, y_{\rho(t,y_t)}); \ t \in J := [0, b], \quad t \neq t_k, \ k = 1, \dots, m, \tag{1.1}$$

$$\Delta y(t_k) = I_k(y_{t_k}); \quad k = 1, \dots, m, \tag{1.2}$$

$$y(0) = \phi(t), \qquad t \in (-\infty, 0]. \tag{1.3}$$

where $0 < \alpha \leq 1$, $F : J \times \mathcal{D} \to \mathcal{P}(E)$ is a given multivalued map with non-empty convex compact values, $\mathcal{D}$ is the phase space defined axiomatically (see Section 2) which contains the mappings from $(-\infty, 0]$ into E, $I_k : \mathcal{D} \to E, \ k = 1, 2, \dots, m$ are appropriate functions to be specified later, $\Delta y(t_k) = y(t_k^+) - y(t_k^-)$, $\rho : J \times \mathcal{D} \to (-\infty, b]$, $0 = t_0 < t_1 < \dots < t_m < t_{m+1} = b$, $\mathcal{P}(E)$ is the collection of all subsets of E, $\phi \in \mathcal{D}$, $A : D(A) \subset E \to E$ is the generator of an $\alpha-$resolvent operator function ($\alpha-$ROF for short) S_α. For any continuous function y defined on $[-r, b] - \{t_1, t_2, ..., t_m\}$ and any $t \in J$, we denote by y_t the element of $\mathcal{D}$ defined by

$$y_t(\theta) = y(t + \theta), \quad \theta \in (-\infty, 0].$$

Recently, fractional differential equations and inclusions have been extensively studied and several results concerning existence and uniqueness were established.

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In the last decade, there has been a significant development in fractional differential equations see [4], [23], the monographs of Kilbas *et al*, Lakshmikantham *et al*[14], Anguraj *et al*[1], because their applicability in various fields like; engineering, physics, electrical net work, control theory of dynamical systems.
For further details, we refer the reader to [3], [13], Miller and Ross [17], Samko *et al* [22], Kilbas and Marzan [12], Momani *et al* [18], Podlubny *et al*[21] and the references therein, see also [17, 19, 22]).
The Cauchy problem for abstract differential equations involving Riemann-Liouville fractional integral have been treated by several searchers like: Cueva and De Souza[5, 6], Benchohra *et al*[2] and references therein.
To our knowledge, there are very few results for impulsive fractional differential equations and inclusions. The results of the present paper extend and complete those obtained by [8] with finite delay. This paper is organized as follow, in section 2 we introduce some preliminaries that will be used in the sequel, in section 3 we give sufficient conditions for the existence of the mild solution of problem (1.1)-(1.3). Finally we illustrate our result by an example.

2. Preliminaries

In this section, we introduce notations, definitions, and preliminaries facts which are used throughout this paper.
For $\psi \in \mathcal{D}$ the norm of ψ is given by

$$\|\psi\|_{\mathcal{D}} = \sup\{|\psi(t)| : t \in (-\infty, 0]\}$$

$\mathcal{B}$ is the Banach space of all bounded linear operators from E into E with the norm

$$\|N\|_{\mathcal{B}} = \sup\{|N(y)| : |y| = 1\}$$

$L^1[J, E]$ denotes the Banach space of measurable functions $u : J \to E$ which are Bochner integrable normed by

$$\|u\|_{L^1} = \int_0^b |u(t)|dt.$$

In order to define the solution of the problem (1.1)-(1.3), we introduce some additional concepts and notations. Let $(X, |\cdot|)$ be a normed space. Denote by

$$\begin{aligned}\mathcal{B}_b \quad &= \Big\{y : (-\infty, b] \to E, y_k \in C(J_k, E);\ y(t_k^-), y(t_k^+) \\ &\qquad \text{exist with} \quad y(t_k) = y(t_k^-),\ y(t) = \phi(t), t \leq 0\Big\}\end{aligned}$$

where y_k is the restriction of y to $J_k = (t_k, t_{k+1}],\quad k = 0, \ldots, m$. Let $\|.\|_b$ be the semi-norm in $\mathcal{B}_b$ defined by

$$\|y\|_b = \|y\|_{\mathcal{D}} + \sup\{|y(s)| : 0 \leq s \leq b\}, \quad y \in \mathcal{B}_b.$$

The axiomatic definition for the phase space $\mathcal{D}$ is similar to those introduced in [11]. Specifically, $\mathcal{D}$ will be a linear space of functions mapping $(-\infty, 0]$ into E endowed with a semi norm $\|.\|_{\mathcal{D}}$, and satisfies the following axioms introduced at first by Hale and Kato in [9]:

(A1) There exist a positive constant H and functions $K(\cdot)$, $M(\cdot) : \mathbb{R}^+ \to \mathbb{R}^+$ with K continuous and M locally bounded, such that for any $b > 0$, if $y : (-\infty, b] \to E$, $y \in \mathcal{D}$, and $y(\cdot)$ is continuous on $[0, b]$, then for every $t \in [0, b]$ the following conditions hold:

(i) y_t is in $\mathcal{D}$;
(ii) $|y(t)| \leq H\|y_t\|_{\mathcal{D}}$;
(iii) $\|y_t\|_{\mathcal{D}} \leq K(t)\sup\{|y(s)| :\ 0 \leq s \leq t\} + M(t)\|y_0\|_{\mathcal{D}}$, and H, K and M are independent of $y(\cdot)$

(A) The space $\mathcal{D}$ is complete. Denote

$$K_b = \sup\{K(t) :\ \ t \in J\} \ \text{ and } \ M_b = \sup\{M(t) : t \in J\}.$$

Let (X, d) be a metric space. The following notations will be used:
$P_{cl} = \{Y \in \mathcal{P}(X) : Y \text{closed}\}$, $P_{bd} = \{Y \in \mathcal{P}(X) : Y \text{bounded}\}$,
$P_{cv} = \{Y \in \mathcal{P}(X) : Y \text{cconvex}\}$, $P_{cp} = \{Y \in \mathcal{P}(X) : Y \text{compact}\}$,
Consider $H_d : \mathcal{P}(X) \times \mathcal{P}(X) \to \mathbb{R}_+ \Cup \{\infty\}$ given by

$$H_d(A, B) = \max\{\sup_{a \in A} d(a, B), \sup_{b \in B} d(b, A)\}$$

In the following, we give some basic notions about fractional calculus and $\alpha-$resolvent operator.

Definition 2.1. The fractional integral operator I^α of order $\alpha > 0$ of a continuous function $f(t)$ is given by

$$I_t^\alpha f(t) := \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) ds.$$

We can write $I_t^\alpha f(t) = f(t) * \psi_\alpha(t)$ where $\psi_\alpha(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)}$ for $t > 0$ and $\psi_\alpha(t) = 0$ for $t \leq 0$ and $\psi_\alpha(t) \to \delta(t)$ (the delta function) as $\alpha \to 0$.

Definition 2.2. the $\alpha-$th Riemann-Liouville fractional-order derivative of f, is defined by:

$$D_a^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_a^t (t-s)^{n-\alpha-1} f(s) ds.$$

Here $n = [\alpha] + 1$ and $[\alpha]$ denotes the integer part of α

Definition 2.3 ([12])**.** For a function f given on the interval $[a, b]$, the Caputo fractional-order derivative of order α of f, is defined by

$$({}^c_{a+}D_t^\alpha f)(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-s)^{n-\alpha-1} f^{(n)}(s) ds,$$

where $n = [\alpha] + 1$.

Therfore; for $0 < \alpha < 1$, The Caputo's fractional derivative for $t \in [0, b]$ is

$$({}^c_0D_t^\alpha f)(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} f'(s) ds,$$

For more detail see [13, 17, 20]).

Definition 2.4. [3] *Let $\alpha > 0$. A function $S_\alpha : \mathbb{R}_+ \to B(X)$ is called an $\alpha-$resolvent operator($\alpha-$ROF) if the following conditions are satisfied:*

(a) *$S_\alpha(.)$ is strongly continuous on $\mathbb{R}_+$ and $S_\alpha(0) = I$,*
(b) *$S_\alpha(s)S_\alpha(t) = S_\alpha(t)S_\alpha(s)$ for all $s, t \geq 0$,*

(c) *the functional equation*

$$S_\alpha(s)I_t^\alpha S_\alpha(t) - I_s^\alpha S_\alpha(s)S_\alpha(t) = I_t^\alpha S_\alpha(t) - I_s^\alpha S_\alpha(s)$$

holds for all $s, t \geq 0$.
The generator A *of* S_α *is defined by:*

$$D(A) := \left\{ x \in X : \lim_{t \to 0^+} \frac{S_\alpha(t)x - x}{\psi_{\alpha+1}(t)} \text{ exists} \right\}$$

And

$$Ax = \lim_{t \to 0^+} \frac{S_\alpha(t)x - x}{\psi_{\alpha+1}(t)}, \quad x \in D(A).$$

Definition 2.5. *An* $\alpha-$*ROF* S_α *is said to be exponentially bounded if there exist constants* $M \geq 0, \omega \geq 0$ *such that:*

$$\|S_\alpha(t)\| \leq Me^{\omega t}, \qquad t \geq 0.$$

In this case we write $A \in \mathcal{C}_\alpha(M, \omega)$

Proposition 2.1. *Let* S_α *be an* $\alpha-$*ROF generated by the operator* A*. The following assertions hold:*

(a) $S_\alpha(t)D(A) \subset D(A)$ *and* $AS_\alpha(t)x = S_\alpha(t)Ax$ *for all* $x \in D(A)$ *and* $t \geq 0$,
(b) *For all* $x \in X$, $I_t^\alpha S_\alpha(t)x \in D(A)$ *and*

$$S_\alpha(t)x = x + AI_t^\alpha S_\alpha(t)x, \qquad t \geq 0,$$

(c) $x \in D(A)$ *and* $Ax = y$ *if and only if*

$$S_\alpha(t)x = x + AI_t^\alpha S_\alpha(t)x, \qquad t \geq 0,$$

(d) A *is closed, densely defined.*

Proposition 2.2. *Let* $\alpha > 0$. $A \in \mathcal{C}_\alpha(M, \omega)$ *if and only if* $(\omega^\alpha, \infty) \subset \rho(A)$ *and there exists a strongly continuous function* $S_\alpha : \mathbb{R}_+ \to B(X)$ *such that:* $\|S_\alpha(t)\| \leq Me^{\omega t}$ *and*

$$\int_0^\infty e^{-\lambda t} S_\alpha(t)x dt = \lambda^{\alpha-1} R(\lambda^\alpha, A)x \qquad \lambda > \omega$$

for all $x \in X$*. Further more,* S_α *is the* $\alpha-$*ROF generated by the operator* A*.*

For more detail see[16]. The following definitions are used in the sequel.

Definition 2.6. *A multivalued operator* $N : J \to P_{cl}(X)$ *is called*

(a) *contraction if and only if there exists* $\gamma > 0$ *such that*

$$H_d(N(x), N(y)) \leq \gamma d(x, y), \quad for \quad each \quad x, y \in X,$$

with $\gamma < 1$.
(b) N *has a fixed point if there exists* $x \in X$ *such that* $x \in N(x)$.

Definition 2.7. *A multivalued map* $F : J \times D \to \mathcal{P}(E)$ *is said to be* L^1-*Carathéodory if*

(i) $t \longmapsto F(t, u)$ *is measurable for each* $u \in D$,
(ii) $u \longmapsto F(t, u)$ *is u.s.c. for almost all* $t \in J$.

For each $y \in C(J,E)$, define the set of selections of F by

$$S_{F,y} = \{v \in L^1(J,E) : v(t) \in F(t, y(t)) \text{ a.e. } t \in J\}.$$

Let us introduce the definition of Caputo's derivative in each interval $(t_k, t_{k+1}]$, $k = 0, \dots, m$ see [24]

$$({}^c_{t_k}D^\alpha_t f)(t) = \frac{1}{\Gamma(1-\alpha)} \int_{t_k}^t (t-s)^{-\alpha} f'(s) ds.$$

3. Main result

Now, we are able to define the mild solution of the initial problem (1.1)-(1.3).

Definition 3.1. *A function $y : (-\infty, b] \to E$ is said to be mild solution of (1.1)-(1.3) if $y(t) = \phi(t)$ for all $t \in (-\infty, 0]$, $\Delta y(t_k) = I_k(y_{t_k})$ $k = 1, \dots, m$, the restriction of $y(\cdot)$ to the interval $[0,b]$ is continuous, and there exist $v(\cdot) \in L^1(J,E)$, such that $v(t) \in F(t, y_{\rho(t,y_t)})$, a.e $t \in [0,b]$, and y satisfies the following integral equation:*

$$y(t) = \begin{cases} S_\alpha(t)\phi(0) + \int_0^t S_\alpha(t-s)v(s)ds & \text{if } t \in [0,t_1], \\ S_\alpha(t-t_k)\prod_{i=1}^{k} S_\alpha(t_i - t_{i-1})\phi(0) \\ + \sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} S_\alpha(t-t_k) \prod_{j=i}^{k-1} S_\alpha(t_{j+1}-t_j). \\ .S_\alpha(t_i - s)v(s)ds + \int_{t_k}^t S_\alpha(t-s)v(s)ds \\ + \sum_{i=1}^{k} S_\alpha(t-t_k)\prod_{j=i}^{k-1} S_\alpha(t_{j+1}-t_j) I_i(y_{t_i}), & \text{if } t \in (t_k, t_{k+1}]. \end{cases} \tag{3.1}$$

Set

$$\mathcal{R}(\rho^-) = \{\rho(s,\varphi) : (s,\varphi) \in J \times \mathcal{D}, \rho(s,\varphi) \le 0\}.$$

Let us assume that $\rho : J \times \mathcal{D} \to (-\infty, b]$ is continuous. Additionally, we introduce the following hypotheses:

- (Hφ) The function $t \to \varphi_t$ is continuous from $\mathcal{R}(\rho^-)$ into $\mathcal{D}$ and there exists a continuous and bounded function $L^\phi : \mathcal{R}(\rho^-) \to (0,\infty)$ such that $\|\phi_t\|_{\mathcal{D}} \le L^\phi(t)\|\phi\|_{\mathcal{D}}$ for every $t \in \mathcal{R}(\rho^-)$.
- (H1) assume that A generates a compact $\alpha-$ROF S_α for $t > 0$ wich is exponentially bounded i.e: There exist constants $M \ge 1, \omega \ge 0$ such that:

$$\|s_\alpha(t)\| \le Me^{\omega t}, \qquad t \ge 0.$$

- (H2) $I_k : E \to E$ are continuous and there exist constants $M^* > 0$, $k = 1, \dots, m$ such that

$$\|I_k(y)\| \le M^* \quad \text{for each} \quad y \in \mathcal{D}.$$

- (H3) $F : J \times C([-r,0],E) \to \mathcal{P}_{cp,cv}(E)$ is Carathéodory and there exist $p \in L^1(J, \mathbb{R}_+)$ and a continuous non decreasing function $\psi : [0,\infty) \to (0,\infty)$ such that:

$\|F(t,x)\|_{\mathcal{P}} = \sup\{|v| : v \in F(t,x)\} \leq p(t)\psi(\|u\|_{\mathcal{D}}$ for a. e.$t \in J$. $x \in \mathcal{D}$.
with

$$\int_0^b e^{-\omega s} p(s)ds < \infty,$$

$$\lim_{u\to+\infty}\sup \frac{\left[(M_b + L^{\phi} + MK_b)\|\phi\|_{\mathcal{D}} + K_b\right]u}{C_i^* + C_2^* \int\limits_0^t e^{-\omega s} p(s)\psi\left(K_b u + (M_b + L^{\phi} + MK_b)\|\phi\|_{\mathcal{D}}\right) ds} > 1, \qquad i = 0, 1. \tag{3.2}$$

where

$$C_0^* = (M_b + L^{\phi} + MK_b)\|\phi\|_{\mathcal{D}}, \tag{3.3}$$

$$C_1^* = K_b C_1 + (M_b + L^{\phi} + MK_b)\|\phi\|_{\mathcal{D}}, \tag{3.4}$$

and

$$C_1 = K_b \frac{M^* e^b}{1-M} + \sum_{i=1}^{k} M^{k-i+2} e^{\omega t} \int_{t_{i-1}}^{t_i} e^{-\omega s} p(s)\psi\left(K_b|x(s)| + (M_b + L^{\phi} + MK_b)||\phi||_{\mathcal{D}}\right) ds. \tag{3.5}$$

$$C_2^* = MK_b e^{\omega b} \tag{3.6}$$

The next result is a consequence of the phase space axioms.

Lemma 3.1. *[*[10]*, Lemma 2.1] If $y : (-\infty, b] \to E$ is a function such that $y_0 = \phi$ and $y|_J \in PC(J : D(A))$, then*

$$\|y_s\|_{\mathcal{D}} \leq (M_b + L^{\phi})\|\phi\|_{\mathcal{D}} + K_b \sup\{\|y(\theta)\|;\, \theta \in [0, \max\{0, s\}]\}, \quad s \in \mathcal{R}(\rho^-) \cup J,$$

where $L^{\phi} = \sup\limits_{t\in\mathcal{R}(\rho^-)} L^{\phi}(t)$, $M_b = \sup\limits_{t\in J} M(t)$ and $K_b = \sup\limits_{t\in J} K(t)$.

The nonlinear alternative of Leray-Schauder type is used to investigate the existence of solutions of problem (1.1)-(1.3). We need to use the following result due to Lazota and Opial [15].

Lemma 3.2. *Let E be a Banach space, and F be an L^1-Carathéodory multivalued map with compact convex values, and let $\Gamma : L^1(J,E) \to C(J,E)$ be a linear continuous mapping. Then the operator*

$$\Gamma \circ S_F : C(J,E) \to P_{cp,cv}(C(J,E))$$

is a closed graph operator in $C(J,E) \times C(J,E)$.

Theorem 3.3. *Assume that (Hφ) and (H1)-(H3) hold. If $\phi(0) \in D(A)$ then the IVP (1.1)-(1.3) has at least one mild solution on $(-\infty, b]$.*

Proof. Transform the problem (1.1)-(1.3) into a fixed point problem. Set $\Omega = PC\left((-\infty, b], E\right])$ Consider the multivalued operator: $N : \Omega \to \mathcal{P}(\Omega)$ defined by

$N(y) = \{h \in \Omega\}$ such that

$$h(t) = \begin{cases} \phi(t), & \text{if } t \in (-\infty, 0], \\ S_\alpha(t)\phi(0) + \int_0^t S_\alpha(t-s)v(s)ds & \text{if } t \in [0, t_1], \\ S_\alpha(t-t_k)\prod_{i=1}^{k} S_\alpha(t_i - t_{i-1})\phi(0) \\ + \sum_{i=1}^{k}\int_{t_{i-1}}^{t_i} S_\alpha(t-t_k)\prod_{j=i}^{k-1} S_\alpha(t_{j+1}-t_j). \\ .S_\alpha(t_i - s)v(s)ds + \int_{t_k}^{t} S_\alpha(t-s)v(s)ds \\ + \sum_{i=1}^{k} S_\alpha(t-t_k)\prod_{j=i}^{k-1} S_\alpha(t_{j+1}-t_j)I_i(y_{t_i}), & \text{if } t \in (t_k, t_{k+1}]. \end{cases} \tag{3.7}$$

In the following, we will introduce an auxiliary multivalued operator $\mathcal{A}$ such that, $\mathcal{A}$ has a fixed point equivalent that the operator N. has one.
Let $\widetilde{\phi}(.) : (-\infty, b] \to E$ be the function defined by

$$\widetilde{\phi}(t) = \begin{cases} \phi(t), & t \in (-\infty, 0], \\ S_\alpha(t)\phi(0), & t \in [0, t_1], \\ S_\alpha(t-t_k)\prod_{i=1}^{k} S_\alpha(t_i - t_{i-1})\phi(0), & t \in (t_k, t_{k+1}]. \end{cases} \tag{3.8}$$

Then $\widetilde{\phi}_0 = \phi$. For each $x \in \mathcal{B}_b$ with $x(0) = 0$, we denote by $\overline{x}$ the function defined by

$$\overline{x}(t) = \begin{cases} 0, & t \in (-\infty, 0], \\ x(t), & t \in J, \end{cases}$$

If $y(.)$ satisfies (3.1), we can decompose it as $y(t) = \widetilde{\phi}(t) + x(t), \quad 0 \le t \le b$, which implies $y_t = x_t + \widetilde{\phi}_t$, for every $0 \le t \le b$ and the function $x(.)$ satisfies

$$x(t) = \begin{cases} \int_0^t S_\alpha(t-s)v(s)ds & \text{if } t \in [0, t_1], \\ \sum_{i=1}^{k}\int_{t_{i-1}}^{t_i} S_\alpha(t-t_k)\prod_{j=i}^{k-1} S_\alpha(t_{j+1}-t_j). \\ .S_\alpha(t_i - s)v(s)ds + \int_{t_k}^{t} S_\alpha(t-s)v(s)ds \\ + \sum_{i=1}^{k} S_\alpha(t-t_k)\prod_{j=i}^{k-1} S_\alpha(t_{j+1}-t_j)I_i(y_{t_i}, & \text{if } t \in (t_k, t_{k+1}]. \end{cases} \tag{3.9}$$

where $v(s) \in S_{F, x_{\rho(s, x_s + \widetilde{\phi}_s)} + \widetilde{\phi}_{\rho(s, x_s + \widetilde{\phi}_s)}}$ Let

$$\mathcal{B}_b^0 = \{x \in \mathcal{B}_b : \ x_0 = 0 \in \mathcal{D}\}.$$

For any $x \in \mathcal{B}_b^0$ we have

$$\|x\|_b = \|x_0\|_{\mathcal{D}} + \sup\{|x(s)| : 0 \leq s \leq b\} = \sup\{|x(s)| : 0 \leq s \leq b\}.$$

Thus $(\mathcal{B}_b^0, \|\cdot\|_b)$ is a Banach space. define the $\mathcal{A} : \mathcal{B}_b^0 \to \mathcal{P}(\mathcal{B}_b^0)$ by:

$$\mathcal{A}(x) =: \left\{h \in \mathcal{B}_b^0\right\}$$

with

$$h(t) = \begin{cases} 0, & \text{if } t \in (-\infty, 0], \\ \displaystyle\int_0^t S_\alpha(t-s)v(s)ds & \text{if } t \in [0, t_1], \\ \displaystyle\sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} S_\alpha(t-t_k) \prod_{j=i}^{k-1} S_\alpha(t_{j+1}-t_j). & \\ .S_\alpha(t_i - s)v(s)ds + \displaystyle\int_{t_k}^t S_\alpha(t-s)v(s)ds & \\ + \displaystyle\sum_{i=1}^{k} S_\alpha(t-t_k) \prod_{j=i}^{k-1} S_\alpha(t_{j+1}-t_j) I_i(y_{t_i}), & \text{if } t \in (t_k, t_{k+1}]. \end{cases} \tag{3.10}$$

Clearly, the operator N has a fixed point is equivalent to $\mathcal{A}$ has one, so it turns to prove that $\mathcal{A}$ has a fixed point. We shall show that the operators $\mathcal{A}$ satisfies all assumptions of the nonlinear alternative of Leray-Schauder type [7]. For better readability, we break the proof into a sequence of steps.

Step 1: *$\mathcal{A}(x)$ is convex for each $x \in \mathcal{B}_b^0$.*

Let $h_1, h_2 \in \mathcal{A}(x)$, then there exist $v_1, v_2 \in S_{F, x_{\rho(s, x_s + \tilde{\phi}_s)} + \widetilde{\phi}_{\rho(s, x_s + \tilde{\phi}_s)}}$ such that for each $t \in J$

$$h_p = \begin{cases} \displaystyle\int_0^t S_\alpha(t-s)v_p(s)ds & \text{if } t \in [0, t_1], \\ \displaystyle\sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} S_\alpha(t-t_k) \prod_{j=i}^{k-1} S_\alpha(t_{j+1}-t_j). & \\ .S_\alpha(t_i - s)v_p(s)ds + \displaystyle\int_{t_k}^t S_\alpha(t-s)v_p(s)ds & \\ + \displaystyle\sum_{i=1}^{k} S_\alpha(t-t_k) \prod_{j=i}^{k-1} S_\alpha(t_{j+1}-t_j) I_i(y_{t_i}), & \text{if } t \in (t_k, t_{k+1}]. \end{cases} \qquad p = 1, 2$$

Let $0 \leq \sigma \leq 1$. Then for each $t \in J$ we have:

$$(\sigma h_1 - (1-\sigma)h_2)(t) = \begin{cases} \displaystyle\int_0^t S_\alpha(t-s)\left[\sigma v_1(s) - (1-\sigma)v_2\right] ds & \text{if } t \in [0, t_1], \\ \displaystyle\sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} S_\alpha(t-t_k) \prod_{j=i}^{k-1} S_\alpha(t_{j+1}-t_j). & \\ .S_\alpha(t_i - s)\left[\sigma v_1(s) - (1-\sigma)v_2\right] ds & \\ + \displaystyle\int_{t_k}^t S_\alpha(t-s)\left[\sigma v_1(s) - (1-\sigma)v_2\right] ds & \\ & \text{if } t \in (t_k, t_{k+1}]. \end{cases}$$

Since $S_{F,y}$ is convex (because F has convex values), we have $\sigma h_1-(1-\sigma)h_2 \in \mathcal{A}(x)$.
Step 2: *$\mathcal{A}$ maps bounded sets into bounded sets in $\mathcal{B}_b^0$.*

Let $B_q=\{x\in\mathcal{B}_b^0 : ||x||_b\leq q, \quad q\in\mathbb{R}^+\}$ a bounded set in $\mathcal{B}_b^0$.

It is equivalent to show that there exists a positive constant l such that for each $x\in B_q$ we have $\|\mathcal{A}(x)\|_b\leq l$. choose $x\in B_q$, then from lemma 3.1 it follows that

$$\|x_{\rho(t,x_t+\widetilde{\phi}_t)}+\widetilde{\phi}_{\rho(t,x_t+\widetilde{\phi}_t)}\|_{\mathcal{D}}\leq K_b q+(M_b+L^{\phi})\|\phi\|_{\mathcal{D}}+K_bM|\phi(0)|=q_*$$

Also, for each $h\in\mathcal{A}(x)$, and each $x\in B_q$, there exists $v\in S_{F,x_{\rho(t,x_t+\tilde{\phi}_t)}+\widetilde{\phi}_{\rho(t,x_t+\tilde{\phi}_t)}}$. such that

$$h(t)=\begin{cases}\displaystyle\int_0^t S_\alpha(t-s)v(s)ds & \text{if } \ t\in[0,t_1],\\ \displaystyle\sum_{i=1}^{k}\int_{t_{i-1}}^{t_i} S_\alpha(t-t_k)\prod_{j=i}^{k-1}S_\alpha(t_{j+1}-t_j). & \\ \displaystyle .S_\alpha(t_i-s)v(s)ds+\int_{t_k}^{t}S_\alpha(t-s)v(s)ds & \\ \displaystyle +\sum_{i=1}^{k}S_\alpha(t-t_k)\prod_{j=i}^{k-1}S_\alpha(t_{j+1}-t_j)I_i(y_{t_i}), & \text{if } \ t\in(t_k,t_{k+1}].\end{cases}$$

Then, for $t\in J$

$$|h(t)|\leq\begin{cases}\displaystyle Me^{\omega t_1}\int_0^t e^{-\omega s}\|v(s)\|ds & \text{if } \ t\in[0,t_1],\\ \displaystyle\sum_{i=1}^{k}\int_{t_{i-1}}^{t_i}Me^{\omega(t-t_k)}\prod_{j=i}^{k-1}Me^{\omega(t_{j+1}-t_j)}. & \\ \displaystyle .Me^{\omega(t_i-s)}\|v(s)\|ds+\int_{t_k}^{t}Me^{\omega(t-s)}\|v(s)\|ds & \\ \displaystyle +\sum_{i=1}^{k-1}Me^{\omega(t-t_k)}\prod_{j=i}^{k-1}Me^{\omega(t_{j+1}-t_j)}\|I_i(y(t_i))\|, & \text{if } \ t\in(t_k,t_{k+1}].\end{cases}$$

wich gives

$$|h(t)|\leq\begin{cases}\displaystyle Me^{\omega t_1}\psi(q*)\int_0^t e^{-\omega s}p(s)ds & \\ =l_1 & \text{if } \ t\in[0,t_1],\\ \displaystyle\sum_{i=1}^{k}M^{k-i+2}e^{\omega(2t_k-t_{i-1})}\psi(q*)\int_{t_{i-1}}^{t_i}e^{\omega(-s)}p(s)ds & \\ \displaystyle +Me^{\omega(t_{k+1})}\psi(q*)\int_{t_k}^{t}e^{\omega(-s)}p(s)ds & \\ \displaystyle +\sum_{i=1}^{k}M^{k-i+1}M^*e^{\omega(t_{k+1}-t_{i-1})} & \\ =l_k & \text{if } \ t\in(t_k,t_{k+1}].\end{cases}$$

This further, implies that

$$\|\mathcal{A}(x)\|_b\leq l.$$

Hence $\mathcal{A}(B_q)$ is bounded.

step 3: *$\mathcal{A}$ maps bounded sets into equi-continuous sets of $\mathcal{B}_b^0$.*

Let τ_1, $\tau_2 \in J$ with $\tau_1 < \tau_2$, let B_q be a bounded set in $\mathcal{B}_b^0$ as in Step 2, and let $x \in B_q$ and $h \in \mathcal{A}(x)$. Then, if $\epsilon > 0$ with $\epsilon < \tau_1 < \tau_2$

$$|h(\tau_2) - h(\tau_1)| \leq \begin{cases} \displaystyle\int_0^{\tau_1-\epsilon} \|S_\alpha(\tau_2 - s) - S_\alpha(\tau_1 - s)\||v(s)|ds \\ \displaystyle+ \int_{\tau_1-\epsilon}^{\tau_1} \|S_\alpha(\tau_2 - s) - S_\alpha(\tau_1 - s)\||v(s)|ds \\ \displaystyle+ \int_{\tau_1}^{\tau_2} \|S_\alpha(\tau_2 - s)\||v(s)|ds \\ \qquad\qquad\qquad\qquad \text{if } \tau_1, \tau_2 \in [0, t_1], \end{cases}$$

and

$$|h(\tau_2) - h(\tau_1)| \leq \begin{cases} \displaystyle\sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} \|S_\alpha(\tau_2 - t_k) - S_\alpha(\tau_1 - t_k)\|. \\ \displaystyle. \prod_{j=i}^{k-1} \|S_\alpha(t_{j+1} - t_j)\|\|S_\alpha(t_i - s)\||v(s)|ds \\ \displaystyle+ \int_{t_k}^{\tau_2} S_\alpha(\tau_2 - s)v(s)ds - \int_{t_k}^{\tau_1} S_\alpha(\tau_1 - s)v(s)ds \\ \displaystyle+ \sum_{i=1}^{k} \|S_\alpha(\tau_2 - t_k) - S_\alpha(\tau_1 - t_k)\| \prod_{j=i}^{k-1} \|S_\alpha(t_{j+1} - t_j)\|. \\ .\|I_i(y(t_i^-))\|, \\ \qquad\qquad\qquad\qquad \text{if } \tau_1, \tau_2 \in (t_k, t_{k+1}]. \end{cases}$$

Which gives

$$|h(\tau_2) - h(\tau_1)| \leq \begin{cases} \displaystyle\psi(q) \int_0^{\tau_1-\epsilon} \|S_\alpha(\tau_2 - s) - S_\alpha(\tau_1 - s)\|p(s)ds \\ \displaystyle+\psi(q) \int_{\tau_1-\epsilon}^{\tau_1} \|S_\alpha(\tau_2 - s) - S_\alpha(\tau_1 - s)\|p(s)ds \\ \displaystyle+Me^{\omega\tau_2}\psi(q) \int_{\tau_1}^{\tau_2} e^{-\omega s}p(s)ds \\ \qquad\qquad\qquad\qquad \text{if } \ \tau_1, \tau_2 \in [0, t_1], \end{cases}$$

and

$$|h(\tau_2) - h(\tau_1)| \leq \begin{cases} \displaystyle\sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} \|S_\alpha(\tau_2 - t_k) - S_\alpha(\tau_1 - t_k)\|. \\ \displaystyle. \prod_{j=i}^{k-1} \|S_\alpha(t_{j+1} - t_j)\|\|S_\alpha(t_i - s)\||v(s)|ds \\ \displaystyle+\psi(q) \int_{t_k}^{\tau_1-\epsilon} \|S_\alpha(\tau_2 - s) - S_\alpha(\tau_1 - s)\|p(s)ds \\ \displaystyle+\psi(q) \int_{\tau_1-\epsilon}^{\tau_1} \|S_\alpha(\tau_2 - s) - S_\alpha(\tau_1 - s)\|p(s)ds \\ \displaystyle+M\psi(q)e^{\omega\tau_2} \int_{\tau_1}^{\tau_2} e^{-\omega s}p(s)ds \\ \displaystyle+ \sum_{i=1}^{k} \|S_\alpha(\tau_2 - t_k) - S_\alpha(\tau_1 - t_k)\| \prod_{j=i}^{k-1} \|S_\alpha(t_{j+1} - t_j)\|. \\ .\|I_i(y(t_i^-))\|, \\ \qquad\qquad\qquad\qquad \text{if } \ \tau_1, \tau_2 \in (t_k, t_{k+1}]. \end{cases}$$

As $\tau_1 \to \tau_2$ and ϵ becomes sufficiently small, the right-hand side of the above inequality tends to zero, since S_α is a strongly continuous operator and the compactness of S_α for $t > 0$ implies the continuity in the uniform operator topology. This proves the equicontinuity for the case where $t \neq t_i, i = 1, ..., m+1$. It remains to examine the equicontinuity at $t = t_i$. First we prove the equicontinuity at $t = t_i^-$, we have for some $x \in Bq$, there exists $v \in S_{F, x_{\rho(t, x_t + \tilde{\phi}_t)} + \tilde{\phi}_{\rho(t, x_t + \tilde{\phi}_t)}}$. such that for each $t \in J$ we have:
if $t \in [0, t_1]$,

$$h(t) = \int_0^t S_\alpha(t-s)v(s)ds$$

if $t \in (t_k, t_{k+1}]$

$$\begin{aligned} h(t) &= \sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} S_\alpha(t-t_k) \prod_{j=i}^{k-1} S_\alpha(t_{j+1}-t_j).S_\alpha(t_i - s)v(s)ds \\ &+ \int_{t_k}^{t} S_\alpha(t-s)v(s)ds + \sum_{i=1}^{k} S_\alpha(t-t_k) \prod_{j=i}^{k-1} S_\alpha(t_{j+1}-t_j) I_i(y(t_i^-)), \end{aligned}$$

Fix $\delta_1 > 0$ such that $\{t_k, k \neq l\} \cap [t_l - \delta_1, t_l + \delta_1] = \emptyset$. For $0 < \rho < \delta_1$, we have

$$|h(t_l - \rho) - h(t_l)| \leq \begin{cases} \psi(q) \displaystyle\int_0^{t_l - \rho} \|S_\alpha(t_l - \rho - s) - S_\alpha(t_l - s)\| p(s) ds \\ +Me^{\omega t_l} \psi(q) \displaystyle\int_{t_- \rho}^{t_l} e^{-\omega s} p(s) ds \qquad \text{if } t_l - \rho, t_l \in [0, t_1], \end{cases}$$

and

$$|h(t_l - \rho) - h(t_l)| \leq \begin{cases} \psi(q) \displaystyle\sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} \|S_\alpha(t_l - \rho - t_k) - S_\alpha(t_l - t_k)\|. \\ .\displaystyle\prod_{j=i}^{k-1} \|S_\alpha(t_{j+1} - t_j)\| \|S_\alpha(t_i - s)\| |p(s)| ds \\ +\psi(q) \displaystyle\int_{t_k}^{t_l - \rho} \|S_\alpha(t_l - \rho - s) - S_\alpha(t_l - s)\| p(s) ds \\ +M\psi(q) e^{\omega t_l} \displaystyle\int_{t_l - \rho}^{t_l} e^{-\omega s} p(s) ds \\ +\displaystyle\sum_{i=1}^{k} \|S_\alpha(t_l - \rho - t_k) - S_\alpha(t_l - t_k)\|. \\ \displaystyle\prod_{j=i}^{k-1} \|S_\alpha(t_{j+1} - t_j)\|.\|I_i(y(t_i^-))\|, \qquad \text{if } t_l - \rho, t_l \in (t_k, t_{k+1}]. \end{cases}$$

Which tends to zero as $\rho \to 0$.
Define

$$\hat{h}_0(t) = h(t), \quad \text{if } t \in [0, t_1]$$

and

$$\hat{h}_i(t) = \begin{cases} h(t), & \text{if } t \in (t_i, t_{i+1}] \\ h(t_i^+), & \text{if } \quad t = t_i \end{cases}$$

Next, we prove equicontinuity at $t = t_i^+$. Fix $\delta_2 > 0$ such that $\{t_k, k \neq i\} \cap [t_i - \delta_2, t_i + \delta_2] = \emptyset$. First we study the equicontinuity at $t = 0^+$.

If $t \in [0, t_1]$ we have

$$\hat{h}_1(t) = \begin{cases} h(t), & \text{if } t \in (0, t_1] \\ 0, & \text{if } t = 0 \end{cases}$$

For $0 < \rho < \delta_2$, we have

$$|\hat{h}_1(\rho) - \hat{h}_1(0)| \leq e^{-\omega\rho}\psi(q)\int_0^{\rho} e^{-\omega s}p(s)ds$$

The right hand-side tends to zero as $\rho \to 0$. (I is the unitary operator)
Now we study the equicontinuity at $t = t_i^+, i \geq 1$ For $0 < \rho < \delta_2$, we have

$$\begin{aligned} |\hat{h}(t_i + \rho) - \hat{h}(t_i)| &\leq \psi(q)\sum_{l=1}^{i}\int_{t_{l-1}}^{t_l} \|S_\alpha(\rho) - I\|. \\ &. \prod_{j=l}^{i-1}\|S_\alpha(t_{j+1} - t_j)\|\|S_\alpha(t_l - s)\||p(s)|ds \\ &+ M\psi(q)e^{\omega(t_i+\rho)}\int_{t_i}^{t_i+\rho} e^{-\omega s}p(s)ds \\ &+ \sum_{l=1}^{i}\|S_\alpha(\rho) - I\|. \\ &. \prod_{j=l}^{i-1}\|S_\alpha(t_{j+1} - t_j)\|.\|I_l(y(t_l^-))\|, \end{aligned}$$

The right hand-side tends to zero as $\rho \to 0$.
The equicontinuity for the cases $\tau_1 < \tau_2 \leq 0$ and $\tau_1 \leq 0 \leq \tau_2$ As a consequence of Steps 1 to 2 together with Arzelá-Ascoli theorem it suffices to show that $\mathcal{A}$ maps B_q into a precompact set in E.
Let $0 < t^* < b$ be fixed and let ϵ be a real number satisfying $0 < \epsilon < t^*$. For $x \in B_q$, we define

$$h_\epsilon(t^*) = \begin{cases} \int_0^{t^*-\epsilon} S_\alpha(t^* - \epsilon - s)v(s)ds & \text{if } t^* \in [0, t_1], \\ \sum_{i=1}^{k}\int_{t_{i-1}}^{t_i} S_\alpha(t^* - t_k)\prod_{j=i}^{k-1} S_\alpha(t_{j+1} - t_j). & \\ .S_\alpha(t_i - s)v(s)ds + \int_{t_k}^{t^*-\epsilon} S_\alpha(t^* - \epsilon - s)v(s)ds & \\ +\sum_{i=1}^{k} S_\alpha(t^* - t_k)\prod_{j=i}^{k-1} S_\alpha(t_{j+1} - t_j)I_i(x(t_i^-)), & \text{if } t^* \in (t_k, t_{k+1}]. \end{cases}$$

where $v \in S_{F, x_{\rho(t,x_t+\tilde{\phi}_t)}+\tilde{\phi}_{\rho(t,x_t+\tilde{\phi}_t)}}$. Since $S_\alpha(t^*)$ is a compact operator, the set

$$H^\epsilon(t^*) = \{h_\epsilon(t^*) : \quad h_\epsilon \in \mathcal{A}(x)\}$$

is precompact in E for every ϵ, $0<\epsilon<t^*$. Moreover, for every $h\in\mathcal{A}(x)$ we have

$$|h(t^*)-h_\epsilon(t^*)|\leq\begin{cases}\psi(q)\int_0^{t^*-\epsilon}\|S_\alpha(t^*)-S_\alpha(t^*-\epsilon)\|p(s)ds\\ +M\psi(q)e^{\omega t^*}\int_{t^*-\epsilon}^{t^*}e^{-\omega s}p(s)ds & \text{if}\quad t^*\in[0,t_1],\\ \psi(q)\int_{t_k}^{t^*-\epsilon}\|S_\alpha(t^*)-S_\alpha(t^*-\epsilon)\|p(s)ds\\ +M\psi(q)e^{\omega t^*}\int_{t^*-\epsilon}^{t^*}e^{-\omega s}p(s)ds\\ & \text{if}\quad t^*\in(t_k,t_{k+1}],\end{cases}$$

Therefore, there are precompact sets arbitrarily close to the set $H(t^*)=\{h(t^*):h\in\mathcal{A}(x)\}$. Hence the set $H(t^*)=\{h(t^*):h\in\mathcal{A}(B_q)\}$ is precompact in E. Hence the operator $\mathcal{A}$ is completely continuous.

Step 4: *$\mathcal{A}$ has a closed graph.*

Let $x^n\to x^*$, $h^n\in\mathcal{A}(x)(x^n)$, and $h^n\to h^*$. We shall show that $h^*\in\mathcal{A}(x^*)$. $h^n\in\mathcal{A}(x^n)$ means that there exists $v^n\in S_{F,x^n_{\rho(t,x_t+\tilde{\phi}_t)}+\tilde{\phi}_{\rho(t,x_t+\tilde{\phi}_t)}}$. such that:

$$h^n(t)=\begin{cases}0, & \text{if}\ \ t\in(-\infty,0],\\ \int_0^t S_\alpha(t-s)v^n(s)ds & \text{if}\ \ t\in[0,t_1],\\ \sum_{i=1}^k\int_{t_{i-1}}^{t_i}S_\alpha(t-t_k)\prod_{j=i}^{k-1}S_\alpha(t_{j+1}-t_j).\\ .S_\alpha(t_i-s)v^n(s)ds+\int_{t_k}^t S_\alpha(t-s)v^n(s)ds\\ +\sum_{i=1}^k S_\alpha(t-t_k)\prod_{j=i}^{k-1}S_\alpha(t_{j+1}-t_j)I_i(x(t_i^-)), & \text{if}\ \ t\in(t_k,t_{k+1}].\end{cases}$$

We must prove that there exists $v^*\in S_{F,x^*_{\rho(t,x_t+\tilde{\phi}_t)}+\tilde{\phi}_{\rho(t,x_t+\tilde{\phi}_t)}}$. such that for each $t\in J$ we have

$$h^*(t)=\begin{cases}\int_0^t S_\alpha(t-s)v^*(s)ds & \text{if}\ \ t\in[0,t_1],\\ \sum_{i=1}^k\int_{t_{i-1}}^{t_i}S_\alpha(t-t_k)\prod_{j=i}^{k-1}S_\alpha(t_{j+1}-t_j).\\ .S_\alpha(t_i-s)v^*(s)ds+\int_{t_k}^t S_\alpha(t-s)v^*(s)ds\\ +\sum_{i=1}^k S_\alpha(t-t_k)\prod_{j=i}^{k-1}S_\alpha(t_{j+1}-t_j)I_i(x(t_i^-)), & \text{if}\ \ t\in(t_k,t_{k+1}].\end{cases}$$

Consider the linear and continuous operator $\mathcal{L} : L^1(J,\mathbb{R}) \to \mathcal{B}_b^0$ defined by

$$(\mathcal{L}v)(t) = \begin{cases} \displaystyle\int_0^t S_\alpha(t-s)v(s)ds & \text{if } \ t \in [0,t_1], \\ \displaystyle\sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} S_\alpha(t-t_k) \prod_{j=i}^{k-1} S_\alpha(t_{j+1}-t_j). & \\ \displaystyle .S_\alpha(t_i-s)v(s)ds + \int_{t_k}^{t} S_\alpha(t-s)v(s)ds & \text{if } \ t \in (t_k,t_{k+1}]. \end{cases}$$

We have, if $t \in [0,t_1]$

$$|h^n(t) - h^*(t)| \quad \leq \quad \|h^n - h^*\|_\infty \to 0, \quad \text{as} \ \ n \mapsto \infty.$$

From Lemma 3.2 it follows that $\mathcal{L} \circ S_F$ is a closed graph operator and from the definition of $\mathcal{L}$ one has

$$h^n(t) \in \mathcal{L} \circ S_{F,x^n}.$$

As $x^n \to x^*$ and $h^n \to h^*$, there exists $v^* \in S_{F,x^*}$ such that

$$h^*(t) = \int_0^t S_\alpha(t-s)v^*(s)ds.$$

If $t \in (t_k, t_{k+1}]$

$$\begin{aligned} &\Bigg|\left(h^n(t) - \sum_{i=1}^{k} S_\alpha(t-t_k) \prod_{j=i}^{k-1} S_\alpha(t_{j+1}-t_j) I_i(x(t_i^-))\right) \\ &- \left(h^*(t) - \sum_{i=1}^{k} S_\alpha(t-t_k) \prod_{j=i}^{k-1} S_\alpha(t_{j+1}-t_j) I_i(x(t_i^-))\right)\Bigg| \\ &= \Big|h^n(t) - h^*(t)\Big| \\ &\leq \|h^n - h^*\|_\infty \to 0, \quad \text{as} \ \ n \mapsto \infty. \end{aligned}$$

From Lemma 3.2 it follows that $\mathcal{L} \circ S_F$ is a closed graph operator and from the definition of $\mathcal{L}$ one has

$$h^n(t) - \sum_{i=1}^{k} S_\alpha(t-t_k) \prod_{j=i}^{k-1} S_\alpha(t_{j+1}-t_j) I_i(x(t_i^-)) \in \mathcal{L} \circ S_{F,x^n}.$$

As $x^n \to x^*$ and $h^n \to h^*$, there is a $v^* \in S_{F,x^*}$ such that

$$\begin{aligned} h^*(t) &- \sum_{i=1}^{k} S_\alpha(t-t_k) \prod_{j=i}^{k-1} S_\alpha(t_{j+1}-t_j) I_i(x(t_i^-)) \\ &= \sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} S_\alpha(t-t_k) \prod_{j=i}^{k-1} S_\alpha(t_{j+1}-t_j). \\ &.S_\alpha(t_i-s)v^*(s)ds + \int_{t_k}^{t} S_\alpha(t-s)v^*(s)ds \end{aligned}$$

Hence the multivalued operator $\mathcal{A}$ is upper semi-continuous therefore, it has a closed graph.

Step 5: *A priori bounds on solutions.*

Now, it remains to show that the set

$$\mathcal{E} = \{x \in \mathcal{B}_b^0 : x \in \lambda \mathcal{A}(x), \quad 0 \leq \lambda \leq 1\}$$

is bounded.

Let Let $x \in \mathcal{E}$ be any element. Then there exist $v \in S_{F, x_{\rho(t,x_t+\tilde{\phi}_t)} + \tilde{\phi}_{\rho(t,x_t+\tilde{\phi}_t)}}$. such that

$$x(t) = \begin{cases} \displaystyle\int_0^t S_\alpha(t-s)v(s)ds, & \text{if } t \in [0, t_1], \\ \displaystyle\sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} S_\alpha(t-t_k) \prod_{j=i}^{k-1} S_\alpha(t_{j+1}-t_j). & \\ \displaystyle .S_\alpha(t_i - s)v(s)ds + \int_{t_k}^{t} S_\alpha(t-s)v(s)ds & \\ \displaystyle + \sum_{i=1}^{k} S_\alpha(t-t_k) \prod_{j=i}^{k-1} S_\alpha(t_{j+1}-t_j) I_i(y(t_i^-)), & \text{if } t \in (t_k, t_{k+1}]. \end{cases}$$

Then from (H1),(H2),(H3)

$$\|x(t)\| \leq \begin{cases} \displaystyle Me^{\omega t} \int_0^t e^{-\omega s} p(s) \psi(\|x_{\rho(t,x_t+\tilde{\phi}_t)} + \tilde{\phi}_{\rho(t,x_t+\tilde{\phi}_t)}\| ds, & \text{if } t \in [0, t_1], \\ \displaystyle\sum_{i=1}^{k} M^{k-i+2} e^{\omega t} \int_{t_{i-1}}^{t_i} e^{-\omega s} p(s) \psi(\|x_{\rho(t,x_t+\tilde{\phi}_t)} + \tilde{\phi}_{\rho(t,x_t+\tilde{\phi}_t)}\| ds. & \\ \displaystyle +Me^{\omega t} \int_{t_k}^{t} e^{-\omega s} p(s) \psi((\|x_{\rho(t,x_t+\tilde{\phi}_t)} + \tilde{\phi}_{\rho(t,x_t+\tilde{\phi}_t)}\| ds & \\ \displaystyle +M^* \sum_{i=1}^{k} M^{k-i+1} e^{\omega(t-t_i)} & \text{if } t \in (t_k, t_{k+1}]. \end{cases}$$

$$\|x(t)\| \leq \begin{cases} \displaystyle Me^{\omega t} \int_0^t e^{-\omega s} p(s) \psi\Big(K_b|x(s)| + (M_b + L^\phi + MK_b)||\phi||_{\mathcal{D}}\Big) ds, & \text{if } t \in [0, t_1], \\ \displaystyle\sum_{i=1}^{k} M^{k-i+2} e^{\omega t} \int_{t_{i-1}}^{t_i} e^{-\omega s} p(s) \psi\left(K_b|x(s)| + (M_b + L^\varphi + MK_b)||\varphi||_{\mathcal{D}}\right) ds. & \\ \displaystyle +Me^{\omega t} \int_{t_k}^{t} e^{-\omega s} p(s) \psi\Big(K_b|x(s)| + (M_b + L^\varphi + MK_b)||\varphi||_{\mathcal{D}}\Big) ds & \\ \displaystyle +M^* \sum_{i=1}^{k} M^{k-i+1} e^{\omega(t-t_i)} & \text{if } t \in (t_k, t_{k+1}]. \end{cases}$$

$$\|x(t)\| \leq \begin{cases} \displaystyle Me^{\omega b} \int_0^t e^{-\omega s} p(s) \psi\Big(K_b|x(s)| + (M_b + L^\phi + MK_b)||\phi||_{\mathcal{D}}\Big) ds, \text{ if } t \in [0, t_1], \\ \displaystyle C_1 + C_2 \int_{t_k}^{t} e^{-\omega s} p(s) \psi\Big(K_b|x(s)| + (M_b + L^\varphi + MK_b)||\varphi||_{\mathcal{D}}\Big) ds \\ \hfill \text{if } t \in (t_k, t_{k+1}]. \end{cases}$$

$$(M_b+L^{\phi}+MK_b)||\phi||_{\mathcal{D}}+K_b\|x(t)\| \leq \begin{cases} C_0^* + C_2^* \displaystyle\int_0^t e^{-\omega s} p(s)\psi\Big(K_b|x(s)| \\ +(M_b + L^{\phi} + MK_b)||\phi||_{\mathcal{D}}\Big)ds, \qquad \text{if} \ \ t \in [0, t_1], \\ C_1^* + C_2^* \displaystyle\int_{t_k}^t e^{-\omega s} p(s)\psi\Big(K_b|x(s)| \\ +(M_b + L^{\phi} + MK_b)||\phi||_{\mathcal{D}}\Big)ds, \ \text{if} \ \ t \in (t_k, t_{k+1}]. \end{cases} \tag{3.11}$$

Thus

$$\frac{(M_b + L^{\phi} + MK_b)||\phi||_{\mathcal{D}} + K_b\|x(t)\|_{\mathcal{B}_b^0}}{C_i^* + C_2^*\psi\left(K_b||x(s)||_{\mathcal{B}_b^0} + (M_b + L^{\phi} + MK_b)||\phi||_{\mathcal{D}}\right)\displaystyle\int_0^b e^{-\omega s}p(s)ds} \leq 1, \quad i = 0, 1 \tag{3.12}$$

From (3.2) it follows that there exists a constant $R > 0$ such that for each $x \in \mathcal{E}$ with $\|x\|_{\mathcal{B}_b^0} > R$ the condition (3.12) is violated. Hence $\|x\|_{\mathcal{B}_b^0} \leq R$ for each $x \in \mathcal{E}$, which means that the set $\mathcal{E}$ is bounded. As a consequence of Theorem of Leray-Schauder, the multivalued operator $\mathcal{A}$ has a fixed point x $\mathcal{B}_b^0$, hence the multivalued operator N has one on the interval $[-r, b]$ which is a mild solution of problem (1.1)-(1.3). □

4. Example

Let $X = L^2(0, \pi)$, $0 < \alpha < 1$. Consider the following fractional order partial differential inclusion of the form:

$$\begin{aligned} &\frac{d^{\alpha}}{dt^{\alpha}}w(t,x) \in \partial_x^2 w(t,x) + k(t)a(t, w(t - \sigma(w(t,0)), x)), \\ &x \in (0,\pi), t \in J := [0,1], \quad t \neq t_k, \ k = 1, \dots, m \end{aligned} \tag{4.1}$$

$$w(t, 0) = w(t, \pi) = 0, \quad t \in [0, 1], \quad t \neq t_k, \ k = 1, \dots, m \tag{4.2}$$

$$w(t, x) = h(t, x), \ t \in (-\infty, 0], x \in [0, \pi] \tag{4.3}$$

$$\Delta w(t_i)(x) = \int_{-\infty}^{t_i} \gamma_i(t_i - s)[\langle -|w(s, x)|, |w(s, x)|\rangle]ds, \tag{4.4}$$

where $h : (-\infty, 0] \times [0, \pi] \longrightarrow \mathbb{R}$, $\gamma_i : [0, \infty) \longrightarrow \mathbb{R}$ are continuous functions, $0 < t_1 < t_2 < \ldots < t_m < 1$, $k : [0, 1] \longrightarrow \mathbb{R}^+$ $a : [0, 1] \times \mathbb{R} \longrightarrow \mathcal{P}_{cv,cp}(\mathbb{R})$, $\sigma : \mathbb{R} \longrightarrow \mathbb{R}^+$ is continuous. We assume the existence of positive constants b_1, b_2 such that

$$|a(t, u)| \leq b_1|x| + b_2 \quad \text{for every} \quad (t, u) \in [0, 1] \times \mathbb{R}$$

Let A be the operator defined as:

$$Au == u'' \quad \text{with} \quad D(A) = \left\{u \in H_0^1(0, \pi) \cap H^2(0, \pi)\right\}$$

The operator A is the infinitesimal generator of an anlytic semi-group S(t).
Set $\gamma > 0$. For the phase space, we choose $\mathcal{D}$ to be defined by:

$$\mathcal{D} = PC^{\gamma} = \left\{\Phi \in PC((-\infty, 0], X) : \lim_{\theta \mapsto -\infty} e^{\gamma^{\theta}} \Phi(\theta) \quad \text{exists in} \quad X\right\}$$

with norm

$$||\phi||_{\gamma} = \sup_{\theta \in (-\infty, 0]} e^{\gamma^{\theta}} |\phi(\theta)|, \ \phi \in PC^{\gamma}$$

For this space, axioms (A1), (A2) are satisfed(see [11]) The problem (4.1)-(4.4) takes the abstract form (1.1)-(1.3) by making the following change of variables.

$$y(t)(x) = w(t,x),\ x \in (0,\pi), t \in J := [0,1],$$

$$\phi(\theta)(x) = h(t,x),\ x \in (0,\pi), \theta \leq 0,$$

$$F(t,\varphi)(x) = k(t)a(t,\varphi(0,x)),\ t \in [0,1],\ x \in [0,\pi], \varphi \in PC^{\gamma} \tag{4.5}$$

$$\rho(t,\varphi) = t - \sigma(\varphi(0,0)) \tag{4.6}$$

$$I_k(y_{t_k}) = \int_{-\infty}^{0} \gamma_k(-s)[\langle -|h(s,x)|, |h(s,x)|\rangle]ds \tag{4.7}$$

Moreover, we have

$$||F(t,\varphi)||_{\mathcal{P}} \leq k(t)(b_1||\varphi||_{\mathcal{D}} + b_2, \quad \text{forall } (t,\varphi) \in J \times \mathcal{D}$$

with

$$\int_1^{\infty} \frac{ds}{\psi(s)} = \int_1^{\infty} \frac{ds}{b_1 s + b_2} = +\infty$$

Theorem 4.1. *Let $\varphi \in \mathcal{B}$ such that H_{φ} holds, the problem (4.1)-(4.4) has at least one mild solurion.*

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