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# GAP FUNCTIONS AND ERROR BOUNDS FOR RANDOM EXTENDED GENERALIZED VARIATIONAL INEQUALITY PROBLEM

A.H. DAR AND M.K. AHMAD

Abstract. In this paper, we introduce a new type of variational inequality called random extended generalized variational inequality problem . We find error bounds for the variational inequality problem with the help of gap functions. The results obtained in this paper improve and generalize some corresponding known results in literature.

## 1. Introduction

One of the basic approaches to study variational inequality is to convert it into an identical optimization problem. The advantage of this approach is that optimization problem may be solved by descent algorithms which possess a global convergence property. Gap functions are used to transform variational inequality into an identical optimization problem. Recently many authors have developed gap functions for different kinds of variational inequality problems; as for example [3, 4, 5, 6, 12, 13, 15, 16, 17, 18, 23, 24, 27, 26, 28, 30, 31, 32, 35]. Besides these, gap functions are used in constructing globally convergent algorithms, in analysing the rate of convergence of some iterative methods and in getting error bounds.

Due to uncertainty in the real world decision problems, variational inequalities in fuzzy setting have become important problems both in theory and practice. It was Chang and Zhu [7] who set in motion the concept of variational inequalities for fuzzy mappings, in 1989. Different types of variational inequalities and complementarity problems for fuzzy mapping were studied by Chang et al. [10], Chang and Salahuddin [11], Anastassios and Salahuddin [2], Ahmad et al. [1], Verma and Salahuddin [33], Lee et al. [25], Park et al. [29], Khan et al. [22], Ding et al. [14] and Lan and Verma [9].

In this paper, we have developed gap functions for random extended generalized variational inequality problem in fuzzy setting. Then we get error bounds for the variational inequality problem with the help of residual vector. Gap functions and error bounds in fuzzy setting were first studied by Khan et al. [24].

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## 2. Preliminaries and Definitions

In this paper, we consider $(\mathrm{S}, \mathbb{T})$ as measurable space where S is a set and $\mathbb{T}$ is a $\sigma$-algebra of subsets of S , and $\mathbb{H}$ as real Hilbert space with inner product and norm designated by $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$ respectively. We use the notions $\mathrm{B}(\mathbb{H})$, $2^{\mathbb{H}}$, $CB(\mathbb{H})$ and $\mathbb{H}(\cdot,\cdot)$ to denote the collection of Borel $\sigma$-fields in $\mathbb{H}$, the class of all non empty subsets of $\mathbb{H}$, the class of all nonempty closed bounded subsets of $\mathbb{H}$, and the Hausdorff metric on $CB(\mathbb{H})$, respectively.

Let $\mathcal{F}(\mathbb{H})$ be the collection of all fuzzy sets over $\mathbb{H}$. A mapping $M : \mathbb{H} \to \mathcal{F}(\mathbb{H})$ is called a fuzzy mapping on $\mathbb{H}$. If $M$ is a fuzzy mapping on $\mathbb{H}$, then for any $z \in \mathbb{H}$, $M(z)$ (denoted by $M_z$ in the sequel) is a fuzzy set on $\mathbb{H}$ and $M_z w$ is the membership function of w in $M_z$. The set $(F)_\lambda = \{z \in \mathbb{H} : F(z) \geq \lambda\}$, $F \in \mathcal{F}(\mathbb{H})$, $\lambda \in [0, 1]$ is called a $\lambda$-cut of $F$.

**Definition 2.1.** [19] *A fuzzy mapping $M : \mathrm{S} \times \mathbb{H} \to \mathcal{F}(\mathbb{H})$ is said to be a random fuzzy mapping, if for any $z \in \mathbb{H}$, $M(., z) : \mathrm{S} \to \mathcal{F}(\mathbb{H})$ is a measurable fuzzy mapping.*

**Definition 2.2.** [19] *A mapping $M : \mathrm{S} \times \mathbb{H} \to 2^{\mathbb{H}}$ is said to be a random set-valued mapping if for any $z \in \mathbb{H}$, $M(., z)$ is measurable. A random set-valued mapping $M : \mathrm{S} \times \mathbb{H} \to CB(\mathbb{H})$ is said to be $\mathbb{H}$- continuous if for any $s \in \mathrm{S}$, $M(., z)$ is continuous in the Hausdorff metric.*

**Definition 2.3.** [19] *A mapping $z : \mathrm{S} \to \mathbb{H}$ is said to be measurable selection of a set-valued measurable mapping $M : \mathrm{S} \to 2^{\mathbb{H}}$ if $z$ is measurable and for any $s \in \mathrm{S}, z(s) \in M(s)$.*

The multi-valued mappings, random multi valued mappings, and fuzzy mappings are some particular cases of the random fuzzy mappings.

Let $M, N : \mathrm{S}\times\mathbb{H} \to \mathcal{F}(\mathbb{H})$ be two random fuzzy mappings fulfilling the condition:
$(a_1)$: There exist two mappings $c, d : \mathbb{H} \to (0, 1]$ such that
$(T_{s,z})_{c(z)} \in CB(\mathbb{H})$, $(N_{s,z})_{d(z)} \in CB(\mathbb{H})$, $\forall(s, z) \in \mathrm{S} \times \mathbb{H}$.

Using these two random fuzzy mappings $M, N$, the random set-valued mappings $\hat{M}$ and $\hat{N}$ can be defined as follows:
$\hat{M} : \mathrm{S} \times \mathbb{H} \to CB(\mathbb{H})$, $z \mapsto (M_{s,z})_{c(z)}$ $\forall(s, z) \in \mathrm{S} \times \mathbb{H}$
$\hat{N} : \mathrm{S} \times \mathbb{H} \to CB(\mathbb{H})$, $z \mapsto (N_{s,z})_{d(z)}$ $\forall(s, z) \in \mathrm{S} \times \mathbb{H}$.

The mappings $\hat{M}$ and $\hat{N}$ are known as random set-valued mappings induced by random fuzzy mappings $M$ and $N$, respectively.

For mappings $c, d : \mathbb{H} \to (0, 1]$, the random fuzzy mappings $M, N : \mathrm{S}\times\mathbb{H} \to \mathcal{F}(\mathbb{H})$ fulfills condition $(a_1)$ and random operator $g : \mathrm{S}\times\mathbb{H} \to \mathbb{H}$ with $Img \cap dom(\partial\phi) \neq \emptyset$, we have considered the following problem:

Find measurable mappings $z, u, v, : \mathrm{S} \to \mathbb{H}$, such that for all $s \in \mathrm{S}$, $w(s) \in \mathbb{H}$ and $z(s) \in \mathbb{H}$, $M_{s,z(s)}(u(s)) \geq c(z(s))$, $N_{s,z(s)}(v(s)) \geq d(z(s))$, $g(s, z(s)) \cap dom(\partial\phi) \neq \emptyset$, and

$$\langle u(s) - v(s), w(s) - g(s, z(s))\rangle + \phi(w(s)) - \phi(g(s, z(s))) \geq 0, \tag{2.1}$$

where $\phi : \mathbb{H} \to R \cup \{+\infty\}$ is a proper, convex and lower semi-continuous function and $\partial\phi$ denotes its sub-differential. This problem is called random extended generalized variational inequality problem (REGVIP). The set of measurable mappings (z,u,v) is known as random solution of the random extended generalized variational inequality (REGVIP) (2.1).

**Special cases**:

If we take $d$ as zero operator, then the problem (2.1) is equivalent to random generalized variational inequality problem (RGVIP), which is finding measurable mappings $z, u : \mathrm{S} \to \mathbb{H}$, such that, $\forall\ s \in \mathrm{S},\ w(s) \in \mathbb{H}$,

$$M_{s,z(s)}(u(s)) \geq c(z(s)),\ \langle u(s), w(s) - g(s, z(s))\rangle + \phi(w(s)) - \phi(g(s, z(s))) \geq 0. \quad (2.2)$$

This problem was studied by Khan et al. [24]. They derived error bounds for problem (2.2).

In the above problem (2.2), if we take $c$ as a zero operator and $M : \mathbb{H} \to \mathbb{H}$ a single valued operator, then the problem becomes the generalized mixed variational inequality problem (GMVIP), which is finding $z \in \mathbb{H}$ such that

$$\langle Mz, w - g(z)\rangle + \phi(w) - \phi(g(z)) \geq 0,\ \forall w \in \mathbb{H}. \quad (2.3)$$

This problem was investigated by Solodov [30]. He obtained error bounds for the variational inequality by using gap functions.

If $g(z) = z, \forall z \in \mathbb{H}$, then problem GMVIP (2.3) becomes mixed variational inequality problem (MVIP), which is to find $z \in \mathbb{H}$ such that

$$\langle Mz, w - z\rangle + \phi(w) - \phi(z) \geq 0,\ \forall w \in \mathbb{H}. \quad (2.4)$$

This problem was investigated by Tang and Huang [32]. They constructed two regularized gap functions for the above MVIP(2.4) and investigated their differentiable properties.

If the funcion $\phi(.)$ is taken as an indicator function of a closed set $K$ in $\mathbb{H}$, then problem MVIP (2.4) collapses to classical variational inequality problem(VIP), which is to find $z \in K$ such that

$$\langle Mz, w - z\rangle \geq 0,\ \forall w \in K. \quad (2.5)$$

This was investigated by many authors, like [4, 16, 28, 32]. They obtained error bounds for VIP (2.5) by using regularized gap functions and the D-gap functions.

Khan et al. [24] constructed the natural residual vector in the random fuzzy setting for RGVIP (2.2) as follows:

$$R^{\phi}_{\lambda(s)}(s, z(s)) = g(s, z(s)) - P^{\phi;z}_{\lambda(s)}[g(s, z(s)) - \lambda(s)u(s)],\ z(s) \in \mathbb{H}.$$

Motivated by [24], we construct a natural residual vector for (2.1) as:

$$R^{\phi}_{\lambda(s)}(s, z(s)) = g(s, z(s)) - J^{\phi}_{\lambda(s)}[g(s, z(s)) - \lambda(s)(u(s) - v(s))],$$

where $\lambda : \mathrm{S} \to (0, \infty)$ is a measurable function and $J^{\phi}_{\lambda(s)} = (I + \lambda(s)\partial\phi)^{-1}$ is the proximal mapping on $\mathbb{H}$.

**Lemma 2.1.** *The measurable function $z : \mathrm{S} \to \mathbb{H}$ is a solution of REGVIP* (2.1) *if and only if $R^{\phi}_{\lambda(s)}(s, z(s)) = 0$, where $\lambda : \mathrm{S} \to (0, +\infty)$ is a measurable function.*

*Proof.* Suppose $R^{\phi}_{\lambda(s)}(s, z(s)) = 0$, this means that

$$g(s, z(s)) = J^{\phi}_{\lambda(s)}[g(s, z(s)) - \lambda(s)(u(s) - v(s))].$$

Now from the definition of proximal mapping, it follows that

$$g(s, z(s)) = arg \min_{w(s) \in \mathbb{H}} \big\{\phi(w(s)) + \frac{1}{2\lambda(s)}\|w(s) - g(s, z(s)) - \lambda(s)(u(s) - v(s))\|^2\big\}.$$

By using the condition of optimality (which are sufficient and necessary by convexity), we can write

$$\begin{aligned} 0 \in \partial\phi(g(s,z(s))) \quad + \quad & \frac{1}{\lambda(s)}\big(g(s,z(s)) - (g(s,z(s)) - \lambda(s)(u(s)-v(s))))\big) \\ & = \partial\phi(g(s,z(s))) + \big(u(s)-v(s)\big), \end{aligned}$$

so one can write $-\big((u(s)-v(s)\big) \in \partial\phi(g(s,z(s)))$.
Finally using the definition of $\partial\phi$, we get

$$\phi(w(s)) \geq \phi(g(s,z(s))) - \langle u(s)-v(s), w(s)-g(s,z(s))\rangle, \forall w(s) \in \mathbb{H}, s \in \mathrm{S}.$$

Which shows that $z(s)$ solves REGVIP (2.1). □

**Definition 2.4.** *A random set-valued mapping $\hat{M} : \mathrm{S} \times \mathbb{H} \to CB(\mathbb{H})$ is said to be strongly g-monotone, if there exists a measurable function $\gamma_1 : \mathrm{S} \to (0,\infty)$ such that* $\langle u_1(s) - u_2(s), g(s,z_1(s)) - g(s,z_2(s))\rangle$
$\geq \gamma_1(s)\|z_1(s) - z_2(s)\|^2, \ \forall z_i(s) \in \hat{M}(t,u_i), \forall z_i(s) \in \mathbb{H}, i = 1,2, \forall s \in \mathrm{S}.$

**Definition 2.5.** *A random operator $g : \mathrm{S} \times \mathbb{H} \to \mathbb{H}$ is called Lipschitz continuous, if there exists a measurable function $L_1 : \mathrm{S} \to (0,+\infty)$ such that*
$\|g(s,z_1(s)) - g(s,z_2(s))\| \leq L_1(s)\|z_1(s) - z_2(s)\|, \ \forall z_i(s) \in \mathbb{H}, i = 1,2, \forall s \in \mathrm{S}.$

**Definition 2.6.** *A random multi- valued mapping $\hat{M} : \mathrm{S} \times \mathbb{H} \to CB(\mathbb{H})$ is said to be $\mathbb{H}$-Lipschitz continuous, if there exists a measurable function $\theta_1 : \mathrm{S} \to (0,+\infty)$ such that*
$H(\hat{M}(s,z(s)), (\hat{M}(s,z_0(s))) \leq \theta_1(s)\|z(s) - z_0(s)\|, \ \forall z(s), z_0(s) \in \mathbb{H}.$

**Definition 2.7.** *A function $G : \mathbb{H} \to \mathbb{R}$ is said to be a gap function for REGVIP* (2.1)*, if it satisfies the following properties:*

1. $G(u) \geq 0$, $\forall u \in \mathbb{H}$*;*
2. $G(u_*) = 0$*, if and only if* $u_* \in \mathbb{H}$ *solves problem* (2.1)

$R^{\phi}_{\lambda(s)}(s,z(s))$ is a gap function for random extended generalized variational inequality for random fuzzy mappings (2.1). This can be verified by using Lemma 2.1.

**Lemma 2.2.** [8] *Let $\hat{M}_1, \hat{M}_2 : \mathrm{S} \to CB(\mathbb{H})$ be two random measurable multi-valued mappings, $\epsilon > 0$ be a constant, and $u_1 : \mathrm{S} \to \mathbb{H}$ be a measurable selection of $\hat{M}_1$, then there exists a measurable selection $u_2 : \mathrm{S} \to \mathbb{H}$ of $\hat{M}_2$ such that for all $s \in \mathrm{S}$,* $\|u_1(s) - u_2(s)\| \leq (1+\epsilon)\mathbb{H}(\hat{M}_1(s), \hat{M}_2(s))$.

**Lemma 2.3.** *Suppose* $(\mathrm{S}, \mathbb{T})$ *be a measurable space and $\mathbb{H}$ be a Hilbert space. Let $g : \mathrm{S}\times\mathbb{H} \to \mathbb{H}$ be a random fuzzy mapping and $\phi : \mathbb{H} \to R\cup\{+\infty\}$ be an extended real valued function. Suppose that the random fuzzy mappings $M, N : S \times \mathbb{H} \to \mathcal{F}(\mathbb{H})$ satisfy the condition $(a_1)$ and random set-valued mappings $\hat{M}, \hat{N} : \mathrm{S}\times\mathbb{H} \to CB(\mathbb{H})$ induced by random fuzzy mappings mappings $M$ and $N$, respectively, be strongly g-monotone with the measurable functions $\gamma_1, \gamma_2 : \mathrm{S} \to (0,+\infty)$ , then the REGVIP* (2.1) *has a unique solution.*

*Proof.* Let $z_1, z_2 : \mathrm{S} \to \mathbb{H}$ be two random solutions of REGVIP (2.1) with $z_1(s) \neq z_2(s) \in \mathbb{H}$. Then we can write

$$\langle u_1(s) - v_1(s), w(s) - g(s,z_1(s))\rangle + \phi(w(s)) - \phi(g(s,z_1(s))) \geq 0, \tag{2.6}$$

$$\langle u_2(s)-v_2(s),w(s)-g(s,z_2(s))\rangle+\phi(w(s))-\phi(g(s,z_2(s)))\geq 0. \tag{2.7}$$

Putting $w(s)=g(s,z_2(s))$ in (2.6) and $w(s)=g(s,z_1(s))$ in (2.7), then after adding these equations, we get

$$\langle u_1(s)-u_2(s),g(s,z_2(s))-g(s,z_1(s))\rangle-\langle v_1(s)-v_2(s),g(s,z_1(s))-g(s,z_2(s))\rangle\geq 0.$$

By using strongly g-monotonicity of $\hat{M}$ and $\hat{N}$ with measurable functions $\gamma_1,\gamma_2:\mathrm{S}\to(0,+\infty)$ $(\gamma_1(s)>\gamma_2(s))$ respectively, we have

$$\begin{aligned}0\leq\quad &\langle u_1(s)-u_2(s),g(s,z_2(s))-g(s,z_1(s))\rangle\\ &-\langle v_1(s)-v_2(s),g(s,z_1(s))-g(s,z_2(s))\rangle\\ &\leq-(\gamma_1(s)-\gamma_2(s))\|z_1(s)-z_2(s)\|^2,\end{aligned}$$

which implies that $z_1(s)=z_2(s)$. □

## 3. Main results

**Theorem 3.1.** *Let $z_0(s)$, $\forall\ s\in\mathrm{S}$ be a random solution of random extended generalized variational inequality problem* (2.1)*. Let $(\mathrm{S},\mathbb{T})$ be a measurable space, and $\mathbb{H}$ be a real Hilbert space. Let the random fuzzy mappings $M,N:\mathrm{S}\times\mathbb{H}\to\mathcal{F}(\mathbb{H})$ satisfy the condition $(a_1)$ and $\hat{M},\hat{N}:\mathrm{S}\times\mathbb{H}\to CB(\mathbb{H})$ be the random multi-valued mappings induced by random fuzzy mappings $M$ and $N$, respectively. If $g:\mathrm{S}\times\mathbb{H}\to\mathbb{H}$ be random mapping and $\phi:\mathbb{H}\to R\cup\{+\infty\}$ be an extended real valued function such that*

1. *for each $s\in\mathrm{S}$, the measurable mappings $\hat{M}$ and $\hat{N}$ are strongly $g-$monotone with measurable functions $\gamma_1,\gamma_2:\mathrm{S}\to(0,\infty)$, respectively;*
2. *for each $s\in\mathrm{S}$, the measurable mappings $\hat{M}$ and $\hat{N}$ are $\mathbb{H}$-Lipschitz continuous with measurable functions $\theta_1,\theta_2:\mathrm{S}\to(0,+\infty)$, respectively;*
3. *for each $s\in\mathrm{S}$, the mapping $g:\mathrm{S}\times\mathbb{H}\to\mathbb{H}$ are Lipschitz continuous with measurable function $L_1:\mathrm{S}\to(0,+\infty)$,*

*then for any $z(s)\in\mathbb{H}$, $s\in\mathrm{S}$, we have*

$$\|z_0(s)-z(s)\|\leq\frac{\lambda(s)(1+\varepsilon)(\theta_1(s)-\theta_2(s))+L_1(s)}{\lambda(s)(\gamma_1(s)-\gamma_2(s))}$$
$$\times\|R^{\phi}_{\lambda(s)}(s,z(s))\|,$$

*where $\gamma_1(s)>\gamma_2(s)$ and $\theta_1(s)>\theta_2(s)$.*

*Proof.* Let $z_0(s)$, $\forall\ s\in\mathrm{S}$ be a random solution of random extended generalized variational inequality problem (REGVIP) (2.1). Then we have

$$\langle u_0(s)-v_0(s),w(s)-g(s,z_0(s))\rangle+\phi(w(s))-\phi(g(s,z_0(s)))\geq 0,\ \forall w(s)\in\mathbb{H}.$$

Taking $w(s)=J^{\phi,z_0}_{\lambda(s)}[g(s,z(s))-\lambda(s)(u(s)-v(s))]$ in the above inequality, we get

$$\begin{aligned}&\langle u_0(s)-v_0(s),J^{\phi,z_0}_{\lambda(s)}[g(s,z(s))-\lambda(s)(u(s)-v(s))]-g(s,z_0(s))\rangle\\ &+\phi(J^{\phi,z_0}_{\lambda(s)}[g(s,z(s))-\lambda(s)(u(s)-v(s))])-\phi(g(s,z_0(s)))\geq 0,\ \forall w(s)\in\mathbb{H}.\end{aligned}\tag{3.1}$$

For $z(s)\in\mathbb{H}$ and measurable function $\lambda:\mathrm{S}\to(0,+\infty)$, we have

$$\begin{aligned}g(s,z(s))-\lambda(s)(u(s)-v(s))\in\quad&(I+\lambda(s)\partial\phi)(I+\lambda(s)\partial\phi)^{-1}(g(s,z(s))-\lambda(s)(u(s)-v(s)))\\ &=(I+\lambda(s)\partial\phi)J^{\phi,z_0}_{\lambda(s)}[g(s,z(s))-\lambda(s)(u(s)-v(s))],\end{aligned}$$

which means that

$$-(u(s)-v(s)) \qquad +\frac{1}{\lambda(s)}[g(s,z(s))-J^{\phi,z_0}_{\lambda(s)}[g(s,z(s))-\lambda(s)(u(s)-v(s))]]$$
$$\in \partial\phi\big(J^{\phi,z_0}_{\lambda(s)}[g(s,z(s))-\lambda(s)(u(s)-v(s))]\big).$$

By the definition of $\partial\phi(.,.)$, we have

$$\begin{aligned}&\Big\langle u(s)-v(s)-\frac{1}{\lambda(s)}\big(g(s,z(s))-J^{\phi,z_0}_{\lambda(s)}[g(s,z(s))-\lambda(s)(u(s)-v(s))]\big),\\ &w(s)-J^{\phi,z_0}_{\lambda(s)}[g(s,z(s))-\lambda(s)(u(s)-v(s))]\Big\rangle+\phi(w(s))\\ &-\phi(J^{\phi,z_0}_{\lambda(s)}[g(s,z(s))-\lambda(s)(u(s)-v(s))])\geq 0.\end{aligned}$$

Substituting $w(s)=g(s,z_0(s))$ in the above inequality, we get

$$\begin{aligned}&\Big\langle u(s)-v(s)-\frac{1}{\lambda(s)}\big(g(s,z(s))-J^{\phi,z_0}_{\lambda(s)}[g(s,z(s))-\lambda(s)(u(s)-v(s))]\big),\\ &g(s,z_0(s))-J^{\phi,z_0}_{\lambda(s)}[g(s,z(s))-\lambda(s)(u(s)-v(s))]\Big\rangle+\phi(g(s,z_0(s)))\\ &-\phi(J^{\phi,z_0}_{\lambda(s)}[g(s,z(s))-\lambda(s)(u(s)-v(s))])\geq 0.\end{aligned}$$

This can be written as

$$\begin{aligned}&\Big\langle-(u(s)-v(s))+\frac{1}{\lambda(s)}\big(g(s,z(s))-J^{\phi,z_0}_{\lambda(s)}[g(s,z(s))-\lambda(s)(u(s)-v(s))]\big),\\ &J^{\phi,z_0}_{\lambda(s)}[g(s,z(s))-\lambda(s)(u(s)-v(s))]-g(s,z_0(s))\Big\rangle+\phi(g(s,z_0(s)))\\ &-\phi(J^{\phi,z_0}_{\lambda(s)}[g(s,z(s))-\lambda(s)(u(s)-v(s))])\geq 0.\end{aligned}\tag{3.2}$$

Adding (3.1) and (3.2), we have

$$\begin{aligned}&\Big\langle u_0(s)-v_0(s)-(u(s)-v(s))+\frac{1}{\lambda(s)}\big(g(s,z(s))-J^{\phi,z_0}_{\lambda(s)}[g(s,z(s))-\lambda(s)(u(s)-v(s))]\big),\\ &J^{\phi,z_0}_{\lambda(s)}[g(s,z(s))-\lambda(s)(u(s)-v(s))]-g(s,z_0(s))\Big\rangle\geq 0,\end{aligned}$$

which implies that

$$\begin{aligned}&\lambda(s)\big\langle u_0(s)-v_0(s)-(u(s)-v(s)),J^{\phi,z_0}_{\lambda(s)}[g(s,z(s))-\lambda(s)(u(s)-v(s))]-g(s,z(s))\big\rangle\\ &+\lambda(s)\langle u_0(s)-v_0(s)-(u(s)-v(s)),g(s,z(s))-g(s,z_0(s))\rangle\\ &+\big\langle g(s,z(s))-J^{\phi,z_0}_{\lambda(s)}[g(s,z(s))-\lambda(s)(u(s)-v(s))],\\ &J^{\phi,z_0}_{\lambda(s)}[g(s,z(s))-\lambda(s)(u(s)-v(s))]-g(s,z(s))\big\rangle\\ &+\big\langle g(s,z(s))-J^{\phi,z_0}_{\lambda(s)}[g(s,z(s))-\lambda(s)(u(s)-v(s))],g(s,z(s))-g(s,z_0(s))\big\rangle\geq 0,\end{aligned}$$

this can be written as

$$\begin{aligned}
&\lambda(s)\big\langle u_0(s)-u(s), J^{\phi,z_0}_{\lambda(s)}[g(s,z(s))-\lambda(s)(u(s)-v(s))]-g(s,z(s))\big\rangle\\
&-\lambda(s)\big\langle v_0(s)-v(s), J^{\phi,z_0}_{\lambda(s)}[g(s,z(s))-\lambda(s)(u(s)-v(s))]-g(s,z(s))\big\rangle\\
&+\lambda(s)\langle u_0(s)-u(s), g(s,z(s))-g(s,z_0(s))\rangle\\
&-\lambda(s)\langle v_0(s)-v(s), g(s,z(s))-g(s,z_0(s))\rangle\\
&+\big\langle g(s,z(s))-J^{\phi,z_0}_{\lambda(s)}[g(s,z(s))-\lambda(s)(u(s)-v(s))],\\
&J^{\phi,z_0}_{\lambda(s)}[g(s,z(s))-\lambda(s)(u(s)-v(s))]-g(s,z(s))\big\rangle\\
&+\big\langle g(s,z(s))-J^{\phi,z_0}_{\lambda(s)}[g(s,z(s))-\lambda(s)(u(s)-v(s))], g(s,z(s))-g(s,z_0(s))\big\rangle\geq 0,
\end{aligned}$$

this implies that

$$\begin{aligned}
&\lambda(s)\big\langle u_0(s)-u(s), J^{\phi,z_0}_{\lambda(s)}[g(s,z(s))-\lambda(s)(u(s)-v(s))]-g(s,z(s))\big\rangle\\
&-\lambda(s)\langle v_0(s)-v(s), J^{\phi,z_0}_{\lambda(s)}[g(s,z(s))-\lambda(s)(u(s)-v(s))]-g(s,z(s))\rangle\\
&+\big\langle g(s,z(s))-J^{\phi,z_0}_{\lambda(s)}[g(s,z(s))-\lambda(s)(u(s)-v(s))], g(s,z(s))-g(s,z_0(s))\big\rangle\\
&\geq \lambda(s)\langle u_0(s)-u(s), g(s,z_0(s))-g(s,z(s))\rangle\\
&-\lambda(s)\langle v_0(s)-v(s), g(s,z_0(s))-g(s,z(s))\rangle\\
&+\big\langle g(s,z(s))-J^{\phi,z_0}_{\lambda(s)}[g(s,z(s))-\lambda(s)(u(s)-v(s))],\\
&g(s,z(s))-J^{\phi,z_0}_{\lambda(s)}[g(s,z(s))-\lambda(s)(u(s)-v(s))]\big\rangle.
\end{aligned}$$

Now using strongly g-monotonicity of $\hat{M}$ and $\hat{N}$ with measurable functions$\gamma_1,\gamma_2$ : S $\to (0,+\infty)$, the above can be written as

$$\begin{aligned}
&\lambda(s)\langle u_0(s)-u(s), J^{\phi,z_0}_{\lambda(s)}[g(s,z(s))-\lambda(s)(u(s)-v(s))]-g(s,z(s))\rangle\\
&-\lambda(s)\langle v_0(s)-v(s), J^{\phi,z_0}_{\lambda(s)}[g(s,z(s))-\lambda(s)(u(s)-v(s))]-g(s,z(s))\rangle\\
&+\langle g(s,z(s))-J^{\phi,z_0}_{\lambda(s)}[g(s,z(s))-\lambda(s)(u(s)-v(s))], g(s,z(s))-g(s,z_0(s))\rangle\\
&\geq \lambda(s)(\gamma_1(s)-\gamma_2(s))\|z_0(s)-z(t\|^2+\|R^{\phi}_{\lambda(s)}(s,z(s))\|^2.
\end{aligned}$$

By Cauchy-Schwartz inequality and triangular inequality, we can write

$$\begin{aligned}
&\lambda(s)\|u_0(s)-u(s)\|\|J^{\phi,z_0}_{\lambda(s)}[g(s,z(s))-\lambda(s)(u(s)-v(s))]-g(s,z(s))\|\\
&-\lambda(s)\|v_0(s)-v(s)\|\|J^{\phi,z_0}_{\lambda(s)}[g(s,z(s))-\lambda(s)(u(s)-v(s))]-g(s,z(s))\|\\
&+\|\langle g(s,z(s))-J^{\phi,z_0}_{\lambda(s)}[g(s,z(s))-\lambda(s)(u(s)-v(s))]\|\|g(s,z(s))-g(s,z_0(s))\|\\
&\geq \lambda(s)(\gamma_1(s)-\gamma_2(s))\|z_0(s)-z(s)\|^2.
\end{aligned}$$

Using the $\mathbb{H}$-Lipschitz continuity of $\hat{M}$ and $\hat{N}$, and the Lipschitz continuity of g, we get

$$\begin{aligned}
&\lambda(s)\theta_1(s)(1+\varepsilon)\|z_0(s)-z(s)\|\|R^{\phi}_{\lambda(s)}\big(s,z(s)\big)\|\\
&-\lambda(s)\theta_2(s)(1+\varepsilon)\|z_0(s)-z(s)\|\|R^{\phi}_{\lambda(s)}\big(s,z(s)\big)\|\\
&\|R^{\phi}_{\lambda(s)}\big(s,z(s)\big)\|L_1(s)\|z_0(s)-z(s)\|\\
&\geq \lambda(s)(\gamma_1(s)-\gamma_2(s))\|z_0(s)-z(s)\|^2.
\end{aligned}$$

$$\|z_0(s) - z(s)\| \le \frac{\lambda(s)(1+\varepsilon)(\theta_1(s) - \theta_2(s)) + L_1(s)}{\lambda(s)(\gamma_1(s) - \gamma_2(s))} \times \|R^{\phi}_{\lambda(s)}(s, z(s))\|,$$

where $\gamma_1(s) > \gamma_2(s)$ and $\theta_1(s) > \theta_2(s)$. □

**Corollary 3.1.** *If the random fuzzy mapping $N : \mathrm{S} \times \mathbb{H} \to \mathcal{F}(\mathbb{H})$ and the random multi-valued mapping $\hat{N} : \mathrm{S} \times \mathbb{H} \to CB(\mathbb{H})$ are taken as zero mappings then Theorem 3.1 gives the error bound for* (2.2) *without using the condition:*

$$\|J^{\phi,z}_{\lambda(s)}(x) - J^{\phi,z_0}_{\lambda(s)}(x)\| \le K(s)\|z(s) - z_0(s)\|,$$

*which was used by Khan et al.* [24].

## 4. GENERALIZED REGULARIZED GAP FUNCTIONS FOR RANDOM EXTENDED GENERALIZED VARIATIONAL INEQUALITY PROBLEM (2.1)

The regularized gap function $g_\lambda : \mathbb{H} \to \mathbb{R}$ is defined as:

$$g_\lambda(z) = \max_{w \in K}\{\langle Tz, z - w\rangle - \frac{\lambda}{2}\|z - w\|^2\}, \textit{where } \lambda > 0 \textit{ is a positive parameter.}$$

This function was studied by Fukushima [16]. The function $g_\lambda$ converts variational inequality (2.6) into an equivalent constrained optimization formulation as $z$ solves variational inequality problem (2.6) if and only if $z$ minimizes $g_\lambda$ on $K$ and $g_\lambda(z) = 0, \forall z \in \mathbb{H}$. Wu et al. [34] further extended the regularized gap function and considered the function $G_\lambda : \mathbb{H} \to \mathbb{R}$ defined by $g_\lambda(z) = \max_{w \in K}\{\langle Tz, z - w\rangle - \alpha F(x, y)\}$. They showed that $G_\lambda$ constitutes an equivalent constrained differentiable optimization reformulation of the VIP (2.6). Khan et al. [24] constructed a generalized regularized gap function for problem RGVIP(2.2). Inspired and motivated by the above work we construct a generalized regularized gap function $G_{\lambda(s)} : \mathrm{S} \times \mathbb{H} \to \mathbb{R}$, associated with the problem (2.1) defined by

$$G_{\lambda(s)}(z(s)) = \max_{w(s) \in \mathbb{H}} \psi_{\lambda(s)}(z(s), w(s)),$$

$$\textit{where } \lambda : \mathrm{S} \to (0, +\infty) \textit{ is a measurable function,}$$

$$\begin{aligned} G_{\lambda(s)}(z(s)) &= \max_{w(s) \in \mathbb{H}}\{\langle u(s) - v(s), g(s, z(s)) - w(s)\rangle + \phi(g(s, z(s))) \\ &\quad - \phi(w(s)) - \lambda(s)F(g(s, z(s)), w(s)\}, \end{aligned}$$

$$\begin{aligned} G_{\lambda(s)}(z(s)) &= \langle u(s) - v(s), g(s, z(s)) - \pi_{\lambda(s)}z(s)\rangle + \phi(g(s, z(s))) \\ &\quad - \phi(\pi_{\lambda(s)}z(s)) - \lambda(s)F(g(s, z(s)), \pi_{\lambda(s)}z(s)), \end{aligned} \tag{4.1}$$

where $\pi_{\lambda(s)}z(s)$ is the unique minimizer of $-\psi_{\lambda(s)}(z(s), .)$ on $\mathbb{H}$ for each $s \in \mathrm{S}$ and the function $F : \mathbb{H} \times \mathbb{H} \to \mathbb{R}$ satisfies the following properties:

- $(p_1)$ $F$ is nonnegative on $\mathbb{H} \times \mathbb{H}$;
- $(p_2)$ $F$ is continuously differentiable on $\mathbb{H} \times \mathbb{H}$;
- $(p_3)$ $F(z(s), .)$ is strongly convex uniformly in $z(s)$, for each $s \in \mathrm{S}$, i.e., there exists a measurable function $\eta : \mathrm{S} \to (0, +\infty)$ such that, for any $s \in \mathrm{S}$, $z(s) \in \mathbb{H}$,
  $F(z(s), w_1(s)) - F(z(s), w_2(s))$
  $\ge \langle \nabla_2 F(z(s), w_2(s)), w_1(s) - w_2(s)\rangle + \eta(s)\|w_1(s) - w_2(s)\|^2, \forall w_1(s), w_2(s) \in$

$\mathbb{H}$,
where $\nabla_2 F$ denotes the partial differential of $F$ with respect to the second variable of $F$;

($p_4$) $F(z(s), w(s)) = 0$ if and only if $z(s) = w(s),\ \forall s \in \mathrm{S}$;

($p_5$) $\nabla_2 F(z(s), .)$ is uniformly Lipschitz continuous, i.e., there exists a measurable function $\alpha : \mathrm{S} \to (0, +\infty)$ such that, for any $s \in \mathrm{S}$, $z(s) \in \mathbb{H}$, $\|\nabla_2 F(z(s), w_1(s)) - \nabla_2 F(z(s), w_2(s))\| \leq \alpha(s)\|w_1(s) - w_2(s)\|,\ w_1(s), w_2(s) \in \mathbb{H}$.

**Lemma 4.1.** [24] *If $F$ satisfies ($p_1$) -($p_4$), then for each $\nabla_2 F(z(s), w(s)) = 0$ if and only if $z(s) = w(s), \forall s \in \mathrm{S}$.*

**Lemma 4.2.** [24] *If the function $F$ satisfies ($p_3$). Then $\nabla_2 F(z(s), .)$ is strongly monotone with modulus $2\eta(s)$ on $\mathbb{H}$, i.e.,*
$\langle \nabla_2 F(z(s), w_1(s)) - \nabla_2 F(z(s), w_2(s)), w_1(s) - w_2(s)\rangle \geq 2\eta(s)\|w_1(s) - w_2(s)\|^2$
$\forall w_1(s), w_2(s) \in \mathbb{H}$.

**Lemma 4.3.** [24] *If the function $F$ satisfies ($p_1$)- ($p_5$) with the associated measurable functions $\alpha, \eta : \mathrm{S} \to (0, +\infty)$. Then*
$F(z(s), w(s)) \leq (\alpha(s) - \eta(s))\|z(s) - w(s)\|^2\ \forall z(s), w(s) \in \mathbb{H}$.

Using these Lemmas, we prove the following Lemma:

**Lemma 4.4.** *If the function $F$ satisfies ($p_1$)- ($p_4$). Then measurable mapping $z : \mathrm{S} \to \mathbb{H}$ is a solution of random extended generalized variational inequality* (2.1) *if and only if $\forall s \in \mathrm{S}, g(s, z(s)) = \pi_{\lambda(s)}(z(s))$.*

*Proof.* Since $\pi_{\lambda(s)}(z(s))$ minimizes $-\psi_{\lambda(s)}(z(s), .)$ on $\mathbb{H}$, and $-\psi_{\lambda(s)}(z(s), .)$ is convex for any $z(s) \in \mathbb{H}$, we have

$$\begin{aligned}
&0 \in \partial\psi_{\lambda(s)}(z(s), w(s)) \\
&= u(s) - v(s) + \partial\phi(\pi_{\lambda(s)}(z(s))) + \lambda(s)\nabla_2 F(g(s, z(s)), \pi_{\lambda(s)}(z(s))) \\
&-(u(s) - v(s) - \lambda(s)\nabla_2 F(g(s, z(s)), \pi_{\lambda(s)}(z(s))) \in \partial\phi(\pi_{\lambda(s)}(z(s))).
\end{aligned}$$

By the definition of $\partial\phi$, we have

$$\begin{aligned}
\phi(z(s)) \quad \geq \quad & \phi(\pi_{\lambda(s)}(z(s))) - \langle u(s) - v(s) + \lambda(s)\nabla_2 F(g(s, z(s)), \pi_{\lambda(s)}(z(s))), \\
& w(s) - \pi_{\lambda(s)}(z(s)))\rangle.
\end{aligned}$$

This implies that

$$\begin{aligned}
&\langle u(s) - v(s), w(s) - \pi_{\lambda(s)}(z(s))\rangle + \phi(w(s)) - \phi(\pi_{\lambda(s)}(z(s))) \\
&\geq \lambda(s)\langle -\nabla_2 F(g(s, z(s)), \pi_{\lambda(s)}(z(s))), w(s) - \pi_{\lambda(s)}(z(s)))\rangle. \qquad (4.2)
\end{aligned}$$

Now by Lemma 4.1, we see that $z(s)$ is a solution of the random extended generalized variational inequality (2.1) if and only if $g(s, z(s)) = \pi_{\lambda(s)}(z(s)))$.

Conversely, $z(s)$ is a solution of the the random extended generalized variational inequality (2.1), then taking $z(s) = \pi_{\lambda(s)}(z(s)))$ in (2.1), we obtain

$$\langle u(s) - v(s), \pi_{\lambda(s)}(z(s))) - g(s, z(s))\rangle + \phi(\pi_{\lambda(s)}(z(s)))) - \phi(g(s, z(s))) \geq 0.$$

Since $g(s, z(s)) \in \mathbb{H}$, so from equation (4.2), we can write

$$\begin{aligned}
&\langle u(s) - v(s), g(s, z(s)) - \pi_{\lambda(s)}(z(s)))\rangle + \phi(g(s, z(s))) - \phi(\pi_{\lambda(s)}(z(s))) \\
&\qquad \geq \lambda(s)\langle -\nabla_2 F(g(s, z(s)), \pi_{\lambda(s)} z(s)), w(s) - \pi_{\lambda(s)}(z(s))\rangle.
\end{aligned}$$

Adding the above two inequalities, we get

$$\lambda(s)\langle \nabla_2 F(g(s,z(s)),\pi_{\lambda(s)}(z(s))), g(s,z(s))-\pi_{\lambda(s)}z(s)\rangle \geq 0.$$

Now by using the strong convexity of $F(g(s,z(s)),.)$, together with $(p_1)$ and $(p_4)$, we get

$$\begin{aligned}&\lambda(s)\langle \nabla_2 F(g(s,z(s)),\pi_{\lambda(s)}z(s)), g(s,z(s))-\pi_{\lambda(s)}(z(s))\rangle\\&+\beta(s)\|g(s,z(s))-\pi_{\lambda(s)}z(s))\|^2\\&\leq F(g(s,z(s)),g(s,z(s)))-F(g(s,z(s)),\pi_{\lambda(s)}z(s))\leq 0.\end{aligned}$$

From the above inequalities, it follows that $g(s,z(s))=\pi_{\lambda(s)}(z(s))$. □

**Theorem 4.1.** *Let $z_0(s)\in\mathbb{H}$ be a solution of random extended generalized variational inequality problem (REGVIP) (2.1) for all $s\in\mathrm{S}$. Suppose that for each $s\in\mathrm{S}$, the multi-valued mappings $\hat{M}$ and $\hat{N}$ are strongly g-monotone with measurable functions $\gamma_1,\gamma_2:\mathrm{S}\to(0,+\infty)$ respectively. Also assume that $\hat{M}$ and $\hat{N}$ are $\mathbb{H}$-Lipschitz with measurable functions $\theta_1,\theta_2:\mathrm{S}\to(0,+\infty)$. If $g:\mathrm{S}\times\mathbb{H}\to\mathbb{H}$ is Lipschitz continuous with measurable function $L_1:\mathrm{S}\to(0,+\infty)$ and the function $F$ satisfies $(p_1)$- $(p_5)$, then*

$$\begin{aligned}\|z(s)-z_0(s)\|\leq&\frac{\big((\theta_1(s)-\theta_2(s))(1+\varepsilon)+\lambda(s)\alpha(s)L_1(s)\big)}{\gamma_1(s)-\gamma_2(s)}\\&\times\|g(s,z(s))-\pi_{\lambda(s)}(z(s)))\|,\end{aligned}$$

*where $\gamma_1(s)>\gamma_2(s)$ and $\theta_1(s)>\theta_2(s)$.*

*Proof.* As $z_0(s)$ is a solution of REGVIP (2.1) and $\pi_{\lambda(s)}(z(s)))\in\mathbb{H}$ for each $z(s)\in\mathbb{H}$, we can write

$$\langle u_0(s)-v_0(s),\pi_{\lambda(s)}(z(s)))-g(s,z_0(s))\rangle+\phi(\pi_{\lambda(s)}(z(s)))-\phi(g(s,z_0(s)))\geq 0. \quad (4.3)$$

Putting $w(s)=g(s,z_0(s))$ in (4.2), we have

$$\begin{aligned}&\langle u(s)-v(s),g(s,z_0(s))-\pi_{\lambda(s)}z(s)\rangle+\phi(g(s,z_0(s)))-\phi(\pi_{\lambda(s)}(z(s))\\&\geq\lambda(s)\langle-\nabla_2F(g(s,z(s)),\pi_{\lambda(s)}z(s)),g(s,z_0(s))-\pi_{\lambda(s)}(z(s))\rangle.\end{aligned}\quad (4.4)$$

Adding the two inequalities (4.3) and (4.4), we get

$$\begin{aligned}&\langle u(s)-v(s)-(u_0(s)-v_0(s)),\pi_{\lambda(s)}(z(s)))-g(s,z_0(s))\rangle\\&\leq\lambda(s)\langle\nabla_2F(g(s,z(s)),\pi_{\lambda(s)}(z(s))),g(s,z_0(s))-\pi_{\lambda(s)}(z(s)))\rangle.\end{aligned}\quad (4.5)$$

Using Lemmas (4.1), (4.2) and $((p_5))$, we get

$$\begin{aligned}&\lambda(s)\langle\nabla_2F(g(s,z(s)),\pi_{\lambda(s)}(z(s)),g(s,z_0(s))-\pi_{\lambda(s)}(z(s)))\rangle\\&=\lambda(s)\langle\nabla_2F(g(s,z(s)),\pi_{\lambda(s)}(z(s)))-\nabla_2F(g(s,z(s)),g(s,z(s))),\\&g(s,z_0(s))-g(s,z(s))\rangle\\&-\lambda(s)\langle\nabla_2F(g(s,z(s)),g(s,z(s)))-\nabla_2F(g(s,z(s)),\pi_{\lambda(s)}(z(s))),\\&g(s,z(s))-\pi_{\lambda(s)}(z(s)))\rangle\\&\leq\lambda(s)\|\nabla_2F(g(s,z(s)),\pi_{\lambda(s)}(z(s)))-\nabla_2F(g(s,z(s)),g(s,z(s)))\|\\&\times\|g(s,z_0(s))-g(s,z(s))\|-2\lambda(s)\eta(s)\|g(s,z(s))-\pi_{\lambda(s)}(z(s)))\|^2\\&\leq\lambda(s)\alpha(s)\|g(s,z(s))-\pi_{\lambda(s)}(z(s)))\|g(s,z_0(s))-g(s,z(s))\|\\&-2\lambda(s)\eta(s)\|g(s,z(s))-\pi_{\lambda(s)}(z(s)))\|^2.\\&\leq\lambda(s)\alpha(s)\|g(s,z(s))-\pi_{\lambda(s)}(z(s)))\|g(s,z_0(s))-g(s,z(s))\|.\end{aligned}$$

Now using the Lipschitz continuity of g, we can write

$$\begin{aligned}&\lambda(s)\langle\nabla_2 F(g(s,z(s)),\pi_{\lambda(s)}(z(s))),g(s,z_0(s))-\pi_{\lambda(s)}(z(s)))\rangle\\&\leq \lambda(s)\alpha(s)L_1(s)\|g(s,z(s))-\pi_{\lambda(s)}(z(s)))\|z_0(s)-z(s)\|.\end{aligned}\tag{4.6}$$

It follows from (4.5) and (4.6) that

$$\begin{aligned}&\langle u(s)-v(s)-(u_0(s)-v_0(s)),\pi_{\lambda(s)}(z(s)))-g(s,z_0(s))\rangle\\&\leq \lambda(s)\alpha(s)L_1(s)\|g(s,z(s))-\pi_{\lambda(s)}(z(s))\|z_0(s)-z(s)\|.\end{aligned}\tag{4.7}$$

Now using the strongly g-monotonicity of $\hat{M}$ and $\hat{N}$ with measurable functions $\gamma_1,\gamma_2:\mathrm{S}\to\mathbb{H}$ , we can write

$$\begin{aligned}&\big(\gamma_1(s)-\gamma_2(s)\big)\|z(s)-z_0(s)\|^2\\&\leq\langle u(s)-v(s)-(u_0(s)-v_0(s)),g(s,z(s))-g(s,z_0(s))\rangle\\&\leq\langle u(s)-v(s)-(u_0(s)-v_0(s)),g(s,z(s))-\pi_{\lambda(s)}(z(s)))\rangle\\&+\langle u(s)-v(s)-(u_0(s)-v_0(s)),\pi_{\lambda(s)}(z(s)))-g(s,z_0(s))\rangle\\&\leq\langle u(s)-u_0(s),g(s,z(s))-\pi_{\lambda(s)}(z(s)))\rangle\\&-\langle v(s)-v_0(s),g(s,z(s))-\pi_{\lambda(s)}(z(s)))\rangle\\&+\big\langle u(s)-v(s)-(u_0(s)-v_0(s)),\pi_{\lambda(s)}(z(s)))-g(s,z_0(s))\big\rangle.\end{aligned}$$

Now from $\mathbb{H}$-Lipschitz continuity of $\hat{M}$ and $\hat{N}$ with measurable functions $\gamma_1,\gamma_2:\mathrm{S}\to\mathbb{H}$ and using (4.7), we have

$$\begin{aligned}&\big(\gamma_1(s)-\gamma_2(s)\big)\|z(s)-z_0(s)\|^2\\&\leq\theta_1(s)(1+\varepsilon)\|z(s)-z_0(s)\|\|g(s,z(s))-\pi_{\lambda(s)}(z(s)))\|\\&-\theta_2(s)(1+\varepsilon)\|z(s)-z_0(s)\|\|g(s,z(s))-\pi_{\lambda(s)}(z(s)))\|\\&+\langle u(s)-v(s)-(u_0(s)-v_0(s)),\pi_{\lambda(s)}(z(s)))-g(s,z_0(s))\rangle\\&\leq\big((\theta_1(s)-\theta_2(s))(1+\varepsilon)+\lambda(s)\alpha(s)L_1(s)\big)\|z(s)-z_0(s)\|\|g(s,z(s))-\pi_{\lambda(s)}(z(s)))\|.\end{aligned}$$

So, we get

$$\begin{aligned}\|z(s)-z_0(s)\|\leq&\frac{\big((\theta_1(s)-\theta_2(s))(1+\varepsilon)+\lambda(s)\alpha(s)L_1(s)\big)}{\gamma_1(s)-\gamma_2(s)}\\&\times\|g(s,z(s))-\pi_{\lambda(s)}(z(s)))\|,\end{aligned}$$

where $\gamma_1(s)>\gamma_2(s)$ and $\theta_1(s)>\theta_2(s)$. □

**Corollary 4.1.** *In the above Theorem 4.1, if the random multi-valued mapping $\hat{N}$ is taken as zero mapping, then we get the result for* (2.2) *proved by* [24].

**Theorem 4.2.** *If the function F satisfies ($p_1$)- ($p_4$), then*

$$G_{\lambda(s)}(z(s))\geq\lambda(s)\eta(s)\|g(s,z(s))-\pi_{\lambda(s)}(z(s)))\|,\forall z(s)\in\mathbb{H},\forall s\in\mathrm{S},$$

*and $z(s)$ is solution of the REGVIP* (2.1) *if and only if $G_{\lambda(s)}(z(s))=0$.*

*Proof.* substituting $w(s)=g(s,z(s))$ in (4.2), we get

$$\begin{aligned}&\langle u(s)-v(s),g(s,z(s))-\pi_{\lambda(s)}(z(s))\rangle+\phi(g(s,z(s)))-\phi(\pi_{\lambda(s)}(z(s)))\\&\quad\geq\lambda(s)\langle-\nabla_2 F(g(s,z(s)),\pi_{\lambda(s)}(z(s))),g(s,z(s))-\pi_{\lambda(s)}(z(s)))\rangle.\end{aligned}$$

So, we can write

$$\begin{aligned} G_{\lambda(s)}(z(s)) \quad &\geq \quad \langle -\nabla_2 F(g(s,z(s)), \pi_{\lambda(s)}(z(s))), g(s,z(s)) - \pi_{\lambda(s)}(z(s))) \rangle \\ &\quad -\lambda(s) F(g(s,z(s)), \pi_{\lambda(s)}(z(s))). \end{aligned}$$

Using $(p_3)$, we have

$$G_{\lambda(s)}(z(s)) \geq \lambda(s)[-F(g(s,z(s)), g(s,z(s))) + \eta(s)\|g(s,z(s)) - \pi_{\lambda(s)}(z(s)))\|^2].$$

Now from $(p_4)$, we get

$$G_{\lambda(s)}(z(s)) \geq \lambda(s)\eta(s)\|g(s,z(s)) - \pi_{\lambda(s)}(z(s)))\|^2.$$

The second part of the Theorem follows from Lemma (4.4). □

Using Theorem 4.2, Theorem 4.1 can be restated as:

**Theorem 4.3.** *Let $z_0(s) \in \mathbb{H}$ be a solution of random extended generalized variational inequality* (2.1) *for all $s \in \mathrm{S}$. Suppose that for each $s \in \mathrm{S}$, the multi-valued mappings $\hat{M}$ and $\hat{N}$ are strongly g-monotone with measurable functions $\gamma_1, \gamma_2 : \mathrm{S} \to (0, +\infty)$, respectively. Also assume that $\hat{M}$ and $\hat{N}$ are $\mathbb{H}$-Lipschitz continuous with measurable functions $\theta_1, \theta_2 : \mathrm{S} \to (0, +\infty)$. If $g : \mathrm{S} \times \mathbb{H} \to \mathbb{H}$ is Lipschitz continuous with measurable function $L_1 : \mathrm{S} \to (0, +\infty)$ and the function $F$ satisfies ($p_1$)- ($p_5$), then*

$$\|z(s) - z_0(s)\| \leq \frac{\big((\theta_1(s) - \theta_2(s))(1+\varepsilon) + \lambda(s)\alpha(s)L_1(s)\big)\sqrt{G_{\lambda(s)}z(s)}}{\big(\gamma_1(s) - \gamma_2(s)\big)\sqrt{\lambda(s)\eta(s)}},$$

*where $\gamma_1(s) > \gamma_2(s)$ and $\theta_1(s) > \theta_2(s)$.*

**Corollary 4.2.** *If in the above theorem (4.2), the random multi-valued mapping $\hat{N}$ is taken as zero mapping, then we get the corresponding result for* (2.2) *proved by Khan et al.* [24].

Also we can obtain the error bound for REGVIP (2.1) without using the Lipschitz continuity of $\hat{M}$ and $\hat{N}$.

**Theorem 4.4.** *Let $z_0(s) \in \mathbb{H}$ be a solution of random extended generalized variational inequality* (2.1) *for all $s \in \mathrm{S}$. Suppose that for each $s \in \mathrm{S}$, the multi-valued mappings $\hat{M}$ and $\hat{N}$ are strongly g-monotone with measurable functions $\gamma_1, \gamma_2 : \mathrm{S} \to (0, +\infty)$, respectively. If $g : \mathrm{S} \times \mathbb{H} \to \mathbb{H}$ is Lipschitz continuous with measurable function $L_1 : \mathrm{S} \to (0, +\infty)$ and the function $F$ satisfies ($p_1$)- ($p_5$), then*

$$\|z(s) - z_0(s)\| \leq \frac{1}{\sqrt{\big(\gamma_1(s) - \gamma_2(s)\big) + \lambda(s)\big(\eta(s) - \alpha(s)\big)L_1^2(s)}} \times \sqrt{G_{\lambda(s)}(z(s))} \ \forall z(s) \in \mathbb{H},$$

*where $\gamma_1(s) > \gamma_2(s)$ and $\eta(s) > \alpha(s)$.*

*Proof.* As $u_0(s)$ is a solution of random extended generalized variational inequality problem (2.1), so by the definition of $G_{\lambda(s)}(z(s))$ for $s \in \mathrm{S}$, $z(s) \in \mathbb{H}$, we have

$$\begin{aligned} G_{\lambda(s)}(z(s)) \quad &\geq \quad \langle u(s) - v(s), g(s,z(s)) - g(s,z_0(s)) \rangle + \phi(g(s,z(s))) - \phi(g(s,z_0(s))) \\ &\quad -\lambda(s) F(g(s,z(s)), g(s,z_0(s))). \end{aligned}$$

Now by using strongly g-monotonicity of $\hat{M}$ *and* $\hat{N}$ with the measurable functions $\gamma_1, \gamma_2 : \mathrm{S} \to (0, +\infty)$, we have

$$\begin{aligned} G_{\lambda(s)}(z(s)) \quad \geq \quad & \langle u_0(s) - v_0(s), g(s, z(s)) - g(s, z_0(s)) \rangle + \big(\gamma_1(s) - \gamma_2(s)\big)\|z(s) - z_0(s)\|^2 \\ & \phi(g(s, z(s))) - \phi(g(s, z_0(s))) - \lambda(s) F(g(s, z(s)), g(s, z_0(s))). \end{aligned} \tag{4.8}$$

As $z_0(s) \in \mathbb{H}$, $\forall s \in \mathrm{S}$ is a solution of (REGVIP)(2.1), we have

$$\langle u_0(s) - v_0(s), w(s) - g(s, z_0(s)) \rangle + \phi(w(s)) - \phi(g(s, z_0(s))) \geq 0.$$

Substituting $w(s) = g(s, z(s))$ in the above inequality, we get

$$\langle u_0(s) - v_0(s), g(s, z(s)) - g(s, z_0(s)) \rangle + \phi(g(s, z(s))) - \phi(g(s, z_0(s))) \geq 0. \tag{4.9}$$

From (4.8) and (4.9), we get

$$G_{\lambda(s)}(z(s)) \geq \big(\gamma_1(s) - \gamma_2(s)\big)\|z(s) - z_0(s)\|^2 - \lambda(s) F(g(s, z(s)), g(s, z_0(s))). \tag{4.10}$$

Now using Lipschitz continuity of g and Lemma 4.3, we get

$$\begin{aligned} -F(g(s, z(s)), g(s, z_0(s))) \quad \geq \quad & \big(\eta(s) - \alpha(s)\big)\|g(s, z(s)) - g(s, z_0(s))\|^2 \\ & \big(\eta(s) - \alpha(s)\big) L_1^2(s)\|z(s) - z_0(s)\|^2. \end{aligned} \tag{4.11}$$

So, it follows from (4.10) and (4.11) that

$$\begin{aligned} \|z(s) - z_0(s)\| \leq & \frac{1}{\sqrt{\big(\gamma_1(s) - \gamma_2(s)\big) + \lambda(s)\big(\eta(s) - \alpha(s)\big) L_1^2(s)}} \\ & \times \sqrt{G_{\lambda(s)}(z(s))} \ \forall z(s) \in \mathbb{H}, \end{aligned}$$

where $\gamma_1(s) > \gamma_2(s)$ and $\eta(s) > \alpha(s)$. □

**Corollary 4.3.** *If in the above Theorem 4.4, the random multi-valued mapping $\hat{N}$ is taken as zero mapping, then we get the corresponding result for* (2.2) *proved by Khan et al.* [24].

## 5. Conclusion

In this paper, we have found error bounds for random extended generalized variational inequality with the help of gap functions. Our approach of obtaining error bounds for the variational inequality is different from the approach used by Khan et al. [24]. The approach can be used to obtain error bounds for some other types of variational inequalities.

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A.H.Dar
Department of Mathematics, Aligarh Muslim University, Aligarh 202002, India
*E-mail address*: daraadil555@gmail.com

M.K.Ahmad
Department of Mathematics, Aligarh Muslim University, Aligarh 202002, India
*E-mail address*: ahmad_kalimuddin@yahoo.co.in