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ON THE STABILITY AND GENERAL SOLUTION OF A SUM FORM FUNCTIONAL EQUATION EMERGING FROM INFORMATION THEORY

DHIRAJ KUMAR SINGH, SHVETA GROVER

ABSTRACT. In this paper we intend to obtain the general solutions of a sum form functional equation containing two unknown mappings followed by discussing the criteria of stability for the same. Some of these solutions are related to entropies of type (α, β) proposed by Behara and Nath [3].

1. INTRODUCTION

For $n = 1, 2, \ldots$: let

$$\Gamma_n = \left\{ (p_1, \ldots, p_n);\ p_i \geq 0,\ i = 1, \ldots, n;\ \sum_{i=1}^{n} p_i = 1 \right\}$$

denote the set of all n-component discrete probability distributions. Let $\mathbb{R}$ denote the set of real numbers; I denote the closed unit interval $[0, 1]$, i.e. $I = \{x \in \mathbb{R} : 0 \leq x \leq 1\} = [0, 1]$.

In this paper, the research methodology includes not only adding new dimensions to the field of research work but it also includes efforts to establish a connect between two existing dimensions that is Functional Equations and Information Theory. Indeed one of the intriguing branches which are explored in the domain of functional equations with reference to information theory is to discover and study those functional equations that are used to characterize several entropies.

An entropy which is referred as uncertainty in information theory (Ash [2]) was introduced by Shannon [15]. For a probability distribution $(p_1, \ldots, p_n) \in \Gamma_n$, the Shannon entropy is defined as:

$$H_n(p_1, \ldots, p_n) = -\sum_{i=1}^{n} p_i \log_2 p_i \tag{1.1}$$

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where $H_n : \Gamma_n \to \mathbb{R}$, $n = 1, 2, \ldots$ and the convention $0 \log_2 0 := 0$ is adopted. Chaundy and McLeod [4] with reference to some statistical thermodynamical problem came across the functional equation

$$\sum_{i=1}^{n} \sum_{j=1}^{m} f\left(p_i q_j\right) = \sum_{i=1}^{n} f\left(p_i\right) + \sum_{j=1}^{m} f(q_j) \tag{1.2}$$

where f is a real valued mapping with domain I; $(p_1, \ldots, p_n) \in \Gamma_n$, $(q_1, \ldots, q_m) \in \Gamma_m$. Moreover, they proved that if $f : I \to \mathbb{R}$ is presumed to be a continuous mapping satisfying the functional equation (1.2) and is valid for all $n,\ m = 1, 2, \ldots$, then f is of the form

$$f(x) = -cx \log_2 x \tag{1.3}$$

where c is an arbitrary real constant. With the help of (1.3) it can be concluded that the functional equation (1.2) plays a key role in characterizing Shannon entropies given by (1.1). This paper [4] added a new aspect in the field of functional equations unfolding from information theory known as "Sum form functional equations emerging from information theory".

Behara and Nath [3] generalized the notion of Shannon entropy given by (1.1) by introducing the entropies of type (α, β). For a probability distribution $(p_1, \ldots, p_n) \in \Gamma_n$, entropy of type (α, β) is defined as:

$$H_n^{(\alpha,\beta)}(p_1, \ldots, p_n) = \begin{cases} (2^{1-\alpha} - 2^{1-\beta})^{-1} \left(\sum\limits_{i=1}^{n} p_i^{\alpha} - \sum\limits_{i=1}^{n} p_i^{\beta} \right) & \text{if } \alpha \neq \beta \\ -2^{\beta-1} \sum\limits_{i=1}^{n} p_i^{\beta} \log_2 p_i & \text{if } \alpha = \beta \end{cases} \tag{1.4}$$

where $H_n^{(\alpha,\beta)}$ is a real valued mapping with domain Γ_n, $n = 1, 2, \ldots$; α and β are fixed positive real powers such that

$$0^{\alpha} := 0, 0^{\beta} := 0, 1^{\alpha} := 1, 1^{\beta} := 1 \tag{1.5}$$

and $0^{\beta} \log_2 0 := 0$. This phenomenon of entropies of type (α, β) represented by (1.4) initiated the study of the functional equation

$$\sum_{i=1}^{n} \sum_{j=1}^{m} f(p_i q_j) = \sum_{i=1}^{n} p_i^{\alpha} \sum_{j=1}^{m} f(q_j) + \sum_{j=1}^{m} q_j^{\beta} \sum_{i=1}^{n} f(p_i) \tag{1.6}$$

where $f : I \to \mathbb{R}$; $(p_1, \ldots, p_n) \in \Gamma_n$, $(q_1, \ldots, q_m) \in \Gamma_m$; α and β are fixed positive real powers which satisfy the conventions stated in (1.5).

Behara and Nath [3] were first to discover the continuous solutions of the functional equation (1.6) assuming that $\alpha > 0$, $\beta > 0$ and for all $n = m = 1, 2, \ldots$. The functional equation (1.6) was also studied by Kannappan [8], [9] who obtained its integrable and measurable solutions by imposing some assumptions on the mapping $f : I \to \mathbb{R}$.

Finally, without imposing any regularity condition on the real valued mapping $f : I \to \mathbb{R}$, Losonczi and Maksa [11] found the general solutions of (1.6) for fixed integers $n \geq 3$, $m \geq 2$ with $\alpha \neq 1$, $\beta \neq 1$. The functional equation (1.6) was readdressed by Kocsis and Maksa [10] who examined the stability of the same. The problem of stability was raised for the first time by S.M. Ulam [17]. For the problem of stability concerning functional equations, we refer to the survey paper of Hyers and Rassias [7].

The objective of this paper is to explore the general solutions of the functional equation

$$\sum_{i=1}^{n}\sum_{j=1}^{m} f\left(p_i q_j\right) = \sum_{i=1}^{n} p_i^{\alpha} \sum_{j=1}^{m} h(q_j) + \sum_{j=1}^{m} q_j^{\beta} \sum_{i=1}^{n} h\left(p_i\right) \tag{A}$$

where f and h are unknown real valued mappings each having the domain I; $(p_1, \ldots, p_n) \in \Gamma_n$, $(q_1, \ldots, q_m) \in \Gamma_m$; $n \geq 3$, $m \geq 2$ be fixed integers; $0 < \alpha \in \mathbb{R}$, $0 < \beta \in \mathbb{R}$, such that $\alpha \neq 1$, $\beta \neq 1$ with (1.5). Now we mention the motivation behind studying (A). As far as we know, Nath and Singh [14] were the first who came across the functional equation (A) while addressing some other functional equation leaving it as an open problem. Equation (A) is a Pexiderized form of (1.6) and it is useful in characterizing entropies of type (α, β). It follows that functional equation (A) is emerging from information theory, thus connecting two aforementioned branches. This provides us the motivation to study functional equation (A). Furthermore, we discuss the problem of stability of functional equation (A) for the fixed integers $n \geq 3$, $m \geq 3$. The problem of stability of the functional equation (A) in the sense of Hyers and Rassias [7] is given along the following lines:

Let $n \geq 3$, $m \geq 3$ be fixed integers and $0 \leq \varepsilon \in \mathbb{R}$ be fixed. Find all the mappings $f : I \to \mathbb{R}$, $h : I \to \mathbb{R}$ satisfying the functional inequality

$$\left|\sum_{i=1}^{n}\sum_{j=1}^{m} f(p_i q_j) - \sum_{i=1}^{n} p_i^{\alpha} \sum_{j=1}^{m} h(q_j) - \sum_{j=1}^{m} q_j^{\beta} \sum_{i=1}^{n} h(p_i)\right| \leq \varepsilon \tag{B}$$

for all $(p_1, \ldots, p_n) \in \Gamma_n$, $(q_1, \ldots, q_m) \in \Gamma_m$.

This paper is divided into five sections. In next section 2, we mention some preliminary results which will be used in the subsequent sections. In section 3, the general solutions of the functional equation (A) are obtained for the fixed integers $n \geq 3$, $m \geq 2$. In section 4, the problem of stability of the functional equation (A) is being examined for the fixed integers $n \geq 3$, $m \geq 3$. In section 5, we discuss the significance of the functional equation (A) from the perspective of the information theory.

2. Some Preliminary Results

In this section, we state some known definitions and results.

A mapping $a : I \to \mathbb{R}$ is said to be additive on I or on the unit triangle

$$\Delta = \{(x, y) : 0 \leq x \leq 1, 0 \leq y \leq 1, 0 \leq x + y \leq 1\}$$

if it satisfies the equation $a(x + y) = a(x) + a(y)$ for all $(x, y) \in \Delta$. A mapping $A : \mathbb{R} \to \mathbb{R}$ is said to be additive on $\mathbb{R}$ if it satisfies the equation $A(x + y) = A(x) + A(y)$ for all $x \in \mathbb{R}$, $y \in \mathbb{R}$. It is known [5] that if a mapping $a : I \to \mathbb{R}$ is additive on I, then it has a unique additive extension $A : \mathbb{R} \to \mathbb{R}$ in the sense that A is additive on $\mathbb{R}$ and $A(x) = a(x)$ for all $x \in I$.

A mapping $\ell : I \to \mathbb{R}$ is said to be logarithmic on I if $\ell(0) = 0$ and $\ell(xy) = \ell(x) + \ell(y)$ for all $x \in]0, 1]$, $y \in]0, 1]$.

Result 2.1 ([12])**.** *Let $n \geq 3$ be a fixed integer and $\psi : I \to \mathbb{R}$ be a real valued mapping on I satisfying the equation $\sum_{i=1}^{n} \psi(p_i) = c$ for all $(p_1, \ldots, p_n) \in \Gamma_n$; c a given real constant. Then there exists an additive mapping $a_1 : \mathbb{R} \to \mathbb{R}$ such that $\psi(p) = a_1(p) - \frac{1}{n}a_1(1) + \frac{c}{n}$ for all $p \in I$.*

Result 2.2 ([11])**.** *Suppose that the mapping $f : I \to \mathbb{R}$ satisfies the functional equation* (1.6) *where $\alpha \neq 1$, $\beta \neq 1$, $0^\alpha = 0^\beta = 0$ and $n \geq 3$, $m \geq 2$ are fixed integers. Then*

$$f(p) = C(p^\alpha - p^\beta) + a(p) \quad \text{if} \quad \alpha \neq \beta, p \in I \tag{2.1}$$

and

$$f(p) = p^\alpha \ell(p) + a(p) \quad \text{if} \quad \alpha = \beta, p \in I; \tag{2.2}$$

where $a : \mathbb{R} \to \mathbb{R}$ is an additive mapping with $a(1) = 0$ and $C \in \mathbb{R}$ is a constant, $\ell : I \to \mathbb{R}$ is a logarithmic mapping. Conversely, the mappings (2.1), (2.2) *satisfy* (1.6).

Result 2.3 ([11])**.** *Let $m \geq 2$ be a fixed integer and $H : I \to \mathbb{R}$ be a real valued mapping on I which satisfies the functional equation*

$$\sum_{j=1}^{m} \left\{ H\left(pq_j\right) - p^\beta H(q_j) - q_j^\beta H(p) \right\} = 0 \tag{C}$$

for all $p \in I$, $(q_1, \ldots, q_m) \in \Gamma_m$; $\beta \neq 1$ being a fixed positive real power satisfying the conventions (1.5)*. If $H(0) = H(1) = 0$, then $H(p) = p^\beta \ell(p)$ for all $p \in I$; $\ell : I \to \mathbb{R}$ is a logarithmic mapping.*

Result 2.4 ([13])**.** *Let $n \geq 3$ be a fixed integer; $0 \leq \varepsilon \in R$ be fixed and $\phi : I \to \mathbb{R}$ be a real valued mapping on I satisfying the functional inequality $\left| \sum_{i=1}^{n} \phi(p_i) \right| \leq \varepsilon$ for all $(p_1, \ldots, p_n) \in \Gamma_n$. Then there exists an additive mapping $a_2 : \mathbb{R} \to \mathbb{R}$ and a bounded mapping $b : \mathbb{R} \to \mathbb{R}$ with $b(0) = 0$, $|b(p)| \leq 18\varepsilon$ such that $\phi(p) - \phi(0) = a_2(p) + b(p)$ for all $p \in I$.*

Result 2.5 ([18])**.** *If a real additive mapping f is bounded on an interval $[a, b]$, then it is linear, i.e. there exists a constant c' such that $f(p) = c'p$ for all $p \in \mathbb{R}$.*

Result 2.6 ([16])**.** *Let $0 \leq \varepsilon' \in \mathbb{R}$ be fixed and $H : I \to \mathbb{R}$ be a real valued mapping on I which satisfies the functional inequality*

$$|H(pq) - p^\beta H(q) - q^\beta H(p)| \leq \varepsilon' \tag{D}$$

for all $p \in I$, $q \in I$; $\beta \neq 1$ being a fixed positive real power satisfying the conventions (1.5)*. Then any solution of (D) is of the form $H(p) = p^\beta \ell(p) + \bar{b}(p)$ for all $p \in I$; $\ell : I \to \mathbb{R}$ is a logarithmic mapping and $\bar{b} : \mathbb{R} \to \mathbb{R}$ is a bounded mapping with $|\bar{b}(p)| \leq 4e\varepsilon'$ where e is the natural base of the logarithmic mapping.*

3. The General Solution Of The Functional Equation (A)

The main result of this section is the following:

Theorem 3.1. *Let $n \geq 3$, $m \geq 2$ be fixed integers; α and β be fixed positive real powers different from 1 satisfying the conventions (1.5) and let $f : I \to \mathbb{R}$, $h : I \to \mathbb{R}$.*
(I) *If $\alpha = \beta$, then the pair (f, h) satisfies (A) if and only if there exist a logarithmic mapping $\ell : I \to \mathbb{R}$, the additive mappings $a_1, a_2 : \mathbb{R} \to \mathbb{R}$ with $(n-m)a_2(1) = 0$ and $\bar{c} \in \mathbb{R}$ such that*

$$\left.\begin{aligned} &(i) \quad f(p) = p^{\beta}\ell(p) + a_1(p) + 2\bar{c}p^{\beta} - \frac{1}{nm}a_1(1), \\ &(ii) \quad h(p) = p^{\beta}\ell(p) + a_2(p) + \bar{c}p^{\beta} - \frac{1}{n}a_2(1). \end{aligned}\right\} \qquad (\alpha_1)$$

(II) *If $\alpha \neq \beta$, then the pair (f, h) satisfies (A) if and only if there exist the additive mappings $a_3, a_4 : \mathbb{R} \to \mathbb{R}$ with $(n-m)a_4(1) = 0$ and $c \in \mathbb{R}$ such that*

$$\left.\begin{aligned} &(i) \quad f(p) = c(p^{\alpha} - p^{\beta}) + a_3(p) - \frac{1}{nm}a_3(1), \\ &(ii) \quad h(p) = c(p^{\alpha} - p^{\beta}) + a_4(p) - \frac{1}{n}a_4(1). \end{aligned}\right\} \qquad (\alpha_2)$$

Proof. Let us put $q_1 = 1$, $q_2 = \cdots = q_m = 0$ in (A). We obtain

$$\sum_{i=1}^{n} \{f(p_i) - [h(1) + (m-1)h(0)]p_i^{\alpha} - h(p_i)\} = n(1-m)f(0).$$

By Result 2.1, there exists an additive mapping $a : \mathbb{R} \to \mathbb{R}$ such that

$$f(p) = h(p) + [h(1) + (m-1)h(0)]p^{\alpha} + a(p) - \frac{1}{n}a(1) + (1-m)f(0) \tag{3.1}$$

for all $p \in I$. The substitution p = 0 in (3.1) and the use of the fact that a(0) = 0 gives

$$a(1) = n(h(0) - mf(0)). \tag{3.2}$$

From (3.1) and (3.2), after performing necessary calculation work, we obtain

$$f(p) = h(p) + [h(1) + (m-1)h(0)]p^{\alpha} + a(p) + f(0) - h(0). \tag{3.3}$$

From (A), (3.3) and (3.2), we get

$$\begin{aligned} \sum_{i=1}^{n}\sum_{j=1}^{m} h(p_i q_j) - \sum_{i=1}^{n} p_i^{\alpha}\left[\sum_{j=1}^{m} h(q_j) - [h(1) + (m-1)h(0)]\sum_{j=1}^{m} q_j^{\alpha}\right] \\ - \sum_{j=1}^{m} q_j^{\beta} \sum_{i=1}^{n} h(p_i) + n(1-m)h(0) = 0 \end{aligned} \tag{3.4}$$

for all $(p_1, \ldots, p_n) \in \Gamma_n$, $(q_1, \ldots, q_m) \in \Gamma_m$; $n \geq 3$, $m \geq 2$ being fixed integers. Now letting $p_1 = 1$, $p_2 = \cdots = p_n = 0$ in (3.4), we obtain equation

$$\begin{aligned} [h(1) + (m-1)h(0)]\sum_{j=1}^{m} q_j^{\alpha} - [h(1) + (n-1)h(0)]\sum_{j=1}^{m} q_j^{\beta} \\ + (n-m)h(0) = 0. \end{aligned} \tag{3.5}$$

Equation (3.5), indicates that the proof depends on the parameters α and β. So we divide our discussion into two cases.

Case 1: $\alpha = \beta$

In this case equation (3.5) reduces to $(n-m)h(0)\left[1-\sum_{j=1}^{m} q_j^{\beta}\right] = 0$. This implies either $(n-m)h(0) = 0$ or $1-\sum_{j=1}^{m} q_j^{\beta}$ vanishes identically on Γ_m. Suppose $1-\sum_{j=1}^{m} q_j^{\beta} = 0$ for all $(q_1,\ldots,q_m) \in \Gamma_m$. In particular for a probability distribution $\left(\frac{1}{2},\frac{1}{2},0,\ldots,0\right) \in \Gamma_m$, we have $\left(\frac{1}{2}\right)^{\beta} = \frac{1}{2}$ which holds only when $\beta = 1$. Since β is assumed to be a fixed positive real power with $\beta \neq 1$, we arrive at a contradiction and hence obtain $(n-m)h(0) = 0$.
Now considering $\alpha = \beta$, equation (3.4) reduces to

$$\sum_{i=1}^{n}\left\{\sum_{j=1}^{m} h(p_i q_j) - p_i^{\beta}\left[\sum_{j=1}^{m} h(q_j) - [h(1)+(m-1)h(0)]\sum_{j=1}^{m} q_j^{\beta}\right]\right.$$
$$\left. - \sum_{j=1}^{m} q_j^{\beta} h(p_i)\right\} = n(m-1)h(0)$$

for all $(p_1,\ldots,p_n) \in \Gamma_n$, $(q_1,\ldots,q_m) \in \Gamma_m$; $n \geq 3$, $m \geq 2$ being fixed integers. By Result 2.1, there exists a mapping $\bar{A} : \mathbb{R} \times \Gamma_m \to \mathbb{R}$, additive in the first variable such that

$$\sum_{j=1}^{m} h(pq_j) - p^{\beta}\left[\sum_{j=1}^{m} h(q_j) - [h(1)+(m-1)h(0)]\sum_{j=1}^{m} q_j^{\beta}\right] - h(p)\sum_{j=1}^{m} q_j^{\beta}$$
$$= \overline{A}(p;q_1,\ldots,q_m) - \frac{1}{n}\overline{A}(1;q_1,\ldots,q_m) + (m-1)h(0) \tag{3.6}$$

for all $p \in I$ and $(q_1,\ldots,q_m) \in \Gamma_m$. The substitution $p = 0$ in (3.6) gives $\overline{A}(1;q_1,\ldots,q_m) = -nh(0)\left[1-\sum_{j=1}^{m} q_j^{\beta}\right]$. Consequently, (3.6) becomes

$$\sum_{j=1}^{m} h(pq_j) - p^{\beta}\left[\sum_{j=1}^{m} h(q_j) - [h(1)+(m-1)h(0)]\sum_{j=1}^{m} q_j^{\beta}\right]$$
$$- (h(p)-h(0))\sum_{j=1}^{m} q_j^{\beta} - mh(0) = \overline{A}(p;q_1,\ldots,q_m). \tag{3.7}$$

Let $x \in I$ and $(r_1,\ldots,r_m) \in \Gamma_m$. Now replacing p by xr_t, $t = 1,\ldots,m$ consecutively in (3.7); summing up the outcoming m equations so obtained

$$\sum_{t=1}^{m}\sum_{j=1}^{m} h(xr_t q_j) - x^{\beta}\sum_{t=1}^{m} r_t^{\beta}\left[\sum_{j=1}^{m} h(q_j) - [h(1)+(m-1)h(0)]\sum_{j=1}^{m} q_j^{\beta}\right]$$
$$- \sum_{t=1}^{m} h(xr_t)\sum_{j=1}^{m} q_j^{\beta} + mh(0)\sum_{j=1}^{m} q_j^{\beta} - m^2 h(0) = \overline{A}(x;q_1,\ldots,q_m) \tag{3.8}$$

for all $x \in I$, $(q_1, \ldots, q_m) \in \Gamma_m$ and $(r_1, \ldots, r_m) \in \Gamma_m$. Now put $p = x$ and $q_1 = r_1, \ldots, q_m = r_m$ in (3.7). We obtain

$$\sum_{t=1}^{m} h(xr_t) = x^\beta \left[\sum_{t=1}^{m} h(r_t) - [h(1) + (m-1)h(0)] \sum_{t=1}^{m} r_t^\beta\right] + (h(x) - h(0)) \sum_{t=1}^{m} r_t^\beta + mh(0) + \overline{A}(x; r_1, \ldots, r_m) \tag{3.9}$$

for all $x \in I$ and $(r_1, \ldots, r_m) \in \Gamma_m$. From equations (3.8) and (3.9), we get

$$\sum_{t=1}^{m}\sum_{j=1}^{m} h(xr_tq_j) - x^\beta \left(\sum_{t=1}^{m} r_t^\beta \sum_{j=1}^{m} h(q_j) + \sum_{j=1}^{m} q_j^\beta \sum_{t=1}^{m} h(r_t)\right) + \Big(2x^\beta[h(1) + (m-1)\,h(0)] - h(x) + h(0)\Big) \sum_{t=1}^{m} r_t^\beta \sum_{j=1}^{m} q_j^\beta - m^2 h(0) = \overline{A}(x; q_1, \ldots, q_m) + \overline{A}(x; r_1, \ldots, r_m) \sum_{j=1}^{m} q_j^\beta$$

for all $x \in I$, $(q_1, \ldots, q_m) \in \Gamma_m$ and $(r_1, \ldots, r_m) \in \Gamma_m$. Apparently, the left hand side of the above equation is symmetric in r_t and q_j, $t = 1, \ldots, m$; $j = 1, \ldots, m$ (Aczél [1]), so should be its right hand side. Hence we get

$$\overline{A}(x; q_1, \ldots, q_m) \left[1 - \sum_{t=1}^{m} r_t^\beta\right] = \overline{A}(x; r_1, \ldots, r_m) \left[1 - \sum_{j=1}^{m} q_j^\beta\right]. \tag{3.10}$$

As explained earlier that for fixed positive real power $\beta \neq 1$, $1 - \sum_{j=1}^{m} q_j^\beta$ does not vanish identically on Γ_m. Thus, there exists a probability distribution $(q_1^*, \ldots, q_m^*) \in \Gamma_m$ such that $1 - \sum_{j=1}^{m} {q_j^*}^\beta \neq 0$. Making use of this in (3.10), we get

$$\overline{A}(x; r_1, \ldots, r_m) = a_2(x) \left[1 - \sum_{t=1}^{m} r_t^\beta\right] \tag{3.11}$$

where $a_2 : R \to R$ defined as $a_2(x) = \left[1 - \sum_{j=1}^{m} {q_j^*}^\beta\right]^{-1} \overline{A}(x; q_1^*, \ldots, q_m^*)$ is an additive mapping with

$$a_2(1) = -nh(0). \tag{3.12}$$

Equations (3.7), (3.11), (3.12) with $(n-m)h(0) = 0$ yields the functional equation (C) where $H : I \to \mathbb{R}$ is defined as

$$H(x) = h(x) - a_2(x) - h(0) - [h(1) + (m-1)\,h(0)]x^\beta \tag{3.13}$$

for all $x \in I$. Clearly $H(0) = 0$ and $H(1) = 0$. Thus by Result 2.3, there exists a logarithmic mapping $\ell : I \to \mathbb{R}$ such that $H(p) = p^\beta \ell(p)$ for all $p \in I$. Hence by taking $\overline{c} := h(1) + (m-1)\,h(0)$, the solution (α_1) of (A) is attained with $(n-m)a_2(1) = 0$ from (3.13), (3.1) (with $\alpha = \beta$) and (3.2), where the additive mapping $a_1 : \mathbb{R} \to \mathbb{R}$ is defined as $a_1(x) = a_2(x) + a(x)$ with $a_1(1) = -nmf(0)$.

Case 2: $\alpha \neq \beta$
In this case, let us put $q_1 = q$, $q_2 = 1 - q, q_3 = \cdots = q_m = 0$ in (3.5). We obtain

$$[h(1)+(m-1)h(0)][q^\alpha+(1-q)^\alpha] - [h(1)+(n-1)h(0)][q^\beta + (1-q)^\beta] + (n-m)h(0) = 0. \tag{3.14}$$

Now, let us put $q = \dfrac{1}{2}$ and $q = \dfrac{1}{4}$ respectively in (3.14). We obtain

$$\left[\frac{1}{2^{\alpha-1}} - \frac{1}{2^{\beta-1}}\right] h(1) + \left[\frac{(m-1)}{2^{\alpha-1}} - \frac{(n-1)}{2^{\beta-1}} + (n-m)\right] h(0) = 0, \tag{3.15}$$

$$\left[\frac{1}{4^\alpha} + \frac{3^\alpha}{4^\alpha} - \frac{1}{4^\beta} - \frac{3^\beta}{4^\beta}\right] h(1) + \left[\frac{(m-1)}{4^\alpha} + \frac{(m-1)3^\alpha}{4^\alpha} - \frac{(n-1)}{4^\beta} - \frac{(n-1)3^\beta}{4^\beta} + (n-m)\right] h(0) = 0. \tag{3.16}$$

Since $\alpha \neq \beta$, so the coefficients of $h(1)$ and $h(0)$ in equations (3.15) and (3.16) are nonzero real numbers. Therefore from (3.15) and (3.16), we have

$$(n-m)h(0)\left[\frac{1}{4^\alpha 2^{\beta-1}} + \frac{3^\alpha}{4^\alpha 2^{\beta-1}} - \frac{1}{4^\beta 2^{\alpha-1}} - \frac{3^\beta}{4^\beta 2^{\alpha-1}} + \frac{1}{2^{\alpha-1}} - \frac{1}{2^{\beta-1}} - \frac{1}{4^\alpha} - \frac{3^\alpha}{4^\alpha} + \frac{1}{4^\beta} + \frac{3^\beta}{4^\beta}\right] = 0.$$

From the above equation, it can be easily observed that either $(n-m)h(0) = 0$ or $\left[\frac{1}{4^\alpha 2^{\beta-1}} + \frac{3^\alpha}{4^\alpha 2^{\beta-1}} - \frac{1}{4^\beta 2^{\alpha-1}} - \frac{3^\beta}{4^\beta 2^{\alpha-1}} + \frac{1}{2^{\alpha-1}} - \frac{1}{2^{\beta-1}} - \frac{1}{4^\alpha} - \frac{3^\alpha}{4^\alpha} + \frac{1}{4^\beta} + \frac{3^\beta}{4^\beta}\right] = 0$. Suppose $\left[\frac{1}{4^\alpha 2^{\beta-1}} + \frac{3^\alpha}{4^\alpha 2^{\beta-1}} - \frac{1}{4^\beta 2^{\alpha-1}} - \frac{3^\beta}{4^\beta 2^{\alpha-1}} + \frac{1}{2^{\alpha-1}} - \frac{1}{2^{\beta-1}} - \frac{1}{4^\alpha} - \frac{3^\alpha}{4^\alpha} + \frac{1}{4^\beta} + \frac{3^\beta}{4^\beta}\right] = 0$ for every pair of fixed positive real powers $\alpha \neq \beta$. In particular, if we take $\alpha = 2$ and $\beta = 4$, then we arrive at a contradiction. So

$$(n-m)h(0) = 0. \tag{3.17}$$

Consequently from (3.14), we obtain the equation

$$[h(1) + (m-1)h(0)][q^\alpha + (1-q)^\alpha - q^\beta - (1-q)^\beta] = 0. \tag{3.18}$$

This implies either $h(1)+(m-1)h(0) = 0$ or $[q^\alpha+(1-q)^\alpha-q^\beta-(1-q)^\beta] = 0$. Suppose $[q^\alpha + (1-q)^\alpha - q^\beta - (1-q)^\beta] = 0$ for all $q \in I$ and every pair of fixed positive real powers $\alpha \neq \beta$. In particular, for $q = \frac{1}{2}$, it follows that $\left(\frac{1}{2}\right)^\alpha = \left(\frac{1}{2}\right)^\beta$ which is true only if $\alpha = \beta$. As a result we get a contradiction, so $h(1) + (m-1)h(0) = 0$ follows. Then (3.4) reduces to

$$\sum_{i=1}^{n}\sum_{j=1}^{m} h(p_i q_j) - \sum_{i=1}^{n} p_i^\alpha \sum_{j=1}^{m} h(q_j) - \sum_{j=1}^{m} q_j^\beta \sum_{i=1}^{n} h(p_i) + n(1-m)h(0) = 0$$

for all $(p_1,\dots,p_n)\in\Gamma_n$, $(q_1,\dots,q_m)\in\Gamma_m$; $n\ge 3$, $m\ge 2$ being fixed integers. Making use of (3.17), the above equation can be written as

$$\sum_{i=1}^{n}\sum_{j=1}^{m}h(p_iq_j)+nh(0)\sum_{i=1}^{n}p_i\sum_{j=1}^{m}q_j-nmh(0)-\sum_{i=1}^{n}p_i^{\alpha}\Bigg[\sum_{j=1}^{m}h\left(q_j\right)+nh(0)$$
$$\times\sum_{j=1}^{m}q_j-mh(0)\Bigg]-\sum_{j=1}^{m}q_j^{\beta}\left[\sum_{i=1}^{n}h\left(p_i\right)+nh(0)\sum_{i=1}^{n}p_i-nh(0)\right]=0. \quad (3.19)$$

Now define a mapping $g: I\to\mathbb{R}$ as

$$g\left(x\right)=h\left(x\right)+nh(0)x-h(0) \quad (3.20)$$

for all $x\in I$. Clearly $g(0)=0$ and $g(1)=0$. Also from (3.19) and (3.20), we obtain functional equation (1.6) (with g in place of f). Hence by Result 2.2, there exists an additive mapping $\overline{a}:\mathbb{R}\to\mathbb{R}$ and $c\in\mathbb{R}$ such that for all $p\in I$, $g(p)=\overline{a}(p)+c(p^{\alpha}-p^{\beta})$ with $\overline{a}(1)=0$. Consequently, the solution (α_2) of functional equation (A) is attained with $(n-m)a_4(1)=0$ from (3.20), (3.1)(with $h(1)+(m-1)h(0)=0$) and (3.2), where the additive mappings $a_4:\mathbb{R}\to\mathbb{R}$ is defined as $a_4(x)=\overline{a}(x)-nh(0)x$ with $a_4(1)=-nh(0)$ and $a_3:\mathbb{R}\to\mathbb{R}$ is defined as $a_3(x)=a_4(x)+a(x)$ with $a_3(1)=-nmf(0)$. This completes the proof. □

Note. We observe that from (3.17) two cases arise, which are $m\neq n$ and $m=n$. Consider the first case $m\neq n$. In this case, from (3.17), we get $h(0)=0$. Consequently (3.18) gives $h(1)[q^{\alpha}+(1-q)^{\alpha}-q^{\beta}-(1-q)^{\beta}]=0$ for all $q\in I$. Proceeding as above, we can obtain $h(1)=0$. Using the fact that $h(1)=0$, $h(0)=0$, equation (3.4) reduces to

$$\sum_{i=1}^{n}\sum_{j=1}^{m}h(p_iq_j)-\sum_{i=1}^{n}p_i^{\alpha}\sum_{j=1}^{m}h(q_j)-\sum_{j=1}^{m}q_j^{\beta}\sum_{i=1}^{n}h(p_i)=0$$

for all $(p_1,\dots,p_n)\in\Gamma_n$, $(q_1,\dots,q_m)\in\Gamma_m$; $n\ge 3$, $m\ge 2$ being fixed integers. By Result 2.2, it follows that $h: I\to\mathbb{R}$ is of the form $h(p)=c(p^{\alpha}-p^{\beta})+a_4(p), a_4(1)=0$, where $a_4:\mathbb{R}\to\mathbb{R}$ is an additive mapping and c is an arbitrary real constant. Further, using this in (3.1) and (3.2)(with $h(1)=0$ and $h(0)=0$), it follows that $f: I\to\mathbb{R}$ is of the form $f(p)=c(p^{\alpha}-p^{\beta})+a_3(p), a_3(1)=-nmf(0)$, where $a_3:\mathbb{R}\to\mathbb{R}$ is an additive mapping defined as $a_3(x)=a_4(x)+a(x)$ and c is an arbitrary real constant. This solution is included in (α_2) of (A).

On the otherhand, if we consider the case $m=n$, then proceeding as in Case 2, the solution (α_2) of (A) follows.

4. The Stability Of The Functional Equation (A)

In this section we discuss the stability of functional equation (A). For this we consider a perturbation of (A) given by functional inequality (B) and our aim is to find that *How do the solutions of inequality* (B) *differ from the solutions of equation* (A)*?*

Indeed in the sense of Hyers and Rassias [7], if the difference between their solutions is only a bounded mapping, we would say functional equation (A) is stable. Following this we establish the stability of (A) and thus prove:

Theorem 4.1. *Let $n \geq 3$, $m \geq 3$ be fixed integers; α and β be fixed positive real powers different from 1 satisfying the conventions* (1.5)*; ε be a nonnegative real constant and let $f : I \to \mathbb{R}$, $h : I \to \mathbb{R}$ be real valued mappings.*
(I) *Suppose $\alpha = \beta$ and the pair (f,h) satisfies (B). Then there exist a logarithmic mapping $\ell : I \to \mathbb{R}$, the additive mappings $a_1, a_2 : \mathbb{R} \to \mathbb{R}$, the bounded mappings $b_1, b_2 : \mathbb{R} \to \mathbb{R}$ and $\bar{c} \in \mathbb{R}$ such that*

$$\left.\begin{array}{ll} (i) & f(p) - f(0) = p^\beta \ell(p) + a_1(p) + 2\bar{c}p^\beta + b_1(p) \\ & \text{with} \\ & |b_1(p)| \leq 4e\big\{m|h(0)| + 36\big[36\varepsilon(m+1) + m(m+2)|h(0)|\big]\big\} + 18\varepsilon, \\ (ii) & h(p) - h(0) = p^\beta \ell(p) + a_2(p) + \bar{c}p^\beta + b_2(p) \\ & \text{with} \\ & |b_2(p)| \leq 4e\big\{m|h(0)| + 36\big[36\varepsilon(m+1) + m(m+2)|h(0)|\big]\big\}. \end{array}\right\} \quad (\beta_1)$$

(II) *Suppose $\alpha \neq \beta$ and the pair (f,h) satisfies (B). Then there exist the additive mappings $a_3, a_4 : \mathbb{R} \to \mathbb{R}$, the bounded mappings $b_3, b_4 : \mathbb{R} \to \mathbb{R}$ and $c, \bar{c} \in \mathbb{R}$ such that*

$$\left.\begin{array}{ll} (i) & f(p) - f(0) = c(p^\alpha - p^\beta) + a_3(p) + b_3(p) \\ & \text{with} \\ & |b_3(p)| \leq \frac{18\varepsilon\left[2+2^{1-\alpha}-2^{1-\beta}\right]+|m-n||h(0)|}{2^{1-\alpha}-2^{1-\beta}} + \bar{c}, \ b_3(0) = 0, \\ (ii) & h(p) - h(0) = c(p^\alpha - p^\beta) + a_4(p) + b_4(p) \\ & \text{with} \\ & |b_4(p)| \leq \frac{36\varepsilon+|m-n||h(0)|}{2^{1-\alpha}-2^{1-\beta}}, \ b_4(0) = 0. \end{array}\right\} \quad (\beta_2)$$

Proof. Let us put $q_1 = 1$, $q_2 = \ldots = q_m = 0$ in (B). We obtain

$$\left| \sum_{i=1}^{n} \left[f(p_i) + (m-1)f(0) - [h(1) + (m-1)h(0)]p_i^\alpha - h(p_i) \right] \right| \leq \varepsilon$$

for all $(p_1, \ldots, p_n) \in \Gamma_n$. By Result 2.4, there exists an additive mapping $A_1 : \mathbb{R} \to \mathbb{R}$ and a bounded mapping $B_1^* : \mathbb{R} \to \mathbb{R}$ with $|B_1^*(p)| \leq 18\varepsilon$ and $B_1^*(0) = 0$, such that for all $p \in I$

$$f(p) - [h(1) + (m-1)h(0)]p^\alpha - h(p) - f(0) + h(0) = A_1(p) + B_1^*(p).$$

From this, one can easily obtain the expression

$$f(p) = h(p) + A_1(p) + B_1(p) + [h(1) + (m-1)h(0)]p^\alpha \tag{4.1}$$

where $B_1 : \mathbb{R} \to \mathbb{R}$ is a bounded mapping defined as $B_1(x) = f(0) - h(0) + B_1^*(x)$. Using (4.1), inequality (B) can be written to the form

$$\left| \sum_{i=1}^{n} \left[\sum_{j=1}^{m} h(p_i q_j) + A_1(1)p_i + \sum_{j=1}^{m} B_1(p_i q_j) + [h(1) + (m-1)h(0)] \right.\right.$$
$$\left.\left. \times p_i^\alpha \sum_{j=1}^{m} q_j^\alpha - p_i^\alpha \sum_{j=1}^{m} h(q_j) - h(p_i) \sum_{j=1}^{m} q_j^\beta \right] \right| \leq \varepsilon.$$

By Result 2.4, there exists a mapping $A_2 : \mathbb{R} \times \Gamma_m \to \mathbb{R}$, additive in the first variable and a mapping $B_2 : \mathbb{R} \times \Gamma_m \to \mathbb{R}$, bounded in the first variable by

18ε with $B_2(0;q_1,\ldots,q_m)=0$, such that

$$\sum_{j=1}^{m} h(pq_j) + A_1(1)p + \sum_{j=1}^{m} B_1(pq_j) + [h(1)+(m-1)h(0)]p^\alpha \sum_{j=1}^{m} q_j^\alpha - p^\alpha \sum_{j=1}^{m} h(q_j) - (h(p)-h(0))\sum_{j=1}^{m} q_j^\beta - mh(0) - mB_1(0) = A_2(p;q_1,\ldots,q_m) + B_2(p;q_1,\ldots,q_m) \tag{4.2}$$

for all $p\in I$, $(q_1,\ldots,q_m)\in\Gamma_m$. Let $x\in I$ and $(r_1,\ldots,r_m)\in\Gamma_m$. Replacing p by xr_t, $t=1,\ldots,m$ consecutively in (4.2) and summing the resulting m equations so obtained, we have

$$\sum_{t=1}^{m}\sum_{j=1}^{m} h(xr_tq_j) + A_1(1)x + \sum_{t=1}^{m}\sum_{j=1}^{m} B_1(xr_tq_j) + x^\alpha[h(1)+(m-1)h(0)] \times \sum_{t=1}^{m} r_t^\alpha \sum_{j=1}^{m} q_j^\alpha - x^\alpha \sum_{t=1}^{m} r_t^\alpha \sum_{j=1}^{m} h(q_j) - \sum_{t=1}^{m} h(xr_t)\sum_{j=1}^{m} q_j^\beta + mh(0)\sum_{j=1}^{m} q_j^\beta - m^2h(0) - m^2B_1(0) = A_2(x;q_1,\ldots,q_m) + \sum_{t=1}^{m} B_2(xr_t;q_1,\ldots,q_m). \tag{4.3}$$

Now for $p=x$ and $q_j=r_t$, $j=1,\ldots,m$; $t=1,\ldots,m$; the functional equation (4.2) gives

$$\sum_{t=1}^{m} h(xr_t) = A_2(x;r_1,\ldots,r_m) + B_2(x;r_1,\ldots,r_m) - A_1(1)x - \sum_{t=1}^{m} B_1(xr_t) - x^\alpha[h(1)+(m-1)h(0)]\sum_{t=1}^{m} r_t^\alpha + x^\alpha \sum_{t=1}^{m} h(r_t) + (h(x)-h(0))\sum_{t=1}^{m} r_t^\beta + mh(0) + mB_1(0).$$

From the above equation, the functional equation (4.3) can be written as

$$\sum_{t=1}^{m}\sum_{j=1}^{m} h(xr_tq_j) + A_1(1)x + \sum_{t=1}^{m}\sum_{j=1}^{m} B_1(xr_tq_j) + x^\alpha[h(1)+(m-1)h(0)] \times \sum_{t=1}^{m} r_t^\alpha \sum_{j=1}^{m} q_j^\alpha - (h(x)-h(0))\sum_{t=1}^{m} r_t^\beta \sum_{j=1}^{m} q_j^\beta - m^2h(0) - m^2B_1(0)$$
$$= x^\alpha\left[\sum_{t=1}^{m} r_t^\alpha \sum_{j=1}^{m} h(q_j) + \sum_{j=1}^{m} q_j^\beta \sum_{t=1}^{m} h(r_t)\right] + \Big[A_2(x;r_1,\ldots,r_m) + B_2(x;r_1,\ldots,r_m) - A_1(1)x - \sum_{t=1}^{m} B_1(xr_t) + mB_1(0)\Big]\sum_{j=1}^{m} q_j^\beta - x^\alpha[h(1)+(m-1)h(0)]\sum_{t=1}^{m} r_t^\alpha \sum_{j=1}^{m} q_j^\beta + A_2(x;q_1,\ldots,q_m) + \sum_{t=1}^{m} B_2(xr_t;q_1,\ldots,q_m)$$

for all $x \in I$, $(q_1, \ldots, q_m) \in \Gamma_m$ and $(r_1, \ldots, r_m) \in \Gamma_m$. The symmetry of the terms in r_t and q_j, $t = 1, \ldots, m$; $j = 1, \ldots, m$ on the left hand side implies the symmetry on the right hand side. As a consequence we get

$$\begin{aligned}
&A_2(x; q_1, \ldots, q_m) \left[1 - \sum_{t=1}^{m} r_t^{\beta}\right] - A_2(x; r_1, \ldots, r_m) \left[1 - \sum_{j=1}^{m} q_j^{\beta}\right] \\
&= \sum_{j=1}^{m} B_2(xq_j; r_1, \ldots, r_m) - \sum_{t=1}^{m} B_2(xr_t; q_1, \ldots, q_m) \\
&+ \left[B_2(x; q_1, \ldots, q_m) - \sum_{j=1}^{m} B_1(xq_j) + mB_1(0) - A_1(1)x\right] \sum_{t=1}^{m} r_t^{\beta} \\
&- \left[B_2(x; r_1, \ldots, r_m) - \sum_{t=1}^{m} B_1(xr_t) + mB_1(0) - A_1(1)x\right] \sum_{j=1}^{m} q_j^{\beta} \\
&- x^{\alpha}[h(1) + (m-1)h(0)] \left[\sum_{j=1}^{m} q_j^{\alpha} \sum_{t=1}^{m} r_t^{\beta} - \sum_{t=1}^{m} r_t^{\alpha} \sum_{j=1}^{m} q_j^{\beta}\right] \\
&+ x^{\alpha} \sum_{t=1}^{m} h(r_t) \left(\sum_{j=1}^{m} q_j^{\alpha} - \sum_{j=1}^{m} q_j^{\beta}\right) - x^{\alpha} \sum_{j=1}^{m} h(q_j) \left(\sum_{t=1}^{m} r_t^{\alpha} - \sum_{t=1}^{m} r_t^{\beta}\right). \qquad (4.4)
\end{aligned}$$

Here, we notice that the equation (4.4), strongly depends on the parameters α and β. Therefore, we divide our discussion into two cases.

Case 1: $\alpha = \beta$
In this case, equation (4.4) results in the following equation

$$\begin{aligned}
&A_2(x; q_1, \ldots, q_m) \left[1 - \sum_{t=1}^{m} r_t^{\beta}\right] - A_2(x; r_1, \ldots, r_m) \left[1 - \sum_{j=1}^{m} q_j^{\beta}\right] \\
&= \sum_{j=1}^{m} B_2(xq_j; r_1, \ldots, r_m) - \sum_{t=1}^{m} B_2(xr_t; q_1, \ldots, q_m) \\
&+ \left[B_2(x; q_1, \ldots, q_m) - \sum_{j=1}^{m} B_1(xq_j) + mB_1(0) - A_1(1)x\right] \sum_{t=1}^{m} r_t^{\beta} \\
&- \left[B_2(x; r_1, \ldots, r_m) - \sum_{t=1}^{m} B_1(xr_t) + mB_1(0) - A_1(1)x\right] \sum_{j=1}^{m} q_j^{\beta}. \qquad (4.5)
\end{aligned}$$

For fixed $(q_1, \ldots, q_m) \in \Gamma_m$ and $(r_1, \ldots, r_m) \in \Gamma_m$, the right hand side of (4.5) is bounded on I while the left hand side is additive in $x \in I$, consequently by applying Result 2.5, it follows that

$$[A_2(x; q_1, \ldots, q_m) - xA_2(1; q_1, \ldots, q_m)] \left[1 - \sum_{t=1}^{m} r_t^{\beta}\right]$$

$$= [A_2(x; r_1, \dots, r_m) - xA_2(1; r_1, \dots, r_m)] \left[1 - \sum_{j=1}^{m} q_j^{\beta}\right]. \tag{4.6}$$

As explained in the previous section 3 that for fixed positive real power $\beta \neq 1$, $1 - \sum_{t=1}^{m} r_t^{\beta}$ does not vanish identically on Γ_m. Hence, there exists a probability distribution $(r_1^*, \dots, r_m^*) \in \Gamma_m$ such that $1 - \sum_{t=1}^{m} {r_t^*}^{\beta} \neq 0$. Equation (4.6) along with this fact results in

$$A_2(x; q_1, \dots, q_m) = a_2(x) \left[1 - \sum_{j=1}^{m} q_j^{\beta}\right] + x\ A_2(1; q_1, \dots, q_m) \tag{4.7}$$

where $a_2 : \mathbb{R} \to \mathbb{R}$ is a mapping defined as

$$a_2(x) = \left[1 - \sum_{t=1}^{m} {r_t^*}^{\beta}\right]^{-1} [A_2(x; r_1^*, \dots, r_m^*) - x\ A_2(1; r_1^*, \dots, r_m^*)].$$

Clearly the mapping $a_2 : \mathbb{R} \to \mathbb{R}$ is an additive mapping with $a_2(1) = 0$. Using $\alpha = \beta$ and $1^{\beta} := 1$ in (4.2), we have

$$A_2\,(1; q_1, \dots, q_m) = A_1(1) + \sum_{j=1}^{m} B_1(q_j) + mh(0) \sum_{j=1}^{m} q_j^{\beta}$$
$$- mh(0) - mB_1(0) - B_2\,(1; q_1, \dots, q_m)\,. \tag{4.8}$$

With the help of (4.2), (4.7), (4.8), $a_2(1) = 0$ and $\alpha = \beta$, we gather that

$$\sum_{j=1}^{m} H(pq_j) - p^{\beta} \sum_{j=1}^{m} H(q_j) - \sum_{j=1}^{m} q_j^{\beta} H(p) = p^{\beta} mh(0)$$
$$+ p\Big[\sum_{j=1}^{m} B_1(q_j) + mh(0) \sum_{j=1}^{m} q_j^{\beta} - mh(0) - mB_1(0) - B_2(1; q_1, \dots, q_m)\Big]$$
$$+ B_2\,(p; q_1, \dots, q_m) - \sum_{j=1}^{m} B_1(pq_j) + mB_1(0) \tag{4.9}$$

where $H : I \to \mathbb{R}$ is a mapping defined as

$$H(x) = h(x) - a_2(x) - [h(1) + (m-1)\,h(0)]x^{\beta} - h(0) \tag{4.10}$$

for all $x \in I$. It follows from the definition of H given by (4.10) that $H(0) = 0$. Apparently, the right hand side of (4.9) is bounded by $36\varepsilon(m+1) + m(m+2)|h(0)|$, consequently by applying Result 2.4, and using $H(0) = 0$, there exists a mapping $A_3 : I \times \mathbb{R} \to \mathbb{R}$, additive in the second variable and a mapping $B_3 : I \times \mathbb{R} \to \mathbb{R}$, bounded in the second variable by $18\big[36\varepsilon(m+1) + m(m+2)|h(0)|\big]$ with $B_3(p, 0) = 0$, such that

$$H(pq) - p^{\beta} H(q) - q^{\beta} H(p) = A_3(p, q) + B_3(p, q)\,. \tag{4.11}$$

Define a mapping $G : I \times I \to \mathbb{R}$ as

$$G(p, q) = H(pq) - p^{\beta} H(q) - q^{\beta} H(p) \tag{4.12}$$

for all $p \in I$, $q \in I$. With the help of (4.12), it can easily be verified that

$$\begin{aligned} H(pqr) - p^\beta q^\beta H(r) - q^\beta r^\beta H(p) - r^\beta p^\beta H(q) &= G(pq, r) + r^\beta G(p, q) \\ &= G(p, qr) + p^\beta G(q, r) \end{aligned} \tag{4.13}$$

for all $p \in I$, $q \in I$ and $r \in I$. From (4.11), (4.12) and (4.13), it follows that

$$\begin{aligned} &A_3(p, qr) + p^\beta A_3(q, r) - A_3(pq, r) \\ &= B_3(pq, r) + r^\beta A_3(p, q) + r^\beta B_3(p, q) - B_3(p, qr) - p^\beta B_3(q, r). \end{aligned} \tag{4.14}$$

The left hand side of (4.14) is additive in $r \in I$, while its right hand side is bounded on I. Consequently by applying Result 2.5, it follows that left hand side is linear, i.e.

$$A_3(p, qr) + p^\beta A_3(q, r) - A_3(pq, r) = r\left[A_3(p, q) + p^\beta A_3(q, 1) - A_3(pq, 1)\right]. \tag{4.15}$$

Now, substituting $r = 1$ in (4.14), we get

$$p^\beta A_3(q, 1) - A_3(pq, 1) = B_3(pq, 1) - p^\beta B_3(q, 1). \tag{4.16}$$

From (4.14), (4.15) and (4.16), we obtain

$$\begin{aligned} \left(r - r^\beta\right) A_3(p, q) &= B_3(pq, r) + r^\beta B_3(p, q) - B_3(p, qr) \\ &\quad - p^\beta B_3(q, r) - rB_3(pq, 1) + rp^\beta B_3(q, 1) \end{aligned} \tag{4.17}$$

for all $p \in I$, $q \in I$ and $r \in I$. Since β is presumed to be a fixed positive real power with $\beta \neq 1$, equation (4.17) yield that the mapping $A_3(p, q)$ is bounded in q on I. Hence by Result 2.5, $A_3(p, q)$ must be linear. Therefore

$$A_3(p, q) = qA_3(p, 1) \tag{4.18}$$

for all $p \in I$, $q \in I$. Also equation (4.16) with the substitution $q = 1$ results in the following

$$A_3(p, 1) = p^\beta A_3(1, 1) - B_3(p, 1) + p^\beta B_3(1, 1) \tag{4.19}$$

for all $p \in I$. Consequently from (4.18) and (4.19), we conclude that the mapping $A_3(p, q)$ is bounded. Moreover we obtain its bound '$m|h(0)| + 18\big[36\varepsilon(m + 1) + m(m + 2)|h(0)|\big]$' as $A_3(p, 1) = -p^\beta H(1) - B_3(p, 1)$ (from (4.11) and (4.19)) and $|H(1)| \leq m|h(0)|$ (from (4.10)). Hence, the mapping G is also bounded and therefore by Result 2.6, on (4.11) we get $H(p) = p^\beta \ell(p) + b_2(p)$, where $\ell : I \to \mathbb{R}$ is a logarithmic mapping and $b_2 : \mathbb{R} \to \mathbb{R}$ is a bounded mapping with $|b_2(p)| \leq 4e\big\{m|h(0)| + 36\big[36\varepsilon(m+1) + m(m+2)|h(0)|\big]\big\}$. On taking $\bar{c} := h(1) + (m - 1)h(0)$, the solution ($\beta_1$) of inequality (B) follows from (4.10) and (4.1) (with $\alpha = \beta$) by defining additive mapping $a_1 : \mathbb{R} \to \mathbb{R}$ as $a_1(x) = a_2(x) + A_1(x)$; a bounded mapping $b_1 : \mathbb{R} \to \mathbb{R}$ as $b_1(x) = b_2(x) + B_1^*(x)$ with $|b_1(x)| \leq 4e\big\{m|h(0)| + 36\big[36\varepsilon(m+1) + m(m+2)|h(0)|\big]\big\} + 18\varepsilon$.

Case 2: $\alpha \neq \beta$

In this case without any loss of generality, we may assume that $n \geq m$. So, letting $p_{m+1} = \cdots = p_n = 0$ in (B) and using (4.1). We get

$$\Bigg| \sum_{i=1}^{m}\sum_{j=1}^{m} h(p_i q_j) - \sum_{i=1}^{m} p_i^{\alpha} \sum_{j=1}^{m} h(q_j) - \sum_{j=1}^{m} q_j^{\beta} \sum_{i=1}^{m} h(p_i) + [h(1) + (m-1)\, h(0)] \sum_{i=1}^{m} p_i^{\alpha} \sum_{j=1}^{m} q_j^{\alpha} + A_1(1) + \sum_{i=1}^{m}\sum_{j=1}^{m} B_1(p_i q_j) + m\,(n-m)\, h(0) - (n-m)\, h(0) \sum_{j=1}^{m} q_j^{\beta} + m\,(n-m)\, B_1(0) \Bigg| \le \varepsilon \tag{4.20}$$

for all $(p_1, \ldots, p_m) \in \Gamma_m$, $(q_1, \ldots, q_m) \in \Gamma_m$. Now on interchanging the places of p_i and q_j, $i = 1, \ldots, m$; $j = 1, \ldots, m$ in the functional inequality (4.20), we have

$$\Bigg| \sum_{i=1}^{m}\sum_{j=1}^{m} h(p_i q_j) - \sum_{j=1}^{m} q_j^{\alpha} \sum_{i=1}^{m} h(p_i) - \sum_{i=1}^{m} p_i^{\beta} \sum_{j=1}^{m} h(q_j) + [h(1) + (m-1)\, h(0)] \sum_{i=1}^{m} p_i^{\alpha} \sum_{j=1}^{m} q_j^{\alpha} + A_1(1) + \sum_{i=1}^{m}\sum_{j=1}^{m} B_1(p_i q_j) + m\,(n-m)\, h(0) - (n-m)\, h(0) \sum_{i=1}^{m} p_i^{\beta} + m(n-m)B_1(0) \Bigg| \le \varepsilon. \tag{4.21}$$

Applying triangle inequality to functional inequalities (4.20) and (4.21), we obtain

$$\Bigg| \left[\sum_{j=1}^{m} q_j^{\alpha} - \sum_{j=1}^{m} q_j^{\beta}\right] \sum_{i=1}^{m} h(p_i) - \left[\sum_{i=1}^{m} p_i^{\alpha} - \sum_{i=1}^{m} p_i^{\beta}\right] \sum_{j=1}^{m} h(q_j) + (n-m)\, h(0) \left[\sum_{i=1}^{m} p_i^{\beta} - \sum_{j=1}^{m} q_j^{\beta}\right] \Bigg| \le 2\varepsilon\,. \tag{4.22}$$

Before proceeding further we assert that $\sum\limits_{j=1}^{m} q_j^{\alpha} - \sum\limits_{j=1}^{m} q_j^{\beta} \not\equiv 0$ on Γ_m. To the contrary suppose, $\sum\limits_{j=1}^{m} q_j^{\alpha} - \sum\limits_{j=1}^{m} q_j^{\beta} = 0$ for all $(q_1, \ldots, q_m) \in \Gamma_m$. In particular for a probability distribution $\left(\frac{1}{m}, \frac{1}{m}, \ldots, \frac{1}{m}\right) \in \Gamma_m$, we obtain $\left(\frac{1}{m}\right)^{\alpha} = \left(\frac{1}{m}\right)^{\beta}$ which is true only if $\alpha = \beta$. However, as per our assumption $\alpha \neq \beta$, we arrive at a contradiction and so our assertion follows. Consequently, there always exists a probability distribution $(q_1^*, \ldots, q_m^*) \in \Gamma_m$ for which $0 \neq \sum\limits_{j=1}^{m} {q_j^*}^{\alpha} - \sum\limits_{j=1}^{m} {q_j^*}^{\beta}$ on Γ_m. Further, in order to obtain a particular bound for the bounded mapping in the subsequent part of the proof we make use of this assertion by choosing $(q_1, \ldots, q_m) = \left(\frac{1}{2}, \frac{1}{2}, 0, \ldots, 0\right) \in \Gamma_m$ in functional inequality (4.22) and obtain

$$\left|\left[2^{1-\alpha}-2^{1-\beta}\right]\sum_{i=1}^{m}h\left(p_i\right)-\left[2h\left(\frac{1}{2}\right)+(m-2)h(0)\right]\sum_{i=1}^{m}p_i^{\alpha}\right.$$
$$\left.+\left[2h\left(\frac{1}{2}\right)+(n-2)h(0)\right]\sum_{i=1}^{m}p_i^{\beta}-(n-m)\,h(0)2^{1-\beta}\right|\leq 2\varepsilon \tag{4.23}$$

for all $(p_1,\ldots,p_m)\in\Gamma_m$. Since for $\alpha\neq\beta$, $2^{1-\alpha}-2^{1-\beta}\neq 0$, the above inequality can written as

$$\left|\sum_{i=1}^{m}\left[h\left(p_i\right)-c_1p_i^{\alpha}+c_2p_i^{\beta}-c_3p_i\right]\right|\leq\frac{2\varepsilon}{2^{1-\alpha}-2^{1-\beta}} \tag{4.24}$$

where $c_1:=\frac{2h\left(\frac{1}{2}\right)+(m-2)h(0)}{2^{1-\alpha}-2^{1-\beta}}\in\mathbb{R}$; $c_2:=\frac{2h\left(\frac{1}{2}\right)+(n-2)h(0)}{2^{1-\alpha}-2^{1-\beta}}\in\mathbb{R}$; $c_3:=\frac{(n-m)h(0)2^{1-\beta}}{2^{1-\alpha}-2^{1-\beta}}\in\mathbb{R}$ and $(p_1,\ldots,p_m)\in\Gamma_m$. By Result 2.4, there exists an additive mapping $A_4:\mathbb{R}\to\mathbb{R}$ and a bounded mapping $B_4:\mathbb{R}\to\mathbb{R}$ where $|B_4(p)|\leq\frac{36\varepsilon}{2^{1-\alpha}-2^{1-\beta}}$ with $B_4(0)=0$, such that

$$h(p)-c_1p^{\alpha}+c_2p^{\beta}-c_3p-h(0)=A_4(p)+B_4(p)$$

for all $p\in I$. Thus, on taking $c:=c_1$ we obtain $(\beta_2)(ii)$ by defining additive mapping $a_4:\mathbb{R}\to\mathbb{R}$ as $a_4(x)=A_4(x)+c_3x$ and bounded mapping $b_4:\mathbb{R}\to\mathbb{R}$ as $b_4(x)=B_4(x)+\frac{(m-n)h(0)}{2^{1-\alpha}-2^{1-\beta}}x^{\beta}$ where $b_4(0)=0$ and $|b_4(x)|\leq\frac{36\varepsilon+|m-n||h(0)|}{2^{1-\alpha}-2^{1-\beta}}$. Further for $\bar{c}:=h(1)+(m-1)h(0)$, we obtain $(\beta_2)(i)$ from (4.1) by defining additive mapping $a_3:\mathbb{R}\to\mathbb{R}$ as $a_3(x)=a_4(x)+A_1(x)$ and bounded mapping $b_3:\mathbb{R}\to\mathbb{R}$ as $b_3(x)=b_4(x)+B_1^*(x)+\bar{c}x^{\alpha}$ where $b_3(0)=0$ and $|b_3(x)|\leq\frac{18\varepsilon\left[2+2^{1-\alpha}-2^{1-\beta}\right]+|m-n||h(0)|}{2^{1-\alpha}-2^{1-\beta}}+\bar{c}$. □

5. Comments

The objective of this section is to discuss the significance of solutions (α_1) and (α_2) of (A) from the perspective of information theory.

The entropies $H_n^{\beta}:\Gamma_n\to\mathbb{R}$, $n=1,2,\ldots$ of degree β $(0<\beta\in\mathbb{R},\beta\neq 1)$ are defined as:

$$H_n^{\beta}(p_1,\ldots,p_n)=(1-2^{1-\beta})^{-1}\left[1-\sum_{i=1}^{n}p_i^{\beta}\right] \tag{5.1}$$

for all $(p_1,\ldots,p_n)\in\Gamma_n$. The nonadditive entropies of degree β given by (5.1) were introduced by Havrda and Charvát [6].

Keeping in mind the form of entropies of type (α,β) for $\alpha=\beta$ given by (1.4), it is desirable to consider the logarithmic mapping $\ell:I\to\mathbb{R}$ as

$$\ell(p)=\begin{cases}\lambda\log_2 p & \text{if } p\in\,]0,1]\\ 0 & \text{if } p=0\end{cases} \tag{5.2}$$

where λ is an arbitrary real constant. With the help of (1.4) and (5.2), the solution (α_1) gives

$$\sum_{i=1}^{n}f(p_i)=2\bar{c}[1+(2^{1-\beta}-1)H_n^{\beta}(p_1,\ldots,p_n)]$$

$$- \lambda 2^{1-\beta} H_n^{(\beta,\beta)}(p_1, \dots, p_n) + n(1-m)f(0)$$

and

$$\sum_{i=1}^{n} h(p_i) = \bar{c}[1 + (2^{1-\beta} - 1)H_n^{\beta}(p_1, \dots, p_n)] - \lambda 2^{1-\beta} H_n^{(\beta,\beta)}(p_1, \dots, p_n).$$

Thus it can be concluded that both the mappings f and h of the solution (α_1) are connected to entropies of type (α, β))(for $\alpha = \beta$) and entropies of degree β if $\lambda \neq 0$ and $\bar{c} \neq 0$. Also if $\lambda = 0$, $\bar{c} \neq 0$, then both the mappings f and h are connected to the entropies of degree β only. Moreover if $\lambda \neq 0$, $\bar{c} = 0$, then both the mappings f and h are connected to the entropies of type (α, β) (for $\alpha = \beta$) only. However if $\lambda = 0$, $\bar{c} = 0$, then the summands $\sum_{i=1}^{n} f(p_i)$ and $\sum_{i=1}^{n} h(p_i)$ do not represent any form of entropies, so this case is not of much importance.

Now, we compute the summands related to the solution (α_2) of (A) and using (1.4), we obtain

$$\sum_{i=1}^{n} f(p_i) = c(2^{1-\alpha} - 2^{1-\beta})H_n^{(\alpha,\beta)}(p_1, \dots, p_n) + n(1-m)f(0)$$

and

$$\sum_{i=1}^{n} h(p_i) = c(2^{1-\alpha} - 2^{1-\beta})H_n^{(\alpha,\beta)}(p_1, \dots, p_n)\,.$$

Thus it can be seen that if $c \neq 0$, then both the mappings f and h of the solution (α_2) are connected to entropies of type (α, β)(for $\alpha \neq \beta$) and if $c = 0$, then the summands $\sum_{i=1}^{n} f(p_i)$ and $\sum_{i=1}^{n} h(p_i)$ do not represent any form of entropies. Consequently the case $c = 0$ is not of much importance.

Summarizing this section we conclude that the functional equation (A) is emerging from information theory as it is related to entropies of type (α, β) and entropies of degree β.

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Dhiraj Kumar Singh
Department of Mathematics, Zakir Husain Delhi College (University of Delhi), Jawaharlal Nehru Marg, Delhi 110002, India
E-mail address: dhiraj426@rediffmail.com, dksingh@zh.du.ac.in

Shveta Grover
Department of Mathematics, University of Delhi, Delhi 110007, India
E-mail address: srkgrover9@gmail.com