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NULL HYPERSURFACES IN BRINKMANN SPACETIMES

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Abstract. We study normalized null hypersurfaces in Brinkmann spacetimes with special attention paid to Plane Fronted Waves. We explore geodesibility criterion and characterization for null hypersurfaces in links with relative position of a global parallel null vector field or under relative harmonicity condition on the local $H-$factor of the $PFW-$metric. We prove that there is no compact null hypersurfaces in Plane Fronted Waves for which the local $H-$factor has sign. We point out that Lorentzian Einstein manifolds with non zero cosmology constant admit no global parallel null vector field, and the same is for Lorentzian space forms with non zero sectional curvature. We establish sufficient conditions for the unicity up to constant factor of parallel null vector field.

1. Introduction

A famous proplem in Physics is to find exact solutions to equations which describe the fundamental interaction of gravitation as a result of spacetime being curved by mass and energy called also Einstein's equation given by

$$\overline{R}ic - \frac{1}{2}S\overline{g} = 8\pi\overline{T}.$$

As a model for electromagnetic and gravitational radiation, Lorentzian metric of the local form

$$ds^2 = 2dudv - H(u, x^i)du^2 + \sum_i (dx^i)^2, \tag{1.1}$$

were proposed ([8]) and some authors refer to these spacetimes as pp-waves. All metrics of the above form admit the parallel lightlike vector field $\frac{\partial}{\partial v}$.

According to [12],[13], Brinkmann spacetime (BST for short) is a Lorentzian manifold admitting a global parallel null vector field V and Plane Fronted Waves (PFW) is a Brinkmann spacetime locally of the form

$$\begin{cases} \overline{M} = \mathcal{M} \times \mathbb{R}^2 & (1.2) \\ \overline{g} = h + 2dudv + H(x, u)du^2 & (1.3) \end{cases}$$

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where $(\mathcal{M}, h)$ is a n-dimensional Riemannian manifold and $x \in \mathcal{M}$, the variables (v, u) are the natural coordinate of $\mathbb{R}^2$ and the smooth scalar field $H : \mathcal{M} \times \mathbb{R} \longrightarrow \mathbb{R}$ called the local $H-$factor is non zero everywhere. With this coordinates V coincides with the coordinate vector field $\partial v = \frac{\partial}{\partial v}$ and the leaves of the distribution $V^{\perp}|_{\mathcal{U}}$ given by $u \equiv constant = \mathcal{M} \times \mathbb{R} \times \{u\}$ are totally geodesic null hypersurfaces. Some questions then arise naturally: Is any totally geodesic null hypersurfaces in PFW, up to isometry, a member of the family $\Pi_{u_0}(u_0 \in \mathbb{R})$? Since the presence of parallel null vector field has been essential in our analysis, it is worth to explore the space of such vector fields such as unicity up to constant factor and algebraic properties. When foliated by a screen structure, is there any relation between the leaves and $\mathcal{M}$?

The aim of this paper is to give some solutions to the above questions by study the geometry of normalized null hypersurface in Brinkmann spacetime with particular attention to Plane Fronted Waves. The paper is organized as follows; In Section 2, we recall some basic notions needed in the rest of the paper. In Section 4, we prove several characterization results (Theorems, 4.1, 4.8, and 4.5.).

2. Normalized null hypersurfaces

Let M be a null hypersurface in a Lorentzian manifold $(\overline{M}^{n+2}, \overline{g})$, $(n \geq 1)$ i.e a hypersurface for which the induced metric tensor $g = \overline{g}_{|_M}$ is degenerate on it. A *screen distribution* on M, is a complementary bundle of $TM^{\perp}$ in TM. It is then a rank n non-degenerate distribution over M. From [3], it is known that for a null hypersurface equipped with a screen distribution, there exists a unique rank 1 vector subbundle $tr(TM)$ of $T\overline{M}$ over M, such that for any non-zero section ξ of $TM^{\perp}$ on a coordinate neighborhood $\mathscr{U} \subset M$, there exists a unique section N of $tr(TM)$ on $\mathscr{U}$ satisfying

$$\overline{g}(N, \xi) = 1, \quad \overline{g}(N, N) = \overline{g}(N, W) = 0, \qquad \forall\, W \in \mathscr{S}(N)|_{\mathscr{U}}) \tag{2.1}$$

where $\mathscr{S}(N)$ denotes the fixed screen distribution.

Then $T\overline{M}$ admits the splitting:

$$T\overline{M}|_M = TM \oplus tr(TM) = \{TM^{\perp} \oplus tr(TM)\} \oplus \mathscr{S}(N). \tag{2.2}$$

We call $tr(TM)$ a *(null) transverse vector bundle* along M. A rigging for M is a vector field ζ defined on some open set containing M such that $\zeta_p \notin T_pM$ for each $p \in M$.

Given a rigging ζ for M, let α denote the 1-form $\overline{g}$-metrically equivalent to ζ, i.e. $\alpha = \overline{g}(\zeta, \ .)$. Take $\omega = i^*\alpha$, being $i : M \hookrightarrow \overline{M}$ the canonical inclusion. Next, consider the tensors

$$\breve{g} = \overline{g} + \alpha \otimes \alpha \quad \text{and} \quad \widetilde{g} = i^{\star}\breve{g}. \tag{2.3}$$

It is easy to show that $\widetilde{g}$ defines a Riemannian metric on the (whole) hypersurface M. The *rigged vector field* of ζ is the $\widetilde{g}$-metrically equivalent vector field to the 1-form ω and it is denoted by ξ. In fact the rigged vector field ξ is the unique lightlike vector field in M such that $\overline{g}(\zeta, \xi) = 1$. Moreover, ξ is $\widetilde{g}$-unitary. To a rigging ζ for M is associated the screen distribution $\mathscr{S}(\zeta)$ given by $\mathscr{S}(\zeta) = TM \cap \zeta^{\perp}$. It is the $\widetilde{g}$-orthogonal subspace to ξ and the corresponding null transverse vector field on M is

$$N = \zeta - \frac{1}{2}\overline{g}(\zeta, \zeta)\xi. \tag{2.4}$$

A null hypersurface M equipped with a rigging ζ is said to be normalized and is denoted (M,ζ) (the latter is called a normalization of the null hypersurface). A normalization (M,ζ) is said to be closed (resp. conformal) if the rigging ζ is closed i.e the 1-form α is closed (resp. ζ is a conformal vector field, i.e there exists a function ρ on the domain of ζ such that $L_\zeta \overline{g} = 2\rho\overline{g}$). We say that ζ is a *null rigging* for M if the restriction of ζ to the null hypersurface M is a null vector field. The screen distribution $\mathscr{S}(\zeta) = \ker\omega$ is integrable whenever ω is closed, in particular if the rigging is closed. Throughout, the ambient Lorentzian metric $\overline{g}$ will also be denoted $\langle\ ,\ \rangle$.

For any vector field X on $\mathcal{U} \subset M$ we have

$$div^g X = \sum_{i=1}^{n} \tilde{g}(\nabla_{\overset{\star}{E}_i} X, \overset{\star}{E}_i) + \tilde{g}(\nabla_\xi X, \xi).$$

If $f \in C^\infty(\mathcal{U})$ we have

$$grad^g f = \nabla f = g^{[ij]}(\overset{\star}{E}_i \cdot f)\ \overset{\star}{E}_j \quad \text{and} \quad \Delta^g f = -\sum_{i=0}^{n} \tilde{g}(\nabla_{\overset{\star}{E}_i} \nabla f, \overset{\star}{E}_i) \tag{2.5}$$

where the (0,2)-tensor $g^{[\cdot,\cdot]}$, inverse of $\tilde{g}$ is called the pseudo-inverse of g [1]. The Gauss and Weingarten formulas are given by

$$\overline{\nabla}_X Y = \nabla_X Y + B(X,Y)N, \qquad \overline{\nabla}_X N = -A_N X + \tau(X)N, \tag{2.6}$$

$$\nabla_X PY = \overset{\star}{\nabla}_X PY + C(X,PY)\xi, \quad \nabla_X \xi = -\overset{\star}{A}_\xi X - \tau(X)\xi, \tag{2.7}$$

for any $X,Y \in \Gamma(TM)$, where $\overline{\nabla}$ denotes the Levi-Civita connection on $(\overline{M},\overline{g})$, ∇ denotes the connection on M induced from $\overline{\nabla}$ through the projection along the null transverse vector field N and $\overset{\star}{\nabla}$ denotes the connection on the screen distribution $\mathscr{S}(\zeta)$ induced from ∇ through the projection morphism P of $\Gamma(TM)$ onto $\Gamma\big(\mathscr{S}(\zeta)\big)$ with respect to the decomposition (2.7). Now the $(0,2)$ tensors B and C are the second fundamental forms on TM and $\mathscr{S}(\zeta)$ respectively, A_N and $\overset{\star}{A}_\xi$ are the shape operators on TM with respect to the rigging ζ and the rigged vector field ξ respectively and τ a 1-form on TM defined by $\tau(X) = \overline{g}(\overline{\nabla}_X N, \xi)$

A null hypersurface M is said to be *totally umbilic* (resp. *totally geodesic*) if there exists a smooth function β on M such that at each $p \in M$ and for all $u,v \in T_pM$, $B(p)(u,v) = \beta(p)\overline{g}(u,v)$ (resp. B vanishes identically on M). These are intrinsic notions on any null hypersurface in the sense that they are independent of the normalization. Remark that M is *totally umbilic* (resp. *totally geodesic*) if and only if $\overset{\star}{A}_\xi = \beta P$ (resp. $\overset{\star}{A}_\xi = 0$).

The (non normalized) null mean curvature is $\mathbf{H}_\xi = tr(\overset{\star}{A}_\xi)$. Let denote by $\overline{R}$ and R the Riemannian curvature tensors of $\overline{\nabla}$ and ∇, respectively. Then the following are the Gauss-Codazzi equations [3].

$$\begin{aligned}
\left\langle \overline{R}(X,Y)Z,\xi \right\rangle &= (\nabla_X B^N)(Y,Z) - (\nabla_Y B^N)(X,Z) && (2.8)\\
&\quad + \tau^N(X)B^N(Y,Z) - \tau^N(Y)B^N(X,Z), && (2.9)\\
\left\langle \overline{R}(X,Y)Z,PW \right\rangle &= \left\langle R(X,Y)Z,PW \right\rangle + B^N(X,Z)C^N(Y,PW) \\
&\quad - B^N(Y,Z)C^N(X,PW), && (2.10)\\
\left\langle \overline{R}(X,Y)\xi,N \right\rangle &= \left\langle R(X,Y)\xi,N \right\rangle = C^N(Y,X) \\
&\quad - C^N(X,Y) && (2.11)\\
&\quad - 2d\tau^N(X,Y), \quad \forall X,Y,Z,W \in \Gamma(TM|_{\mathscr{U}}). && (2.12)\\
\left\langle \overline{R}(X,Y)PZ,N \right\rangle &= \left\langle (\nabla_X A_N)Y,PZ \right\rangle - \left\langle (\nabla_Y A_N)X,PZ \right\rangle && (2.13)\\
&\quad + \tau^N(Y)\left\langle A_N X,PZ \right\rangle - \tau^N(X)\left\langle A_N Y,PZ \right\rangle. && (2.14)
\end{aligned}$$

for every X, Y and Z in $\Gamma(TM)$.

Lemma 2.1. *For every $X, Y, Z \in \Gamma(TM)$ we have*

$$(\nabla_X B)(Y,Z) - (\nabla_Y B)(X,Z) = g\left((\nabla_X \overset{*}{A}_\xi)Y, Z\right) - g\left((\nabla_Y \overset{*}{A}_\xi)X, Z\right). \quad (2.15)$$

Lemma 2.2. [4] *Let (M,ζ) be a normalized null hypersurface and ζ a rigging for it. If ζ is conformal, then $\nabla_\xi \xi = 0$, that is $\tau(\xi) = 0$.*

3. Null Hypersurfaces in Brinkmann spacetime

Throughout, V denotes a fixed global parallel null vector field in a Brinkmann spacetime $(\overline{M}, \overline{g})$. [12]The orthogonal complenent distribution

$$V_p^\perp = \{X \in T_p\overline{M}, \quad \overline{g}(V_p, X) = 0\}, \ p \in M$$

defines a parallel sub-distribution of the tangent bundle of codimension one and it is integrable.

Indeed, for $X, Y \in V^\perp$,

$$\overline{g}([X,Y],V) = \overline{g}(\overline{\nabla}_X Y - \overline{\nabla}_Y X, V) = -\overline{g}(Y, \overline{\nabla}_X V) + \overline{g}(X, \overline{\nabla}_Y V) = 0.$$

This induces a totally geodesic null foliation on the BST.

Let (M,ζ) be a normaized null hypersurface in the BST. The global null vector field V has the following decomposition along M:

$$V = V_{\mathscr{S}} + \nu\xi + \mu N, \quad (3.1)$$

where μ and ν are smooth functions on M and $V_{\mathscr{S}} \in \Gamma(\mathscr{S}(\zeta))$. Then the following holds.

Theorem 3.1. *The functions μ and ν satisfy the partial differential equations of first order,*

$$\xi \cdot \mu = -\mu\tau(\xi), \quad (3.2)$$

$$\xi \cdot \nu = \nu\tau(\xi) - g(A_N\xi, V_{\mathscr{S}}). \quad (3.3)$$

In particular, if ζ is conformal,

$$\xi \cdot \mu = 0, \quad (3.4)$$

$$\xi \cdot \nu = -g(A_N\xi, V_{\mathscr{S}}). \quad (3.5)$$

Proof. For every $X \in \Gamma(TM)$, by straightforward computation, we have,

$$\begin{aligned} 0 = \overline{\nabla}_X V &= \overset{\star}{\nabla}_X V_{\mathscr{S}} - \overline{g}(V,N) \overset{\star}{A}_\xi X - \overline{g}(V,\xi) A_N X \\ &+\Big(C(X,V_{\mathscr{S}}) + X\cdot \overline{g}(V,N) - \overline{g}(V,N)\tau(X)\Big)\xi \\ &+\Big(X\cdot \overline{g}(V,\xi) + \overline{g}(V,\xi)\tau(X) + B(X,V_{\mathscr{S}})\Big)N. \end{aligned} \tag{3.6}$$

Using this and the fact that

$$0 = g(V,V) = g(V_{\mathscr{S}},V_{\mathscr{S}}) + 2\mu\nu. \tag{3.7}$$

we get

$$\begin{cases} \overset{\star}{\nabla}_X V_{\mathscr{S}} = \nu \overset{\star}{A}_\xi X + \mu A_N X, & (3.8) \\ X\cdot\nu = \nu\tau(X) - C(X,V_{\mathscr{S}}), & (3.9) \\ X\cdot\mu = -\mu\tau(X) - B(X,V_{\mathscr{S}}), & (3.10) \\ g(V_{\mathscr{S}},V_{\mathscr{S}}) + 2\mu\nu = 0. & (3.11) \end{cases}$$

Then the results follow setting $X = \xi$. □

Proposition 3.2. *Let (M,ζ) be a normalized null hypersurface of $n+2$-dimensional BST $(\overline{M},\overline{g})$ such that the global null vector field V is tangent to M. Then M is totally geodesic and τ is exact.*

Proof. Since V is tangent to M, we have that $\mu = 0$, and it follows from (3.11) that $V_{\mathscr{S}} = 0$ as the screen in Riemannian. Using this in (3.8), and (3.9), together with the fact that $\mu = 0$, we have $\overset{\star}{A}_\xi = 0$, and $\tau = d\ln(\nu)$. □

Lemma 3.3. *Let (M,ζ) be a normalized null hypersurface of $n+2$-dimensional BST such that decomposition* (3.1) *is given. Then,*

$$\overline{Ric}(V_{\mathscr{S}},\xi) = -\mu\overline{Ric}(N,\xi) - \nu\overline{Ric}(\xi,\xi). \tag{3.12}$$

Proof. Since V is parallel, we have,

$$\begin{aligned} \overline{Ric}(V_{\mathscr{S}},\xi) &= \overline{Ric}(V - \mu N - \nu\xi, \xi) \\ &= \overline{Ric}(V,\xi) - \mu\overline{Ric}(N,\xi) - \nu\overline{Ric}(\xi,\xi) \\ &= -\mu\overline{Ric}(N,\xi) - \nu\overline{Ric}(\xi,\xi). \end{aligned} \tag{3.13}$$

□

Proposition 3.4. *Let M be a null hypersurface of $n+2$-dimensional BST $(\overline{M},\overline{g})$ furnished with a closed rigging ζ. If M is totally geodesic then either V is tangent to M or $\overline{Ric}(N,\xi) = 0$.*

Proof. From [6](see item 3 in Corollary 32), M is totally geodesic imply $\overline{Ric}(V_{\mathscr{S}},\xi) = 0$. Using this in item (a) lemma 3.3, we have $\mu\overline{Ric}(N,\xi) = 0$. That is either $\mu = 0$ or $\overline{Ric}(N,\xi) = 0$. That is $V = \nu\xi$ or $\overline{Ric}(N,\xi) = 0$. □

As V is parallel, we have the following nonexistence result concerning to Einstein BST or BST of constant sectional curvature.

Proposition 3.5. *There is no Lorentzian Einstein manifold(respectively, Lorentzian space form) with non zero cosmology constant $\wedge$ (respectively, with non zero sectional curvature k) admitting a parallel null vector field V.*

Proof. Suppose that such a Lorentzian manifold exists. Being V parallel, we have

$$0 \quad = \quad \overline{R}ic(V, \xi - N) = \wedge\Big(\overline{g}(V,\xi) - \overline{g}(V,N)\Big) = \wedge(\mu - \nu).$$

Thus $\wedge = 0$ along the null hypersurface. Indeed, if $\wedge \neq 0$ then $\mu = \nu$ and from (3.11), we have $0 \leq \overline{g}(V_{\mathscr{S}}, V_{\mathscr{S}}) = -2\mu^2 = -2\nu^2$ which is not possible for $\nu \neq 0$. But $\mu = \nu = 0$ leads to $V = V_{\mathscr{S}}$ which is also not possible since $V_{\mathscr{S}}$ is spacelike and $V \neq 0$. Since $\wedge$ is constant on $\overline{M}$ and vanishes along the null hypersurface, then it is vanished also along $\overline{M}$.

Similar, we prove for Lorentzian space form case. □

Lemma 3.6. *Let (M, ζ) be a normalized null hypersurface of $n+2$-dimensional BST $(\overline{M}, \overline{g})$. If $\tau = 0$ along the screen, then ν, and μ satisfy the following*

$$\overset{*}{\nabla}\nu = -A_N V_{\mathscr{S}} \text{ and } \overset{*}{\nabla}\mu = -\overset{*}{A}_\xi V_{\mathscr{S}}. \tag{3.14}$$

Proof. From equation (3.9) and (3.10), we have for every $X \in S(\zeta)$

$$\begin{cases} X\cdot\nu = \nu\tau(X) - g(A_N V_{\mathscr{S}}, X) \\ X\cdot\mu = \mu\tau(X) - g(\overset{*}{A}_\xi V_{\mathscr{S}}, X) \end{cases} \Rightarrow \begin{cases} g(\overset{*}{\nabla}\nu, X) = \nu\tau(X) - g(A_N V_{\mathscr{S}}, X), \\ g(\overset{*}{\nabla}\mu, X) = \mu\tau(X) - g(\overset{*}{A}_\xi V_{\mathscr{S}}, X), \end{cases}$$

$$\Rightarrow \begin{cases} g(\overset{*}{\nabla}\nu + A_N V_{\mathscr{S}}, X) = \nu\tau(X), \\ g(\overset{*}{\nabla}\mu + \overset{*}{A}_\xi V_{\mathscr{S}}, X) = \mu\tau(X). \end{cases} \tag{3.15}$$

Setting $\tau = 0$ in (3.15), we have the result. □

Let $\{\overset{\star}{E}_0 = \xi, \overset{\star}{E}_1, ..., \overset{\star}{E}_n\}$ be a quasi-orthonormal frame field for (M, ζ). From equation (3.8), we obtain for all $i = 1, ..., n$

$$\overset{\star}{\nabla}_{\overset{\star}{E}_i} V_{\mathscr{S}} = \nu \overset{\star}{A}_\xi \overset{\star}{E}_i + \mu A_N \overset{\star}{E}_i\ . \tag{3.16}$$

Thus we have the following.

Proposition 3.7. *Let M be a normalized null hypersurface of BST $(\overline{M}, \overline{g})$ admitting a rigging vector field ζ. If $\tau = 0$ along the screen, then*

(1)

$$\begin{aligned} \Delta^\star\mu \quad &= \quad -V_{\mathscr{S}}\cdot\mathbf{H}_\xi + \overline{R}ic(V_{\mathscr{S}}, \xi) \\ &\quad -\nu tr(\overset{\star}{A}{}^2_\xi) - \mu tr(A_N \circ \overset{\star}{A}_\xi) + \overline{g}\Big(\overset{\star}{A}_\xi V_{\mathscr{S}}, A_N\xi\Big). \end{aligned} \tag{3.17}$$

(2)

$$\begin{aligned} \Delta^\star\nu \quad &= \quad -V_{\mathscr{S}}\cdot\mathbf{H}_N + \overline{R}ic(V_{\mathscr{S}}, N) \\ &\quad +\mu tr(A_N^2) + \nu(tr(A_N \circ \overset{\star}{A}_\xi) + \overline{g}(\overline{R}(V_{\mathscr{S}}, N)\xi, N) \end{aligned} \tag{3.18}$$

(3)

$$\Delta^g\mu \quad = \quad -\sum_{j=1}^{n}\sum_{i=1}^{n} g\Big(g^{[i,0]}(\overset{\star}{E}_i\cdot\mu)\ \overset{\star}{A}_\xi\overset{\star}{E}_j, \overset{\star}{E}_j\ \Big) + \overset{\star}{\Delta}\mu + \sum_{i=1}^{n}\xi\cdot\Big(g^{[i,0]}(\overset{\star}{E}_i\cdot\mu)\Big) + C(\xi, \overset{\star}{\nabla}\mu).$$

Proof. (1) Since $\overset{\star}{A}_\xi$ is diagonalizable, from the quasi-orthonormal frame field $\{\overset{\star}{E}_0= \xi, \overset{\star}{E}_1, ..., \overset{\star}{E}_n\}$, we have,

$$\begin{aligned}
-div(\overset{\star}{\nabla}\mu) &= -\Delta^\star \mu = div^{\overset{\star}{\nabla}}(\overset{\star}{A}_\xi V_{\mathscr{S}}) = \sum_{i=1}^n \overline{g}\Big(\overset{\star}{\nabla}_{\overset{\star}{E}_i}(\overset{\star}{A}_\xi V_{\mathscr{S}}), \overset{\star}{E}_i\Big) \\
&\overset{(3.8)}{=} \sum_{i=1}^n \overline{g}\Big((\overset{\star}{\nabla}_{\overset{\star}{E}_i}\overset{\star}{A}_\xi)V_{\mathscr{S}}, \overset{\star}{E}_i\Big) + \sum_{i=1}^n \overline{g}\Big(\overset{\star}{A}_\xi(\mu A_N \overset{\star}{E}_i + \nu \overset{\star}{A}_\xi \overset{\star}{E}_i), \overset{\star}{E}_i\Big) \\
&= \sum_{i=1}^n \overline{g}\Big((\overset{\star}{\nabla}_{\overset{\star}{E}_i}\overset{\star}{A}_\xi)V_{\mathscr{S}} + C(\overset{\star}{\nabla}_{\overset{\star}{E}_i}\overset{\star}{A}_\xi, V_{\mathscr{S}})\xi, \overset{\star}{E}_i\Big) \\
&\quad + \mu tr(\overset{\star}{A}_\xi \circ A_N) + \nu tr(\overset{\star}{A}{}^2_\xi) \\
&= \sum_{i=1}^n \overline{g}\Big((\nabla_{\overset{\star}{E}_i}\overset{\star}{A}_\xi)V_{\mathscr{S}}, \overset{\star}{E}_i\Big) + \mu tr(\overset{\star}{A}_\xi \circ A_N) + \nu tr(\overset{\star}{A}{}^2_\xi).
\end{aligned}$$

Now, Gauss-Codazzi equations lead to

$$\begin{aligned}
-\overset{\star}{\Delta}\mu &= \sum_{i=1}^n \overline{g}\Big((\nabla_{V_{\mathscr{S}}}A_N)\overset{\star}{E}_i, \overset{\star}{E}_i\Big) + \sum_{i=1}^n \overline{g}\Big(\overline{R}(\overset{\star}{E}_i, V_{\mathscr{S}})\overset{\star}{E}_i, N\Big) + \mu tr(\overset{\star}{A}_\xi \circ A_N) + \nu tr(\overset{\star}{A}{}^2_\xi) \\
&= \sum_{i=1}^n \overline{g}\Big((\nabla_{V_{\mathscr{S}}}\overset{\star}{A}_\xi)\overset{\star}{E}_i, \overset{\star}{E}_i\Big) - \overline{R}ic(V_{\mathscr{S}}, \xi) + \overline{g}(\overline{R}(\xi, V_{\mathscr{S}})\xi, N) \\
&\quad + \overline{g}(\overline{R}(\xi,\xi)V_{\mathscr{S}}, N) + \mu tr(\overset{\star}{A}_\xi \circ A_N) + \nu tr(\overset{\star}{A}{}^2_\xi) \\
&= V_{\mathscr{S}} \cdot \mathbf{H}_\xi - \overline{R}ic(V_{\mathscr{S}}, \xi) + \nu tr(\overset{\star}{A}{}^2_\xi) + \mu tr(A_N \circ \overset{\star}{A}_\xi) - \overline{g}\Big(\overset{\star}{A}_\xi V_{\mathscr{S}}, A_N\xi\Big).
\end{aligned}$$

(2) Similarly, we have

$$\begin{aligned}
-div(\overset{\star}{\nabla}\nu) &= -\Delta^\star \nu = div^{\overset{\star}{\nabla}}(A_N V_{\mathscr{S}}) = \sum_{i=1}^n \overline{g}\Big(\overset{\star}{\nabla}_{\overset{\star}{E}_i}(A_N V_{\mathscr{S}}), \overset{\star}{E}_i\Big) \\
&\overset{(3.8)}{=} \sum_{i=1}^n \overline{g}\Big((\overset{\star}{\nabla}_{\overset{\star}{E}_i}A_N)V_{\mathscr{S}}, \overset{\star}{E}_i\Big) + \sum_{i=1}^n \overline{g}\Big(A_N(\mu A_N \overset{\star}{E}_i + \nu \overset{\star}{A}_\xi \overset{\star}{E}_i), \overset{\star}{E}_i\Big) \\
&= \sum_{i=1}^n \overline{g}\Big((\overset{\star}{\nabla}_{\overset{\star}{E}_i}A_N)V_{\mathscr{S}} + C(\overset{\star}{\nabla}_{\overset{\star}{E}_i}A_N, V_{\mathscr{S}})\xi, \overset{\star}{E}_i\Big) \\
&\quad + \mu tr(A_N^2) + \nu(A_N \circ \overset{\star}{A}_\xi) \\
&= \sum_{i=1}^n \overline{g}\Big((\nabla_{\overset{\star}{E}_i}A_N)V_{\mathscr{S}}, \overset{\star}{E}_i\Big) + \mu tr(A_N^2) + \nu(tr(A_N \circ \overset{\star}{A}_\xi),
\end{aligned}$$

and by Gauss-Codazzi equations

$$\begin{aligned} -\overset{\star}{\Delta}\nu &= \sum_{i=1}^{n}\overline{g}\Big((\nabla_{V_{\mathscr{S}}}\overset{\star}{A}_\xi)\overset{\star}{E}_i,\overset{\star}{E}_i\Big)+\sum_{i=1}^{n}\overline{g}\Big(\overline{R}(\overset{\star}{E}_i,V_{\mathscr{S}})\overset{\star}{E}_i,\xi\Big)+\mu tr(A_N^2)+\nu(tr(A_N\circ\overset{\star}{A}_\xi) \\ &= \sum_{i=1}^{n}\overline{g}\Big((\nabla_{V_{\mathscr{S}}}\overset{\star}{A}_\xi)\overset{\star}{E}_i,\overset{\star}{E}_i\Big)-\overline{R}ic(V_{\mathscr{S}},N) \\ &\quad+\overline{g}(\overline{R}(N,N)V_{\mathscr{S}},\xi)+\overline{g}(\overline{R}(V_{\mathscr{S}},N)\xi,N)+\mu tr(A_N^2)+\nu(tr(A_N\circ\overset{\star}{A}_\xi) \\ &= V_{\mathscr{S}}\cdot\mathbf{H}_N-\overline{R}ic(V_{\mathscr{S}},N)+\mu tr(A_N^2)+\nu(tr(A_N\circ\overset{\star}{A}_\xi)+\overline{g}(\overline{R}(V_{\mathscr{S}},N)\xi,N). \end{aligned}$$

(3) Also,

$$\nabla u=\sum_{i=0}^{n}g^{[i,0]}(\overset{\star}{E}_i\cdot\mu)\xi+\sum_{i=1}^{n}g^{[0,i]}(\xi\cdot\mu)\overset{\star}{E}_i+\overset{\star}{\nabla}\mu. \tag{3.19}$$

$$\begin{aligned} \Delta^g\mu &= tr(\nabla u)=\sum_{j=0}^{n}\tilde{g}(\nabla_{\overset{\star}{E}_j}\nabla u,\overset{\star}{E}_j)=\sum_{j=1}^{n}\overline{g}(\nabla_{\overset{\star}{E}_j}\nabla u,\overset{\star}{E}_j)+\overline{g}(\nabla_\xi\nabla u,N) \\ &= -\sum_{j=1}^{n}\sum_{i=0}^{n}g\Big(g^{[i,0]}(\overset{\star}{E}_i\cdot\mu)\overset{\star}{A}_\xi\overset{\star}{E}_j,\overset{\star}{E}_j\Big)+\sum_{j=1}^{n}\sum_{i=1}^{n}g\Big(\overset{\star}{E}_j\cdot(g^{[i,0]}\xi\cdot\mu)\overset{\star}{E}_i+g^{[0,i]}(\xi\cdot\mu)\overset{\star}{\nabla}_{\overset{\star}{E}_j}\overset{\star}{E}_i,\overset{\star}{E}_j\Big) \\ &\quad+\overset{\star}{\Delta}\mu+\sum_{i=0}^{n}\Big[\xi\cdot\Big(g^{[i,0]}(\overset{\star}{E}_i\cdot\mu)\Big)-g^{[i,0]}(\overset{\star}{E}_i\cdot\mu)\tau(\xi)-g^{[0,i]}(\xi\cdot\mu)\tau(\overset{\star}{E}_i)\Big]+C(\xi,\overset{\star}{\nabla}\mu) \\ &\overset{(3.4)}{=} -\sum_{j=1}^{n}\sum_{i=1}^{n}g\Big(g^{[i,0]}(\overset{\star}{E}_i\cdot\mu)\overset{\star}{A}_\xi\overset{\star}{E}_j,\overset{\star}{E}_j\Big)+\overset{\star}{\Delta}\mu+\sum_{i=1}^{n}\xi\cdot\Big(g^{[i,0]}(\overset{\star}{E}_i\cdot\mu)\Big)+C(\xi,\overset{\star}{\nabla}\mu). \end{aligned} \tag{3.20}$$

□

4. Special attention to PFW

Fix $u_0\in\mathbb{R}$ and consider the hypersurface

$$\Pi_{u_0}:=\{(x,v,u)\in\bar{M}:u=u_0\}=\mathcal{M}\times\mathbb{R}\times\{u_0\}. \tag{4.1}$$

Π_{u_0} is totally geodesic, since locally $V=\partial v$.

A direct computation shows that the non-necessarily null Christoffel' symbols of (1.3) are

$$\begin{aligned} \Gamma_{ij}^k &= \Gamma_{ij}^{k(R)}\ \Gamma_{uj}^v=\Gamma_{ju}^v=\frac{1}{2}\frac{\partial H}{\partial x^i}(x,u), \\ \Gamma_{uu}^k &= -\frac{1}{2}\sum_{m=1}^{n}g_R^{km}\frac{\partial H}{\partial x^m}(x,u),\ \Gamma_{uu}^v=\frac{1}{2}\frac{\partial H}{\partial u}(x,u), \end{aligned}$$

where $\Gamma_{ij}^{k(R)}$ are Christoffel' symbols associated to the Riemannian metric on $\mathcal{M}$. From this, the connection is given by:

$$\overline{\nabla}_{\partial x_i}\partial x_j=\sum_{k=1}^{n}\Gamma_{ij}^k\partial x_k,\overline{\nabla}_{\partial x_i}\partial u=\Gamma_{ui}^v\partial v,\overline{\nabla}_{\partial u}\partial u=\sum_{k=1}^{n}\Gamma_{uu}^k\partial k+\Gamma_{uu}^v\partial v, \tag{4.2}$$

for every $i,j.k\in\{1,...n\}$.

Remark. [7] *Special classes of PFW are:*

(a) *pp-waves that is PFW for which h is flat.*
(b) *plane waves, PFW for which h is flat and H is a quadratic polynomial in the coordinates on $\mathcal{M}$ with u-dependent coefficients.*
(c) *Cahen-Wallach spaces, which are Lorentzian symmetric spaces. Here h is flat and H is a quadratic polynomial in the coordinates on $\mathcal{M}$ with constant coefficients.*

The Ricci curvature of the PFW is given by [5]

$$\bar{R}ic = \sum_{i,j=1}^{n} R_{ij}^{(R)} dx^i \otimes dx^j - \frac{1}{2}\Delta_x H du \otimes du = Ric^R - \frac{1}{2}\Delta_x H du \otimes du, \qquad (4.3)$$

where $R_{ij}^{(R)}$ is the components of the Ricci curvature associated to the Riemannian metric on $\mathcal{M}$, and $\Delta_x H$ the Laplacian of H with respect to $x = (x_1, ..., x_n)$. Thus, $\bar{R}ic$ is null if and only if both, the Riemannian Ricci tensor Ric^R and the spatial Laplacian $\Delta_x H$, vanish.

Our first result establishes sufficient conditions to guarantee that a totally geodesic null hypersurface in special classes $pp-$waves is, in fact, a member of the family $\Pi_{u_0}(u_0 \in \mathbb{R})$.

Theorem 4.1. *Let M be a totally geodesic null hypersurface, in one of the special classes $pp-$waves, plane waves or Cahen-Wallach generically denoted by $(\overline{M}, \overline{g})$ such that $\Delta_x H \neq 0$. Then M is a member of the family $\Pi_{u_0}(u_0 \in \mathbb{R})$.*

Proof. With respect to the quasi frame field $\{\partial i, \partial v, \partial u\}$, $1 \leq \ i \leq \ n$,

$$\xi \quad = \quad \xi^0 + \xi^v \partial v + \xi^u \partial u.$$

Since (M, ζ) is totally geodesic, we get

$$Ric^R(\xi^0, \xi^0) = \frac{1}{2}\Delta_x(H)\overline{g}(\partial v, \xi)^2 = \frac{1}{2}\Delta_x(H)\mu^2. \qquad (4.4)$$

But $Ric^R(\xi^0, \xi^0) = 0$ taking into account the flatness of h. Therefore, from (4.4), we have $\mu = 0$, and from (3.11), we have that $V_{\mathscr{S}} = 0$ since the screen is Riemannian. That is $\partial v = \nu\xi$ which implies that, for every $x \in M$, $\partial v_x^{\perp} = \xi_x^{\perp} = T_x M$. Hence, M is a leaf of the foliation determined by $\partial v^{\perp}$ and therefore is a member of the family $\Pi_{u_0}(u_0 \in \mathbb{R})$. □

Corollary 4.2. *Let M be a totally geodesic null hypersurface in PFW for which $(\mathcal{M}, h)$ is ricci flat, and $\Delta_x H \neq 0$. Then M is a member of the family $\Pi_{u_0}(u_0 \in \mathbb{R})$.*

Example 4.1. *Let Consider the PFW $(\mathbb{R}^4, \bar{g})$, where $\bar{g}$ is given by:*

$$\bar{g} = \exp(u + x - y)du^2 + 2dudv + \exp(x - y)[dx^2 + dy^2] \qquad (4.5)$$

***Christoffel Symbols and Curvatures**. A direct computation shows that the non-necessarily null Christoffel' symbols are*

$$\begin{aligned}
\Gamma_{xx}^x &= \Gamma_{xx}^y = \Gamma_{xy}^y = -\Gamma_{yy}^x = -\Gamma_{yy}^y = -\Gamma_{xy}^x = -1, \\
\Gamma_{ux}^v &= \Gamma_{xu}^v = -\Gamma_{uy}^v = -\Gamma_{yu}^v = \exp(u + x - y), \\
\Gamma_{uu}^x &= -\Gamma_{uu}^y = \frac{1}{2}\exp(u), \\
\Gamma_{uu}^v &= \frac{1}{2}\exp(u + x - y).
\end{aligned}$$

The only non-null components of the Ricci curvature of the metric are

$$Ric(\partial u, \partial u) = -\frac{1}{2}\Delta_x H(x,u) = \exp(u).$$

$$Ric = -\frac{1}{2}\Delta_x H du \otimes du = \exp(u) du \otimes du. \tag{4.6}$$

As $\exp(u) \neq 0$*, we have that* $\Delta_x H \neq 0$*. Therefore according to Theorem 4.1, each null hypersurface here is a member of the family* $\Pi_{u_0} (u_0 \in \mathbb{R})$*.*

4.2. **Non uniqueness of parallel null vector in PFW.** Localy, for every $X, Z \in \Gamma(TM)$, we have the following:

$$\mathcal{Z} = \mathcal{Z}^u \partial u + \mathcal{Z}^v \partial v + \sum_{i=1}^{n} \mathcal{Z}^i \partial i \;\; X = X^u \partial u + X^v \partial v + \sum_{i=1}^{n} X^i \partial i$$

where for $i = 1, ..., n$, $\mathcal{Z}^i$ and X^i are Riemannian part of $\mathcal{Z}$ and X respectively. By straightforward computation we have:

$$\begin{aligned} \bar{\nabla}_X \mathcal{Z} &= \Big[X \cdot \mathcal{Z}^u\Big]\partial u + \Big[X \cdot \mathcal{Z}^v + X^u\Big(\Gamma^v_{uu} + \sum_{i=1}^{n} X^i \Gamma^v_{iu}\Big) + \sum_{i=1}^{n} X^i \mathcal{Z}^u \Gamma^v_{ui}\Big]\partial v \\ &\quad + \sum_{k=1}^{n}\Big[X \cdot \mathcal{Z}^k + \sum_{i,j=1}^{n} X^i \mathcal{Z}^j \Gamma^k_{ij}\Big]\partial k. \end{aligned} \tag{4.7}$$

In other hand, $\mathcal{Z}$ is null vector if $0 = \bar{g}(\mathcal{Z}, \mathcal{Z}) = \bar{g}(\mathcal{Z}^i, \mathcal{Z}^i) + 2\mathcal{Z}^u \mathcal{Z}^v + (\mathcal{Z}^u)^2 H(x,u)$. Thus, $\mathcal{Z}$ is a parallel null vector field if and only if it satisfies the following system of equations:

$$\begin{cases} \bar{g}(\mathcal{Z}, \mathcal{Z}) + 2\mathcal{Z}^u \mathcal{Z}^v + (\mathcal{Z}^u)^2 H(x,u) = 0, \\ X \cdot \mathcal{Z}^u = 0, \\ X \cdot \mathcal{Z}^v + X^u\Big(\Gamma^v_{uu} + \Gamma^v_{uu} + \sum_{i=1}^{n} X^i \Gamma^v_{iu}\Big) + \sum_{i=1}^{n} X^i \mathcal{Z}^u \Gamma^v_{ui} = 0, \\ X \cdot \mathcal{Z}^k + \sum_{i,j=1}^{n} X^i \mathcal{Z}^j \Gamma^k_{ij} = 0. \end{cases} \tag{4.8}$$

It is easy to see that when H is constant, the vector field $\partial u - \frac{1}{2} H \partial v$ satisfies the above system.

Proposition 4.3. *Let* $\overline{M} = \mathcal{M} \times \mathbb{R}^2$ *be an autonomous PFW (ie.* $H(x,u) \equiv H(x)$*) with satisfies the timelike energie condition such that* $\mathcal{M}$ *is compact. Then* ∂v *and* $\partial u - \frac{1}{2}H(x)\partial v$ *are two parallel null vector fields. Moreover the distribution determined by* $(\partial u - \frac{1}{2}H(x)\partial v)^{\perp}$ *is a totally geodesic null foliation on* $\overline{M}$*.*

Proof. From (4.3), timelike convergence condition $\overline{R}(T,T) \geq 0$ for all timelike vector T holds if and only if $Ric^R(T_0, T_0) \geq 0 \; \forall T_0 \in T\mathcal{M}$, and $\Delta_x H \leq 0$. From the second inequality together with the compactness of $\mathcal{M}$, we have that H is constant. Therefore $\partial u - \frac{1}{2}H(x)\partial v$ is a parallel null vector, which gives the result. $\square$

The following propositions give conditions ensuring the uniqueness of parallel null vector fields.

Proposition 4.4. *Let* $(\overline{M}, \overline{g})$ *be a pp−wave for which the Riemannian curvature of h is different from zero almost everywhere. Then there exists up to constant factor only one parallel null vector field.*

Proof. The only non-vanishing Riemannian curvature terms of $\overline{g}$, up to symmetries are

$$\overline{R}(\partial i, \partial u, \partial u, \partial j) = \frac{1}{2}\frac{\partial^2 H}{\partial x_i \partial x_j}. \tag{4.9}$$

$$\text{If } \mathcal{Z} \quad = \quad \mathcal{Z}^u\partial u + \mathcal{Z}^v\partial v + \sum_{i=1}^{n} l^i\partial_i = \overline{g}(\mathcal{Z}, \partial v)\partial u + (\overline{g}(\mathcal{Z}, u) - \overline{g}(\mathcal{Z}, \partial v)H)\partial v + \mathcal{Z}^0$$

is another parallel null vector field, then from (4.9), it satisfies the following system of equations

$$0 \quad = \quad \overline{g}(\ \overline{R}(\partial x_i, \partial u)\mathcal{Z}, \partial j) = \mathcal{Z}^u \frac{1}{2}\frac{\partial^2 H}{\partial x_i \partial x_w}(x, u) \Rightarrow \mathcal{Z}^u = 0 \tag{4.10}$$

$$0 \quad = \quad \overline{g}(\mathcal{Z}, \mathcal{Z}) = (\mathcal{Z}^u)^2 H(x, u) + 2\mathcal{Z}^u\mathcal{Z}^v + \overline{g}(\mathcal{Z}^0, \mathcal{Z}^0). \tag{4.11}$$

It follows from (4.10) that $\overline{g}(\mathcal{Z}^0, \mathcal{Z}^0) = 0$, that is $\mathcal{Z}^0 = 0$. Thus $\mathcal{Z} = \overline{g}(\mathcal{Z}, \partial u)\partial v$. □

More generally, we have the following.

Proposition 4.5. *In PFW such that* $\det(\text{Hess}\, H) \neq 0$ *at one point). Then there exists up to constant factor only one parallel null vector.*

Proof. If we suppose that ∂w is another parallel null vector fields in PFW which is independent to ∂v at some point x, then according to the sign of the constant function $\overline{g}(\partial v, \partial w)$, a parallel timelike vector field L could be constructed as a linear combination of ∂v and ∂w. So, without loss of generality, let us suppose that $\overline{g}(\partial v, \partial w) > 0$, then $T = \partial v - \partial w$ is a timelike parallel vector field, and from this, the metric would split around x as a product manifolds. That is M is decompossable and which is absurde since from [10, Proposition 3.18] when $\det(\text{Hess}\, H) \neq 0$, PFW is indecomposable. □

Let us mention the following elementary fact.

Theorem 4.6. *There is no compact null hypersurface in PFW for which H has sign.*

Proof. The vector field $\overline{\nabla} v = \partial u - H(x, u)\partial v$ is a nowhere vanishing timelike gradient field in PFW as $\overline{g}(\overline{\nabla} v, \overline{\nabla} v) = -H(x, u) < 0$. Assume existence of compact null hypersurface M and write $\overline{\nabla} v = a\xi + bN + X$ according to decomposition (2.2) meaning that $X \in \mathscr{S}(N)$ and $a, b \in C^\infty(M)$. Then, being $i : M \hookrightarrow \overline{M}$ the inclusion map, we get $\widetilde{\nabla}(v \circ i) = b\xi + X$ and by compactness there is a point in M such that $\widetilde{\nabla}(v \circ i)$ vanishes. At such a point, $\overline{\nabla} v$ is null which is a contradiction. □

From now on, we let (M, ζ) be a normalized null hypersurface with integrable screen distribution $\mathscr{S}(\zeta)$. From [3] M is locally a product manifold $L \times \overset{\circ}{M}$ where L is a null curve and $\overset{\circ}{M}$ is a leaf of $S(\zeta)$ as a codimension two spacelike submanifold of $\bar{M}$.

From (2.6) and (2.7), we have

$$\bar{\nabla}_X Y = \overset{\circ}{\nabla}_X Y + C(X, Y)\xi + B(X, Y)N, \tag{4.12}$$

for all $X, Y \in \mathscr{S}(\zeta)$. Here, $\overset{\circ}{\nabla}$ denotes the induced connection of $\overset{\circ}{M}$ from ∇. It follows that its second fundamental form is

$$\mathbb{I}^{\zeta}(X,Y) = C(X,Y)\xi + B(X,Y)N, \quad X, Y \in \mathscr{S}(\zeta).$$

Taking traces from this expression we obtain that the mean curvature field is given by

$$\mathbf{H} = tr(A_N)\xi + tr(\overset{\star}{A}_{\xi})N = \mathbf{H}_N\xi + \mathbf{H}_{\xi}N, \tag{4.13}$$

where $\mathbf{H}_{\xi}$ (resp. $\mathbf{H}_N$) is the (non normalized) null mean curvature of (M,ζ)(resp. the trace of A_N). Let $A_{\overset{\perp}{V}_{\mathscr{S}}}$ the shape operator associated to $\overset{\perp}{V}_{\mathscr{S}} := \nu\xi + \mu N$, a globally defined normal vector field on the compact leaf $\overset{\circ}{M}$. Then

$$\bar{g}(A_{\overset{\perp}{V}_{\mathscr{S}}} X, Y) = \bar{g}(\mathbb{I}^{\zeta}(X,Y), \overset{\perp}{V}_{\mathscr{S}}) = \mu\bar{g}(A_N X, Y) + \nu\bar{g}(\overset{\star}{A}_{\xi} X, Y). \tag{4.14}$$

We will always be considering an immersion $\Psi : \overset{\circ}{M} \to \overline{M}$, but we will treat Ψ locally as an embedding. We consider now the function $f_u : \overset{\circ}{M} \longrightarrow \mathbb{R}$ defined by

$$f_u = \rho_u \circ \Psi, \tag{4.15}$$

where ρ_u denotes the map $\bar{M} \ni (x,v,u) \mapsto u \in \mathbb{R}$. For every $X \in \mathscr{S}(\zeta)$, we get

$$df_u(a)(X) \quad = \quad d\rho_u\Big(\Psi(a)\Big) \circ d\Psi(a)(X) = d\rho_u\Big(\Psi(a)\Big)(\Psi_{*a}X).$$

So, indentifying X with $\Psi_* X$, we have

$$g(\overset{\star}{\nabla} f_u, X) = \bar{g}(\overline{\nabla}\rho_u, \Psi_* X) = \bar{g}(\partial v, \Psi_* X) = \bar{g}(V_{\mathscr{S}} + \nu\xi + \mu N, \Psi_* X) = \bar{g}(V_{\mathscr{S}}, \Psi_* X) \tag{4.16}$$

Therefore,

$$\overset{\star}{\nabla} f_u = V_{\mathscr{S}}. \tag{4.17}$$

Also,

$$\begin{aligned}
\overset{\star}{\Delta} f_u \quad &= \quad div\overset{\star}{\nabla} f_v = div(V_{\mathscr{S}}) = tr(\overset{\star}{\nabla} V_{\mathscr{S}}) = \sum_{i=1}^{n} \bar{g}(\overset{\star}{\nabla}_{\overset{\star}{E}_i} V_{\mathscr{S}}, \overset{\star}{E}_i) = \sum_{i=1}^{n} \bar{g}(\overline{\nabla}_{\overset{\star}{E}_i} V_{\mathscr{S}}, \overset{\star}{E}_i) \\
&= \quad \sum_{i=1}^{n} \bar{g}(\overline{\nabla}_{\overset{\star}{E}_i} \partial v - \overset{\perp}{V}_{\mathscr{S}}, \overset{\star}{E}_i) = -\sum_{i=1}^{n} \bar{g}(\overline{\nabla}_{\overset{\star}{E}_i} \overset{\perp}{V}_{\mathscr{S}}, \overset{\star}{E}_i) = \sum_{i=1}^{n} \bar{g}(A_{\overset{\perp}{V}_{\mathscr{S}}}, E_i).
\end{aligned} \tag{4.18}$$

Then, from (4.14) and (4.18),

$$\overset{\star}{\Delta} f_u = tr(A_{\overset{\perp}{V}_{\mathscr{S}}}) = \bar{g}(\mathbf{H}, \partial v) = \mu\mathbf{H}_N + \nu\mathbf{H}_{\xi}. \tag{4.19}$$

Let $A_{\overset{\perp}{E}}$ denote the shape oprator associated to $\overset{\perp}{E} := \overset{\star}{\nu}\,\xi + \overset{\star}{\mu}\,N$, a globally defined normal vector field on $\overset{\circ}{M}$ where $\overset{*}{\mu} = \bar{g}(\partial u, \xi)$ and $\overset{*}{\nu} = \bar{g}(\partial u, N)$. Then

$$\bar{g}(A_{\overset{\perp}{E}} X, Y) = \bar{g}(\mathbb{I}^{\zeta}(X,Y), \overset{\perp}{E}) = \overset{\star}{\mu}\,\bar{g}(A_N X, Y) + \overset{\star}{\nu}\,\bar{g}(\overset{\star}{A}_{\xi} X, Y). \tag{4.20}$$

Consider also the function $f_v : M \longrightarrow \mathbb{R}$ defined by

$$f_v = \rho_v \circ \Psi \tag{4.21}$$

where $\rho_{(v)}$ denotes the map $\overline{M} \ni (x,v,u) \mapsto v \in \mathbb{R}$. As before, we get

$$\overset{\star}{\nabla} f_v = E \tag{4.22}$$

where $\partial u = E + E^{\perp} = E + \overset{\star}{\nu}\, \xi + \overset{\star}{\mu}\, N$.

Also,

$$\begin{aligned} \overset{\star}{\Delta} f_v &= div \overset{\star}{\nabla} f_v = div(E) = tr(\overset{\star}{\nabla} E) = \sum_{i=1}^{n} \overline{g}(\overline{\nabla}_{\overset{\star}{E}_i} E, \overset{\star}{E}_i) = \sum_{i=1}^{n} \overline{g}(\overline{\nabla}_{\overset{\star}{E}_i} \partial u - \overset{\perp}{E}, \overset{\star}{E}_i) \\ &= \sum_{i=1}^{n} \overline{g}(\overline{\nabla}_{\overset{\star}{E}_i} \partial u, \overset{\star}{E}_i) - \sum_{i=1}^{n} \overline{g}(\overline{\nabla}_{\overset{\star}{E}_i} \overset{\perp}{E}, \overset{\star}{E}_i) = \sum_{i=1}^{n} \overline{g}(\overline{\nabla}_{\overset{\star}{E}_i} \partial u, \overset{\star}{E}_i) + \sum_{i=1}^{n} \overline{g}(A_{\overset{\perp}{E}}, \overset{\star}{E}_i) \\ &= \sum_{i=1}^{n} \overline{g}(\overline{\nabla}_{\overset{\star}{E}_i} \partial u, \overset{\star}{E}_i) + \overline{g}(\mathbf{H}, E^{\perp}) = \sum_{i=1}^{n} \overline{g}(\overline{\nabla}_{\overset{\star}{E}_i} \partial u, \overset{\star}{E}_i) + \overset{*}{\mu}\, \mathbf{H}_N + \overset{*}{\nu}\, \mathbf{H}_{\xi}. \end{aligned} \tag{4.23}$$

From this, we have the following proposition.

Proposition 4.7. *Let (M, ζ) be a screen integrable normalized null hypersurface in PFW with vanishing rotational one form τ, f_u and f_v given by equation (4.15) and (4.21). Then,*

$$\Delta^g f_u = -\sum_{j=1}^{n}\sum_{i=1}^{n} g\Big(g^{[i,0]}(\overset{\star}{E}_i \cdot f_u)\ \overset{\star}{A}_{\xi} \overset{\star}{E}_j, \overset{\star}{E}_j\Big) + \sum_{i=1}^{n} \xi \cdot \Big(g^{[i,0]}(\overset{\star}{E}_i \cdot f_u)\Big) + C(\xi, V_{\zeta}) + \mu \mathbf{H}_N + \nu \mathbf{H}_{\xi}$$

$$\begin{aligned} \Delta^g f_v &= -\sum_{j=1}^{n}\sum_{i=1}^{n} g\Big(g^{[i,0]}(\overset{\star}{E}_i \cdot f_v)\ \overset{\star}{A}_{\xi} \overset{\star}{E}_j, \overset{\star}{E}_j\Big) + \sum_{i=1}^{n} \xi \cdot \Big(g^{[i,0]}(\overset{\star}{E}_i \cdot f_v)\Big) \\ &\quad + C(\xi, E) + \sum_{i=1}^{n} \overline{g}(\overline{\nabla}_{\overset{\star}{E}_i} \partial u, \overset{\star}{E}_i) + \overset{*}{\mu}\, \mathbf{H}_N + \overset{*}{\nu}\, \mathbf{H}_{\xi} \end{aligned} \tag{4.24}$$

Theorem 4.8. *Let M be a totally geodesic null hypersurface with integrable screen distribution all of whose leaves are compact in a PFW for which $\mathbf{H}_N$ has sign. Then, the following are satisfied*

(a) *$\overset{\circ}{M}$ is a codimension two minimal submanifold of $\overline{M}$.*
(b) $\Psi(\overset{\circ}{M}) \subset \Big\{(x, v_0, u_0), x \in \mathcal{M}\Big\} = \mathcal{M} \times \{v_0\} \times \{u_0\}$.
(c) *$\overset{\circ}{M}$ is locally isometric to $\mathcal{M}$.*

Moreover, either M is a member of the family $\Pi_{u_0} (u_0 \in \mathbb{R})$ or $\overset{\circ}{M}$ is codimension two totally geodesic submanifold of $\overline{M}$.

Proof. Since M is totally geodesic, then the assumption on $\mathbf{H}_N$ together with equation (4.19), assures that either $\overset{\star}{\Delta} f_u \geq 0$ or $\overset{\star}{\Delta} f_u \leq 0$ on $\overset{\circ}{M}$. By compactness, $\overset{\star}{\Delta} f_u = 0$. That is f_u is constant on $\overset{\circ}{M}$

Now, since f_u is constant on $\overset{\circ}{M}$, then $\Phi(\overset{\circ}{M}) \subset \Pi_{u_0}$. Thus identifying $\overset{\star}{E}_i$ $(1 \leq i \leq n)$ to

$\Phi_*(\overset{\star}{E}_i) = F_i + b\partial v \in \Gamma(T\Pi_{u_0})$, and using (4.2), we have that

$$\bar{g}(\bar{\nabla}_{\overset{\star}{E}_i} \partial u, \overset{\star}{E}_i) = 0.$$

From this, (4.23) leads to $\overset{\star}{\Delta} f_v = \overset{*}{\mu} \mathbf{H}_N$. That is $\mathbf{H}_N \xi = \overset{\star}{\Delta} f_v \partial v$. Hence, we must have $\mathbf{H}_N = 0$. Indeed, if we suppose that $\mathbf{H}_N \neq 0$, then we will have either $\overset{\star}{\Delta} f_v \geq 0$ or $\overset{\star}{\Delta} f_v \leq 0$ on $\overset{\circ}{M}$. By compactness, $\overset{\star}{\Delta} f_u = 0$, which is absurde. Therefore, $\overset{\circ}{M}$ is a codimension two minimal submanifold of $\overline{M}$, and f_v is constant on $\overset{\circ}{M}$. Which proves item (a) and (b).

Now, we may write Ψ locally as

$$\Psi : \overset{\circ}{M} \to \overline{M}\ , a \mapsto (\phi(a), v_0, u_0).$$

where

$$\phi : \overset{\circ}{M} \to \mathcal{M}\ , a \mapsto \phi(a) = (x_1(a), x_2(a), ..., x_n(a))$$

From this, for every $X \in T_a \overset{\circ}{M}$, we compute

$$\phi_* X = (U_1, U_2, ..., U_n).$$

Hence, for every $X, Y \in T_a \overset{\circ}{M}$, we get

$$\begin{aligned} h(d\phi(X), d\phi(Y)) &= h(d\phi(X), d\phi(Y)) + 2\bar{g}(d\Psi(X), \partial v)\bar{g}(d\Psi(X), \partial u) \\ &\quad + H(x,u)\bar{g}(d\Psi(X), \partial v)\bar{g}(d\Psi(Y), \partial v) \\ &= \bar{g}(d\Psi(X), d\Psi(Y)) = g(X,Y) \end{aligned}$$

In other words, $\phi^* h = g$, which means that $\phi : \overset{\circ}{M} \to \mathcal{M}$ is a local isometry.

Now as f_u is constant on $\overset{\circ}{M}$, from (4.17), we have $V_{\mathscr{S}} = 0$. Hence, using (3.11), we get either $\mu = 0$ or $\nu = 0$. That is either $\partial v = \nu \xi$ or $\partial v = \mu N$.

(i) If $\partial v = \nu \xi$, then from Theorem 4.1, M will be a member of the family $\Pi_{u_0} (u_0 \in \mathbb{R})$.

(ii) If not if $\partial v = \mu N$, then $A_N = 0$. Using the fact that M is totally geodesic, we have that $\overset{\circ}{M}$ is a codimension two totally submanifold of M which complete the proof of the Theorem.

□

Corollary 4.9. *Let (M, ζ) be a totally geodesic null hypersurface with integrable screen distribution all of whose leaves are compact in a PFW for which $\mathbf{H}_N$ has sign. Then*

(i) $\Delta^g f_u = 0$, $\Delta^g f_v = 0$.

(ii) *ϕ is a Riemannian covering map*

Proof. (i) Using the fact that $\mathbf{H}_\xi = 0$, f_u and f_v constant on $\overset{\circ}{M}$, in Eq.(4.24)-(4.24) the claim follows.

(ii) From [9, p.4] every local isometry between complete Riemannian manifolds is a covering map.

□

Corollary 4.10. *Let (M, ζ) be a toally geodesic null hypersurface with integrable screen distribution all of whose leaves are compact in a PFW for which $\mathbf{H}_N$ has sign, and $\Delta_x H \neq 0$. Then,*

(a) *M is a member of the family $\Pi_{u_0} (u_0 \in \mathbb{R})$.*

(b) $\bar{R}ic(N, N) = \frac{1}{2} \Delta_x H \bar{g}(\partial v, N)^2$.

Proof. As $\partial v = \nu\xi$ or $\partial v = \mu N$, we have that $\bar{g}(X, \xi) = \bar{g}(X, N) = 0$, for every $X \in \Gamma(T\Pi_{u_0})$. From (4.3), we have $\frac{1}{2}\Delta_x H\bar{g}(\partial v, \xi)^2 = 0$. That is $\partial v = \nu\xi$. Using again (4.3), we have $\bar{R}ic(N, N) = \frac{1}{2}\Delta_x H\bar{g}(\partial v, N)^2$. Which give the result. □

When $\mathbf{H}_N = 0$, Theorem 4.8 still true and we have the following example.

Example 4.3. *Take* $\overline{M} = (\mathbb{S}^2 \times \mathbb{R}^2, \overline{g} = g_s + 2dudv + H(x,u)du^2$. *The family* $\Pi_{u_0}(u_0 \in \mathbb{R})$ *is an screen integrable totally geodesic null hypersurface with compact totally geodesic leaves given by* $\{\mathbb{S}^2 \times \{v_0\} \times \{u_0\},\ v_0, u_0 \in \mathbb{R}\}$. *If we suppose that* H *is constant, then* $(\partial u - \frac{1}{2}H\partial v)^{\perp}$ *and* $\partial u^{\perp}$ *will give rise to two screen integrable totally geodesic null foliations, both with the same compact totally geodesic screen leaves.*

In what follows, $l^1(M)$ stands for the space of Lebesgue integrable functions on M

Lemma 4.11. *(see* [11]*) Let* X *be a smooth vector field on n-dimensional complete, non-compact oriented Riemannian manifold* M^n, *such that* $div_M X$ *does not change sign on* M^n. *If* $|X| \in l^1(M)$, *then* $div_M X = 0$.

Theorem 4.12. *Let* (M, ζ) *be a totally geodesic null hypersurface with integrable screen distribution all of whose leaves are complete noncompact in a* PFW *for which* $\mathbf{H}_N$ *has sign, and* $|V_{\mathscr{S}}|$ *is Lebesgue integrable. Then either* M *is a member of the family* $\Pi_{u_0}(u_0 \in \mathbb{R})$*or* $\overset{\circ}{M}$ *is totally geodesic in* M.

Proof. Since M is totally geodesic, and $|V_{\mathscr{S}}|$ is Lebesgue integrable, equation (4.15) together with lemma 4.11 give $0 = div(V_{\mathscr{S}}) = \mu\mathbf{H}_N = 0$. That is either $\mu = 0$ or $\mathbf{H}_N = 0$. Which gives the result. □

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