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TOTALLY - MEASURABILITY ON SIGNED MEASURABLE SPACES FOR QUASI-NORMED SPACES VALUED FUNCTIONS

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Abstract. In this paper our aim is to identify the properties of totally measurable functions with values in a quasi normed space, defined in a measurable space. We are focused on the case when the measurable space is equipped with a signed measure and defining the concept of convergence according to outer measure of a sequence of functions we have proof a convergence theorem which is one of the results obtained.

1. Introduction

The notion of totally measurable functions in case of finitely purely atomic measure and atomic multimeasure spaces is studied in [1] and [2]. Authors in [1], under the assumptions that X is a Banach space, measure is a set multifunction of finite variation valued in $\mathcal{P}(X)$ and the functions are scalar have achieved some results on totally measurable. The paper [2] presents some results on finitely purely atomic measure spaces. The idea is similar to that of [1], but the functions are valued in a Banach space X, namely vector valued functions and the measure is real valued and positive.
This paper research's focus is totally - measurability of quasi - normed spaces valued functions, when the measure is assumed to be a signed measure. The primary aim is to extend, if it is possible, the properties observed in [1] and [2] and further by introducing the concept of convergence according to outer measure, to study a convergence theorem, (Proposition 3.3). Presentation of the absolute variation of a signed measure as $|m| = m^+ + m^-$ ([4], Definition 10.6, Theorem10.5), allows us to extend the concept of the m - totally measurable function in the case of signed measure and further to show the truth of Remark 3.11.4 ([2]) in this case. Using the concept of convergence according to outer measure and quasi-norm properties presented in [5], we have given an equivalent definition of total measurability (Definition 3.3). After that, using this definition and the Riesz's Theorem, we have shown that total measurability brings measurability. In closing this paper, the Egoroff's Theorem allows us to shown that in the case of finite measurable spaces according to $|m|$, the concepts of total measurability and measurability coincide. If

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the condition of being finite is removed, we have noticed by way of example that this compatibility does not apply.

2. Preliminaries

Let T be a nonempty set, $\mathcal{P}(T)$ the family of all subsets of T and Σ a σ - algebra of subsets of T. A partition of T is a finite family P = (A$_i$) $_{i=1,...,n}$ in Σ such that $A_i \cap A_j = \emptyset$, $i \neq j$ and $\cup_{i=1}^{n} A_i$ = T.

Definition 2.1. *([2] Definition 3.8)*
(i) If $P = (A_i)_{\ i=1,...,n}$, P' = $(B_j)_{\ j=1,...,m}$ are two partitions of T, then P' is said to be finer than P (denoted by $P \leq P'$ or $P' \geq P$) if for every $j \in \{1, ..., m\}$ there exists $i_j \in \{1, ..., n\}$ so that $B_j \subseteq A_{i_j}$.
(ii) The common refinement of two partitions $P = (A_i)_{i=1,...,n}$ and P' = $(B_j)_{j=1,...,m}$ is the partition $P \wedge P' = (A_i \cap A_j)_{i=1,...,n;j=1,...,m}$.

Let m : $\Sigma \to$ [-$\infty, +\infty$] be an arbitrary set function, with $m(\emptyset) = 0$.

Definition 2.2. *([4] Definition 10.1)*
The set function m is said to be a signed measure if
1.For every $A \in \Sigma$, $m(A) \neq -\infty$ or for every $A \in \Sigma$, $m(A) \neq +\infty$.
2.For every sequence of sets $(A_n)_{n\in\mathbb{N}}$ in Σ such that, $A_{n_1} \cap A_{n_2} = \emptyset$ if $n_1 \neq n_2$, $m(\cup_{n\in\mathbb{N}} A_n) = \sum_{n\in\mathbb{N}} m(A_n)$ (σ - additivity property).

Definition 2.3. *([2] Definition 2.1)*
The set function $m : \Sigma \to [0, +\infty]$ with $m(\emptyset) = 0$ is said to be:
(i) monotone measure if $m(A) \leq m(B)$ for every $A, B \in \Sigma$ with $A \subseteq B$.
(ii) null-additive measure if $m(A \cup B) = m(A)$, for every $A, B \in \Sigma$ with $m(B) = 0$.
(iii) σ - null - additive measure if $m(\cup_{n\in\mathbb{N}} A_n)$ = 0 as soon as $A_n \in \Sigma$ and $m(A_n) = 0$ for all $n \in \mathbb{N}$.
(iv) subadditive measure if $m(A \cup B) \leq m(A) + m(B)$ for every $A, B \in \Sigma$.
(v) finitely additive measure if $m(A \cup B) = m(A) + m(B)$, for every $A, B \in \Sigma$, with $A \cap B = \emptyset$.
(vi) σ - subadditive measure if $m(\cup_{n=1}^{\infty} A_n) \leq \sum_{n=1}^{\infty} m(A_n)$, for every $(A_n)_{n\in\mathbb{N}} \subset \Sigma$, so that $\cup_{n=1}^{\infty} A_n \in \Sigma$.
(vii) σ - additive measure if $m(\cup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} m(A_n)$, for every $(A_n)_{n\in\mathbb{N}} \subset \Sigma$, so that $\cup_{n=1}^{\infty} A_n \in \Sigma$ and $A_i \cap A_j = \emptyset$ for $i \neq j$, $i, j \in \{1, ..., n\}$.

Remark 2.1. *([3])*
If $m : \Sigma \to [0, +\infty]$ is monotone and subadditive, then m is null - additive. A subadditive monotone measure is sometimes called a submeasure.

Definition 2.4. *([2] Definition 3.1)*
Let $m : \Sigma \to [0, +\infty]$ be an arbitrary set functions, with $m(\emptyset) = 0$.
(i) A set $A \in \Sigma$ is said to be an atom of m if $m(A) > 0$ and for every $B \in \Sigma$, with

$B \subset A$, we have $m(B) = 0$ or $m(A \setminus B) = 0$.
(ii) m is said to be finitely purely atomic (and T a finitely purely atomic space) if there is a finite disjoint family $(A_i)_{i=1}^{n} \subset \Sigma$ of atoms of m so that $T = \cup_{i=1}^{n} A_i$.

Lemma 2.1. *([1], Remark 3.7)*
Let $m : \Sigma \to [0, +\infty]$ be a non-negative set function, with $m(\emptyset) = 0$ and let $A \in \Sigma$ be an atom of m.
(i) If m is monotone measure and the set $B \in \Sigma$ is so that $B \subseteq A$ and $m(B) > 0$, then B is also an atom of m and $m(A \setminus B) = 0$. Moreover, if m is null - additive, then $m(B) = m(A)$.
(ii) If m is monotone and null-additive measure, then for every finite partition $(B_i)_{i=1}^{n}$ of Σ, there exists a unique $i_0 \in \{1, 2..., n\}$ so that $m(B_{i_0}) = m(A)$ and $m(B_i) = 0$ for every $i \in \{1, 2..., n\}$, $i \neq i_0$.

Definition 2.5. *([4], Definition 3.1)*
The set function $m^\star : P(T) \to [0, +\infty]$ with $m^\star(\emptyset) = 0$ called outer measure on T if it is monotone and σ - subadditive measure.

So, an outer measure is a submeasure on T.

Definition 2.6. *([5], Definition 1.1)*
Let X be a vector space. A function $\| \, . \, \|: X \to [0, +\infty)$ is said to be quasi - norm on X if the following conditions hold:
(i) $\| x \| = 0 \Leftrightarrow x = 0$.
(ii) for every $x \in X$ and for every $\lambda \in \mathbb{R}$, $\| \lambda x \| = |\lambda| \, \| x \|$.
(iii) for every $x, y \in X$, $\| x + y \| \leqslant K(\| x \| + \| y \|)$ where $K \geq 1$ is a constant independent from variables x and y.
The smallest of constant K, such that the above conditions hold, is called the modulus of concavity of quasi - norm $\| \, . \, \|$.

If the vector space X is equipped with a quasi - norm $\| \, . \, \|$ on X, then $(X, \| \, . \, \|)$ is called quasi - normed space.

Let $m : \Sigma \to [-\infty, +\infty]$ be an arbitrary set functions, with $m(\emptyset) = 0$.
In the same way to Definition 3.9 to [2], we can give the following definition.

Definition 2.7. *A vector function $f : T \to X$ is said to be:*
(i) m-totally-measurable (on T) if for every $\varepsilon > 0$, there exists a partition of T, $(A_i)_{i=0}^{n} \subset \Sigma$, with $\{A_1, A_2, ..., A_n\} \subset \Sigma \setminus \{\emptyset\}$, such that the following two conditions hold:
(1) $|m|(A_0) = sup\{\sum_{j=1}^{l} |m(A_j)|\} < \varepsilon$; where $(A_j)_{j=1}^{l}$ is a partition of A_0 and supremum is extended over all finite partitions of set A_0.
$|m|$ is called absolute variation of m.
(2) $sup_{t,s \in A_i} \| f(t) - f(s) \| < \varepsilon$, for all $i \in \{1, 2, ..., n\}$.
(ii) The vector function f is called m-totally-measurable on $B \in \Sigma$ if the restriction $f|_B$ is m-totally-measurable on (B, Σ_B, m_B), where $\Sigma_B = \{A \cap B : A \in \Sigma\}$ and $m_B = m|_{\Sigma_B}$.

Recall that :
([4], Definition 10.3) (i) A set $P \in \Sigma$ is called a positive set, according to signed measure m, if for every $P' \in \Sigma$ such that $P' \subseteq P$, $m(P') \geq 0$.
A set $N \in \Sigma$ is called a negative set, according to signed measure m, if for every $N' \in \Sigma$ such that $N' \subseteq N$, $m(N') \leq 0$.
A set $Q \in \Sigma$ is called a null set, according to signed measure m, if for every $Q' \in \Sigma$ such that $Q' \subseteq Q$, $m(Q') = 0$.
([4], Definition 10.4, Theorem 10.3) (ii) As claimed by the Hanh decomposition of signed measure m, we can write $m = m^+ - m^-$ where $\forall A \in \Sigma$, $m^+(A) = m(A \cap P)$, $m^-(A) = -m(A \cap N)$, P and N are respectively positive and negative set of m and $P \cup N = T$, $P \cap N = \emptyset$.
([4], Definition 10.6, Theorem10.5) (iii) $m^+, m^-, |m| = m^+ + m^-$ are $\sigma-$ additive, monotone, non negative measures on T.
([4], Definition 6.9, Proposition 6.22 (ii)) (iv) A function $f : T \to X$ is called a simple function if $f(x) = \sum_{i=1}^{n} a_i \chi_{A_i}(x)$, where χ_{A_i} are characteristic functions on a finite partition of T.

Let X be a quasi - normed space. Now we are giving an example of m-totally-measurable functions on T.

Example 1.
Every simple function $f : T \to X$ is m-totally-measurable function on T.
Proof
The proof is immediate, if we take $A_0 = \emptyset$ and A_i for $i = 1, ..., n$ the sets of partition above. So, $|m|(A_0) = |m|(\emptyset) = 0 < \varepsilon$, for every $\varepsilon > 0$.
On the other hand, $sup_{t,s \in A_i} \parallel f(t) - f(s) \parallel = sup_{t,s \in A_i} \parallel a_i - a_i \parallel = 0 < \varepsilon$, for every $\varepsilon > 0$.

Remark 2.2. *If $f : T \to X$ is m-totally-measurable, then f is m^+-, m^-- and $|m|$-totally-measurable.*
Proof
If $|m|(A_0) < \varepsilon$ then, $0 \leq (m^+ + m^-)(A_0) = m^+(A_0) + m^-(A_0) < \varepsilon$.
That implies, $m^+(A_0), m^-(A_0) < \varepsilon$.
Therefore, $|m^+|(A_0) = sup\{\sum_{k=1}^{n} m^+(A_k) : \cup_{k=1}^{n}(A_k) = A_0\} \leq m^+(A_0) < \varepsilon$
(this implies from the fact that m^+ is σ- additive, monotone , non negative measure on T) .

Remark 2.3. *If the vector function $f : T \to X$ is both m^+- and m^--totally-measurable, then f is m-totally-measurable.*
The proof is immediately from the equality (1) $|m|(A_0) = m^+(A_0) + m^-(A_0) < \varepsilon + \varepsilon = 2\varepsilon = \varepsilon'$.

Remark 2.4. *([2], Remark 3.11.2) (i) If the vector function $f : T \to X$ is m-totally-measurable on T, then f is m-totally-measurable on every $A \in \Sigma$. (The same proposition hold in case when m is a signed measure, because of equality (1), that* *hold for every $A_0 \in \Sigma_A$, and the fact that m^+, m^-, are σ- additive, monotone , non negative measures on T).*

([2], Remark 3.11.4) (ii) [1] If m is null - additive and monotone measure, and $A \subset T$ is an atom for m, then a function $f : T \to X$ is m-totally-measurable on A if and only if :

$$inf_{U\in\mathcal{U}} sup_{t,s\in U} \parallel f(t) - f(s) \parallel = 0 \ ,$$

where $\mathcal{U}$ is the family of all atoms contained in A.

3. Totally measurable functions and convergence theorems

The same proposition with Remark 2.4 (ii) can formulated for signed measure.

Remark 3.1. *If m is a signed measure and $A \subset T$ is an atom for $|m|$, then a function $f : T \to X$ is m-totally-measurable on A if and only if :*

$$inf_{U\in\mathcal{U}} sup_{t,s\in U} \parallel f(t) - f(s) \parallel = 0$$

where $\mathcal{U}$ is the family of all atoms contained in A.

Proof

Suppose that the function $f : T \to X$ is m-totally-measurable on A. From Definition 2.7, for every $\varepsilon > 0$, there exists a partition $(A_i)_{i=0}^n \subset \Sigma_A$ with $\{A_1, A_2, ..., A_n\} \subset \Sigma_A \setminus \{\emptyset\}$ such that:

1. $|m|(A_0) < \varepsilon$

2. $sup_{t,s\in A_i} \parallel f(t) - f(s) \parallel < \varepsilon, \forall i \in \{1, 2, .., n\}$

Thus, if $U \subset A$ and U is an atom for $|m|$, then $|m|(U) > 0$ and for every $B \subset U, B \in \Sigma$ either $|m|(B) = 0$ or $|m|(U \setminus B) = 0$.

On the other hand, A is also an atom for $|m|$. Therefore, $|m|(A \setminus U) = 0$ and so, the monotony of $|m|$ imply that $U \subseteq \cup_{i=1}^n A_i$ and $(A \setminus U) \subseteq A_0$ (for some $\varepsilon > 0$). Furthermore, since the collection $(A_i)_{i=0}^n \subset \Sigma_A$ is a partition of T, only one of sets, let say A_k where $k \in \{1, 2, ..., n\}$, has a positive measure $|m|$ and the other sets has measure $|m|$ zero.

If $U \subseteq A_k$, then $sup_{t,s\in U} \parallel f(t) - f(s) \parallel < \varepsilon$.

Otherwise, denote $U_1 = U \cap A_k$. Thus $|m|(U) = |m|(U_1)$.

For every $B \subset U_1 \subset U$, we have $|m|(B) = 0$ or $|m|(U \setminus B) = 0$. If $|m|(B) \neq 0$, then $|m|(U \setminus B) = 0$ and $(U_1 \setminus B) \subset (U \setminus B)$ that imply $0 \leq |m|(U_1 \setminus B) \leq |m|(U \setminus B) = 0$. So, the set U_1 is an atom for $|m|$ and $U_1 \subset A_k$. This completes the proof.

Conversely, suppose that $inf_{U\in\mathcal{U}} sup_{t,s\in U} \parallel f(t) - f(s) \parallel = 0$, where $\mathcal{U}$ is the family of all atoms contained in A. From Definition of infinum, for every $\varepsilon > 0$ exists an atom $U \in \mathcal{U}$ such that $sup_{t,s\in U} \parallel f(t) - f(s) \parallel < \varepsilon$.

The set $U \subset A$ is an atom for $|m|$ and A is also an atom for $|m|$, so $|m|(A \setminus U) = 0$. Denote $A_0 = A \setminus U, A_1 = U$. The family $\{A_0, A_1\}$ is a partition of A and $|m|(A_0) = 0$, $sup_{t,s\in A_1=U} \parallel f(t) - f(s) \parallel < \varepsilon$. Thus, the partition $\{A_0, A_1\}$ is such that the conditions of Definition 2.7 holds. So, the function f is m-totally-measurable on A.

Let $m : \Sigma \to [-\infty, +\infty]$ a signed measure. The following proposition hold.

Proposition 3.1. *If $f, g : T \to X$ are m-totally-measurable functions and $k \in \mathbb{R}$ then kf and $f \pm g$ are also m- totally-measurable.*

Proof

The first claim is immediately according to Definition 2.7. Let proof the second claim.
For every $\varepsilon > 0$, there exists partitions $P_1 = \{A_i\}_{i=0}^n$ and $P_2 = \{B_j\}_{j=0}^m$ of T such that $|m|(A_0) < \frac{\varepsilon}{2K}$, $|m|(B_0) < \frac{\varepsilon}{2K}$, where K is modulus of concavity of quasi-norm $\| . \|$ on X, and $sup_{t,s\in A_i} \| f(t) - f(s) \| < \frac{\varepsilon}{2K}$ for $i \in \{1, ..., n\}$, $sup_{t,s\in B_j} \| f(t) - f(s) \| < \frac{\varepsilon}{2K}$ for $j \in \{1, ..., m\}$.
Define another partition P_3 of T as following.
Take $C_{00} = A_0 \cup B_0$, $C_{ij} = A_i \cap B_j$ for $i \in \{1, ..., n\}, j \in \{1, ..., m\}$.
We can see easily that the family $P_3 = \{C_{ij}\}_{i=0,j=0}^{n,m}$ is a partition of T and $|m|(C_{00}) \leq |m|(A_0) + |m|(B_0) < \frac{\varepsilon}{2K} + \frac{\varepsilon}{2K} = \frac{\varepsilon}{K} \leq \varepsilon$.
Let see the second condition.
For every $i \in \{1, ..., n\}$, $j \in \{1, ..., m\}$ and $t, s \in C_{ij}$ have:
$\| f(t) + g(t) - (f(s) + g(s)) \| \leq K \| f(t) - f(s) \| + K \| g(t) - g(s) \|$
$\leq K sup_{t,s\in A_i} \| f(t) - f(s) \| + K sup_{t,s\in B_j} \| f(t) - f(s) \| < \varepsilon$.
So, $sup_{t,s\in C_{ij}} \| f(t) - f(s) \| \leq \varepsilon$. Thus for partition P_3 the conditions of Definition 2.7 holds. This complete the proof.
The third claim is clear from equality $f - g = f + (-g)$ and first claim.

Now let formulate the following interesting proposition:

Proposition 3.2. *If $(f_n : T \to X)_{n\in\mathbb{N}}$ is a sequence of m-totally-measurable functions on T that is uniformly converge on function $f : T \to X$ for every $t \in T$, then the function f is also m-totally-measurable on T.*

Proof

Since $(f_n : T \to X)_{n\in\mathbb{N}}$ is uniformly converge on function $f : T \to X$ for every $t \in T$, then exists a $n_0 \in \mathbb{N}$ such that, for every $n \geq n_0$ and for every $t \in T$, $\| f_n(t) - f(t) \| < \varepsilon$.
Thus, $\| f(t) - f(s) \| \leq K(\| f_n(t) - f(t) \| + \| f_n(t) - f(s) \|) \leq K \| f_n(t) - f(t) \| + K^2 \| f_n(t) - f_n(s) \| + K^2 \| f_n(s) - f(s) \| < K\varepsilon + K^2\varepsilon + K^2 \| f_n(t) - f_n(s) \|$ for every $t, s \in T$ and $n \geq n_0$.
From Definition 2.7,we can write that, for some $n \geq n_0$ and for every $\varepsilon > 0$, there exists a partition of T, $(A_i)_{i=0}^{m(n)} \subset \Sigma$, with $\{A_1, A_2, ..., A_{m(n)}\} \subset \Sigma \setminus \{\emptyset\}$, such that the following two conditions hold:

(1) $|m|(A_0) < \varepsilon$; (2) $sup_{t,s\in A_i} \| f_n(t) - f_n(s) \| < \varepsilon, \forall i \in \{1, 2, ..., m(n)\}$.

So, the above inequalities imply that $\| f(t) - f(s) \| < K\varepsilon + 2K^2\varepsilon = \varepsilon'$ for every $t, s \in A_i$, $i \in \{1, 2, ..., m(n)\}$.
The result in above proposition also holds when then sequence $(f_n(t))_{n\in\mathbb{N}}$ converges uniformly to f(t) almost everywhere according to $|m|$ for $t \in T$. In this case, take the partition $(A'_i)_{i=0}^n$ of T such that $A'_0 = A_0 \cup B$, $A'_i = A_i \setminus B$ for every $i = 1, ..., n$, where B is denoted a subset of T with $|m|(B) = 0$.

Let us give the following two definitions and let us see what is their impact.

Definition 3.1. *(The convergence according to outer measure) .*
Let $m : \Sigma \to [0, +\infty]$ be a positive monotone measure on T. The sequence of functions $(f_n : T \to X)_{n \in \mathbb{N}}$ converges according to outer measure $m^\star$ to a function $f : T \to X$ if $lim_{n \to +\infty} m^\star\{t \in T :\parallel f_n(t) - f(t) \parallel \geq \sigma\} = 0$ for every $\sigma > 0$.

Definition 3.2. *The function $m^\star : \mathcal{P}(T) \to [0, +\infty]$ such that, for every $A \subseteq T$, $m^\star(A) = inf\{m(B) : B \in \Sigma, A \subseteq B\}$, called outer measure on T generated from m.*

Remark 3.2. *If $m : \Sigma \to [0, +\infty]$ is a positive monotone measure on T and $m^\star$ is the outer measure generated from m, then for every $A \in \Sigma, m(A) = m^\star(A)$.*
Proof
For every $A \in \Sigma$, $m(A) \in \{m(B) : B \in \Sigma, A \subseteq B\}$ implies that $m^\star(A) \leq m(A)$.
On the other hand, for every $B \in \Sigma$ with $A \subset B$ we have $m(A) \leq m(B)$ implies that $m(A) \leq inf\{m(B) : B \in \Sigma, A \subseteq B\} = m^\star(A)$.

Let $m : \Sigma \to [-\infty, +\infty]$ be a signed measure and $|m|^\star$ be the outer measure generated from $|m|$.

Proposition 3.3. *If $(f_n : T \to X)_{n \in \mathbb{N}}$ is a sequence of m-totally-measurable functions converges according to $|m|^\star$ to a function $f : T \to X$, then the function f is also m-totally-measurable.*
Proof
Let $(f_n : T \to X)_{n \in \mathbb{N}}$ be a sequence of m-totally-measurable functions that converges according to $|m|^\star$ to a function $f : T \to X$.
For every $\varepsilon > 0$, there exists a natural number n_0 such that, for every $n \geq n_0$ we have $|m|^\star\{t \in T :\parallel f_n(t) - f(t) \parallel \geq \varepsilon\} < \varepsilon$.
Denote $A_\varepsilon^{(n)} = \{t \in T :\parallel f_n(t) - f(t) \parallel \geq \varepsilon\}$ and $B_\varepsilon^{(n)} = T \setminus A_\varepsilon^{(n)}$. For every $n \geq n_0$, $|m|^\star(A_\varepsilon^{(n)}) < \varepsilon$ and for every $t, s \in B_\varepsilon^{(n)}$ we have $\parallel f_n(t) - f(t) \parallel < \varepsilon$. Fix a natural number $n \geq n_0$ and from inequality $\parallel f(t) - f(s) \parallel \leq K \parallel f_n(t) - f(t) \parallel + K^2 \parallel f_n(t) - f_n(s) \parallel + K^2 \parallel f_n(s) - f(s) \parallel$ (see proof of above proposition) we can write:

$$\parallel f(t) - f(s) \parallel \leq K\varepsilon + K^2\varepsilon + K^2 \parallel f_n(t) - f_n(s) \parallel,$$

for every $t, s \in B_\varepsilon^{(n)}$.
Since f_n is m-totally-measurable on T, then find a partition $(A_i)_0^n$ of T such that, $(A_i)_{i=0}^n \subset \Sigma$, with $\{(A_i)_{i=1}^n\} \subset \Sigma \setminus \{\emptyset\}$, $|m|(A_0) < \varepsilon$ and $sup_{t,s \in A_i} \parallel f_n(t) - f_n(s) \parallel < \varepsilon$ for $i = 1, ..., n$. Denote B_0 the smallest set in Σ such that $(A_\varepsilon^{(n)} \cup A_0) \subseteq B_0$ (this set is $B_0 = \cap_{B \in \Sigma}\{B : (A_\varepsilon^{(n)} \cup A_0) \subseteq B\}$) and $B_i = (T \setminus B_0) \cap A_i$ for every $i = 1, ..., n$. So, the collection $(B_i)_{i=0}^n$ is a partition of T such that:

$$|m|(B_0) = |m|^\star(B_0) \leq |m|^\star(A_\varepsilon^{(n)}) + |m|^\star(A_0) = |m|^\star(A_\varepsilon^{(n)}) + |m|(A_0) < 2\varepsilon$$
$$\text{and } sup_{t,s \in B_i} \parallel f_n(t) - f_n(s) \parallel \leq sup_{t,s \in A_i} \parallel f_n(t) - f_n(s) \parallel < \varepsilon.$$

Thus $B_i \subseteq B_\varepsilon^n, A_i$ implies that:

$$sup_{t,s \in B_i} \parallel f(t) - f(s) \parallel \leq (K + K^2)\varepsilon + K^2 sup_{t,s \in A_i} \parallel f_n(t) - f_n(s) \parallel < (K + 2K^2)\varepsilon = \varepsilon'$$

This completes the proof.

We can formulate another equivalent definition of m-totally-measurable function.

Definition 3.3. *The vector function $f : T \to X$ is called m-totally-measurable on T, if there exists a sequence of simple functions that converge according to out measure $|m|^\star$ to f.*

Remark 3.3. *Definition 2.7 and Definition 3.3 of m-totally-measurable functions, are equivalent.*

Proof

Suppose that Definition 2.7 holds. Take the numerical sequence $\varepsilon_n = \frac{1}{2^n}$ and define the simple functions sequence $\varphi_n(t) = \sum_{i=0}^{k(n)} a_i^n \chi_{A_i^n}(t)$ for every $n \in \mathbb{N}$, where a_i is equal with a whatever $f(t)$ for $t \in A_i^n$ and $\{A_i^n\}_{i=0}^{k(n)}$ is the partition of T corresponding to ε_n.
Thus, $|m|(A_0^n) < \frac{1}{2^n}$ and $sup_{t,s\in A_i^n} \parallel f(t) - f(s) \parallel < \frac{1}{2^n}$ for all $i \in \{1, 2, ..., k(n)\}$.
Let proof that, the $\varphi_n(t)$ sequence converges according to out measure $|m|^\star$ to f.
For every $\sigma > 0$ exists a $n_0 \in \mathbb{N}$ such that, $\sigma > \frac{1}{2^n}$ for every $n \geq n_0$.
Denote $A_\sigma^n = \{t \in T : \parallel \varphi_n(t) - f(t) \parallel \geq \sigma\}$. Since $\varphi_n(t) = f(s)$ for some $s \in A_i^n$ we have $A_\sigma^n \subseteq A_0^n$ for every $n \geq n_0$, that implies $|m|^\star(A_\sigma^n) \leq |m|^\star(A_0^n) = |m|(A_0^n) < \frac{1}{2^n}$. So, $lim_{n\to+\infty}|m|^\star(A_\sigma^n) = 0$, that is our claim.
Conversely, suppose that exists a simple functions sequence $\varphi_n : T \to X$ that converges according to out measure $|m|^\star$ to f. Let proof that the function f is m-totally-measurable on T according to definition 2.7. Take $\sigma = \frac{\varepsilon}{2K}$, where K is modulus of concavity of quasi-norm on X. For every $\frac{\varepsilon}{2} > 0$ exists $n_0 \in \mathbb{N}$ such that, for every $n \geq n_0$, $|m|^\star\{t \in T : \parallel \varphi_n(t) - f(t) \parallel \geq \frac{\varepsilon}{2K}\} < \frac{\varepsilon}{2}$. Fix some $n \geq n_0$ and denote $B_0 = \{t \in T : \parallel \varphi_n(t) - f(t) \parallel \geq \frac{\varepsilon}{2K}\}$. It is clear that $|m|^\star(B_0) < \frac{\varepsilon}{2}$ and form Definition 3.2 we can write: For every $\frac{\varepsilon}{2} > 0$ exists a set $B_0' \in \Sigma$ such that, $B_0 \subseteq B_0'$ and $|m|^\star(B_0) \leq |m|(B_0') < |m|^\star(B_0) + \frac{\varepsilon}{2} < \varepsilon$.
Denote $(B_0')^c = T\backslash B_0' = \{t \in T : \parallel \varphi_n(t) - f(t) \parallel < \frac{\varepsilon}{2K}\}$. If $\varphi_n(t) = \sum_{i=0}^{k(n)} a_i^n \chi_{A_i^n}(t)$, then $(B_0')^c = (B_0')^c \cap (\cup_{i=0}^{k(n)} A_i^n) = \cup_{i=0}^{k(n)}((B_0')^c \cap A_i^n) = \cup_{i=0}^{k(n)} C_i$ where $C_i = (B_0')^c \cap A_i^n$ for every $i = 0, ..., n$.
It is clear that $C_i \in \Sigma$ and for every $t \in C_i, \parallel \varphi_n(t) - f(t) \parallel = \parallel a_i^n - f(t) \parallel \leq \frac{\varepsilon}{2K}\}$.
Since, for every $t, s \in C_i, \parallel f(t) - f(s) \parallel \leq K \parallel a_i^n - f(t) \parallel + K \parallel a_i^n - f(s) \parallel < \varepsilon$, then $sup_{t,s\in C_i} \parallel f(t) - f(s) \parallel \leq \varepsilon$. Thus take the partition $B_0', C_0, C_1, ..., C_n$ of T that is in accordance with the terms of Definition 2.7.

Remark 3.4. *Every m-totally-measurable function $f : T \to X$ is also m-measurable function.*

Proof

From Definition 3.3, exists a simple functions sequence $\varphi_n : T \to X$ that converges according to measure $|m|^\star$ to function f.
Since $|m| = m^+ + m^-$ from Definition 3.2 we can write $(m^+)^\star + (m^-)^\star \leq |m|^\star$.
So from $lim_{n\to+\infty}|m|^\star\{\parallel \varphi_n(t) - f(t) \parallel \geq \sigma\} = 0$ we conclude that the sequence φ_n converges according to, both measures $(m^+)^\star$ and $(m^-)^\star$, to function f.
The well known Theorem of Riesz ([4], Theorem 9.2) implies that, exists a subsequence φ_{n_k} of φ_n that converges pointwise to f. (This result is true in both spaces (T, m^+) and (T, m^-))

So, the function f is both m^+-measurable and m^--measurable on T. This implies that f is m-measurable on T.
Egoroff's Theorem and the fact that uniformly convergence of a function sequence implies the according to measure convergence, we can conclude that:

If $|m|(T) < +\infty$, then every m-measurable functions sequence that converge to a m-measurable function f converge according to measure $|m|$ to function f also.
Finally:

Remark 3.5. *If $(X, \| \, . \, \|)$ is a normed space and the measure $|m|$ is finite, then every m - measurable function $f : T \to X$ is also m-totally-measurable.*
So, the notions m - measurable function and m-totally-measurable function coincides in conditions of Remark 3.5.
Take the function $f : [0, 1] \to [0, 1]$ such that f(x) =1 in $A \subset C, A \notin \mathcal{B}(\mathbb{R})$ where C is Cantor set and $\mathcal{B}(\mathbb{R})$ is collections of Borel set in $\mathbb{R}$, and f(x) = 2 in $[0, 1] \setminus A$. It is not Borel measurable, because $f^{-1}(-\infty, 2) = A \notin \mathcal{B}(R)$, but the function f is Borel totally-measurable, because we can find a partition $\{C, [0, 1] \setminus C\}$ of [0,1] such that $|m|(C) = \lambda(C) = 0 < \varepsilon$, where λ is denoted the Lebesgue measure in $\mathbb{R}$. Furthermore, for every $t, s \in [0, 1] \setminus C$ we have $f(t) = f(s) = 2$ thus $|f(t) - f(s)| = 0 < \varepsilon$ that imply $sup_{t,s\in[0,1]\setminus C}|f(t) - f(s)| = 0 < \varepsilon$.
So, we conclude that the set of m-measurable functions is a subset of m-totally-mesurable functions set.

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