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STABILITY ANALYSIS FOR A RICKER MODEL WITH A CONTROL TERM

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Abstract. In this paper, we consider a class of Ricker models and study its stability, including the controllable stability and the controllable periodicity. Via analysis methods, we obtain a couple of criteria to guarantee our model is controllably stable and controllably periodic. Two examples are also provided to demonstrated our results.

To the best of our knowledge, this is the first time to discuss the controllable stability for a Ricker model.

1. Introduction

This paper is devoted to consider the stability for a population model given by

$$x(t+1) = \gamma(t)x(t) + u(t)x(t-\tau)e^{r(t)-x(t-\tau)},\ t \in \mathbb{Z}_0, \tag{1.1}$$

where $\mathbb{Z}_0$ is the set of nonnegative integers, $\gamma(t) \in (0,1)$ and $r(t)$ denotes, respectively, the survival rate (see [5]) and the intrinsic growth (see [8]) for each $t \in \mathbb{Z}_0$, $\tau \in \mathbb{Z}_0$ is the time delay, and $u \in l^\infty$ expresses a control term, here l^∞ stands for the set of real sequences $\{w(t)\}_{t\geq 0}$ with $\sup_{0\leq t<\infty} |w(t)| < \infty$.

The origination of our considerations stems from the papers [5, 7, 9, 12] and their references. Precisely speaking, in [5, 12] the authors studied, respectively, the periodic problem of population models

$$x(t+1) = \gamma(t)x(t) + f(t, x(t))$$

and

$$x(t+1) = \gamma(t)x(t) + f(t, x(\sigma(t))).$$

In [9] Perván et al. considered the global convergence for the discrete Ricker delay model

$$x(t+1) = x(t)e^{r-x(t-1)},$$

and Liz [7] discussed the qualitative behavior of Ricker equation

$$x(t+1) = x(t)e^{r(t)-x(t)}.$$

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We observe that the studies are scarce for the Ricker model (1.1) with control term u. This motivates us to consider the present paper. We shall see later the extra control term can be used to dominate the status of (1.1).

On the other hand, for stability problems of continuous or discrete models, the practical stability (or finite-time stability) has garnered much attentions from the researchers, see, for example, [1, 4, 10, 11] and the references therein. Roughly speaking, a model is said to be practically stable if the model's states do not exceed a predesigned bound in a finite (or, an infinite) time interval. This type of stability is different from the Lyapunov stability [10] and more suitable for the practical purposes [11]. Inspired by the practical stability, we introduce a new stability to (1.1), and will call it controllable stability in the sequel.

Our main objective in this paper is twofold. The first goal is to consider whether we can find a constrained condition (of u) such that the states $x(t)$ of (1.1) will have been staying in a prescribed bound during a desired time interval. Our second task is to consider whether we can seek out a control condition such that for a given integer $\eta > 0$, the model (1.1) is asymptotically η-periodic.

For the purposes above, we assume that
(H1) there exist $\gamma_1, \gamma_2 \in (0, 1)$ and $r_1, r_2 \in (0, \infty)$ such that

$$\inf_{t\in\mathbb{Z}_+} \gamma(t) = \gamma_1, \ \sup_{t\in\mathbb{Z}_+} \gamma(t) = \gamma_2 \text{ and } \inf_{t\in\mathbb{Z}_+} r(t) = r_1, \ \sup_{t\in\mathbb{Z}_+} r(t) = r_2;$$

(H2) the functions $f(t, \omega)$ and $F(\omega)$ are defined, respectively, by

$$f(t, \omega) = \omega e^{r(t)-\omega} \text{ and } F(\omega) = \omega e^{r_2-\omega} \text{ for } \omega \geq 0 \text{ and } t \in \mathbb{Z}_0;$$

(H3) for $w = \{w(t)\}_{t\geq 0} \in l^\infty$, we denote the norm by $||w|| = \sup_{0\leq t<\infty} |w(t)|$. Specially, $||w||_T = \sup_{T\leq t<\infty} |w(t)|$ for $w = \{w(T),\ w(T + 1),\ w(T + 2), \ldots\}$;
(H4) for a given integer $\eta > 0$ and two real numbers $\mu_2 > \mu_1 > 0$, the set $l^\infty_\eta[\mu_1, \mu_2] \subset l^\infty$ is defined by

$$l^\infty_\eta[\mu_1, \mu_2] = \{w := \{w(t)\}_{t\geq 0} \in l^\infty : \mu_1 \leq w(t) \leq \mu_2 \text{ and } w \text{ is } \eta\text{-periodic}\}.$$

Note that, (1.1) defines a set of real sequences. Indeed, given a $u \in l^\infty$ and initial values $x(-\tau) = x_{-\tau}, x(-\tau + 1) = x_{-\tau+1}, \ldots, x(0) = x_0$, we may calculate

$$\begin{aligned} x(1) &= \gamma(0)x(0) + u(0)x(-\tau)e^{r(0)-x(-\tau)}, \\ x(2) &= \gamma(1)x(1) + u(1)x(-\tau + 1)e^{r(1)-x(-\tau+1)}, \\ \cdots & \quad \cdots\cdots \end{aligned}$$

in a unique manner, and this sequence $x(t) := \{x(t)\}_{t\geq-\tau}$ is called a solution of (1.1).

Let $\mathcal{D}([-\tau, 0], \mathcal{S})$ denote the set of the functions $\varphi : \{-\tau, -\tau+1, \ldots, 0\} \to \mathcal{S} \subset \mathbb{R}$ with the norm $|\varphi| = \max_{-\tau\leq t\leq 0} |\varphi(t)|$. In general, we denote $\mathcal{D}([-\tau, 0], \mathbb{R})$ by $\mathcal{D}[-\tau, 0]$. In what follows, by $x(t; \varphi, u)$ we denote the solution of (1.1), with the control $u \in l^\infty$ and the initial conditions $x(-\tau) = \varphi(-\tau), x(-\tau + 1) = \varphi(-\tau + 1), \ldots, x(0) = \varphi(0)$ for $\varphi \in \mathcal{D}[-\tau, 0]$. If the context is clear, we will replace $x(t; \varphi, u)$

by $x(t)$. Then, the solution $x(t;\varphi,u)$ of (1.1) satisfies that

$$x(t;\varphi,u)=\varphi(0)\prod_{k=0}^{t-1}\gamma(k)+\sum_{k=0}^{t-1}\left[\prod_{s=k}^{t-2}\gamma(s+1)\right]f(k,x(k-\tau))u(k),\ t=0,1,2,\dots, \tag{1.2}$$

here the function f is defined as in (H2).

We remark that the solution $x(t;\varphi,u)$ of (1.1) is positive when both the initial value function φ and the control u are positive. Since (1.1) describes a type of population models, in the sequel we only focus on the positive initial values φ and the positive controls u.

Definition 1. *We say (1.1) to be controllably stable with respect to* $\{[\beta_1,\beta_2];[\mu_1,\mu_2]\}$ *if for given* $\beta_2>\beta_1>0$, *there exist* $\mu_2>\mu_1>0$ *such that any control* $u\in l^\infty$ *with* $\mu_1\le u(t)\le\mu_2$ *for* $t\in[0,\infty)$, *and any initial value* $\varphi\in\mathcal{D}([-\tau,0],[\beta_1,\beta_2])$, *imply that*

$$\beta_1\le x(t;\varphi,u)\le\beta_2 \ \ \textit{for all}\ \ t\in\mathbb{Z}_0.$$

Definition 2. *We say (1.1) to be controllably permanent with respect to* $\{T,\beta;\alpha,\mu\}$, *if for given* $T>\tau$ *and* $\beta>0$, *there exist two real numbers* $\alpha>0$ *and* $\mu>0$ *such that*

$$|x(t;\varphi,u)|\le\beta \ \textit{for}\ \ \varphi\in\mathcal{D}([-\tau,0],(0,\alpha]),||u||_T\le\mu\ \ \textit{and}\ \ t\ge T.$$

Definition 3. *We say (1.1) to be controllably periodic with respect to* $\{\eta,[\beta_1,\beta_2];l_\eta^\infty[\mu_1,\mu_2]\}$, *if for given integer* $\eta>0$ *and two real numbers* $\beta_2>\beta_1>0$, *there exists an* η-*periodic set* $l_\eta^\infty[\mu_1,\mu_2]\subset l^\infty$ *such that for each* $u\in l_\eta^\infty[\mu_1,\mu_2]$,

$$x(t;\varphi,u)\ \textit{is asymptotically}\ \eta\textit{-periodic for any}\ \varphi\in\mathcal{D}([-\tau,0],[\beta_1,\beta_2]).$$

Here we remark that our controllable stability is different from the practical stability [10], in where the relation

$$\beta_1\le x(t;\varphi,u)\le\beta_2$$

depends only on the initial value φ and the time t. We remark further that we have not found some references involving the controllable periodicity.

2. Controllable Stability

This section is devoted to consider the stability of (1.1), including the controllable stability and the controllable permanence.

Theorem 2.1. *Suppose that*

$$\beta_1e^{r_1-\beta_1}\le\beta_2e^{r_1-\beta_2}\ \ \textit{for}\ \ 0<\beta_1<\beta_2. \tag{2.1}$$

Then, under the assumptions (H1)–(H3), (1.1) is controllably stable with respect to $\{[\beta_1,\beta_2];[\mu_1,\mu_2]\}$, *here*

$$\mu_1=(1-\gamma_1)e^{\beta_1-r_1}<\mu_2=\beta_2(1-\gamma_2)e^{1-r_2}.$$

Proof. It follows from (1.2) that

$$x(t;\varphi,u)\le\gamma_2^t|\varphi(0)|+\sum_{k=0}^{t-1}\gamma_2^{t-1-k}|F(k,x(k-\tau))u(k)|,\ t\in\mathbb{Z}_0, \tag{2.2}$$

where F as in assumption (H2). Note that $F(\omega) \le e^{r_2-1}$ for $\omega \ge 0$. Hence, from (2.2) we obtain

$$\begin{aligned} x(t;\varphi,u) &\le \gamma_2^t|\varphi(0)| + e^{r_2-1}||u||\sum_{k=0}^{t-1}\gamma_2^{t-1-k} \\ &\le \gamma_2^t||\varphi|| + e^{r_2-1}||u||\frac{1-\gamma_2^t}{1-\gamma_2} \\ &\le \beta_2 \ \text{ for } \ t \in \mathbb{Z}_0 \ \text{ and } \ \varphi \in \mathcal{D}([-\tau,0],(0,\beta_2]), \end{aligned} \tag{2.3}$$

where we have used $||u|| \le \mu_2$ for the last step.

Similarly, by the assumption $\inf_{t\ge 0} u(t) \ge \mu_1 = (1-\gamma_1)e^{\beta_1-r_1}$, together with the character of the function $g(w) = we^{r_1-w}$ and the condition (2.1), we have

$$\begin{aligned} x(t;\varphi,u) &\ge \gamma_1^t\varphi(0) + (1-\gamma_1)e^{\beta_1-r_1}\sum_{k=0}^{t-1}\gamma_1^{t-k-1}f(k,x(k-\tau)) \\ &\ge \gamma_1^t\beta_1 + (1-\gamma_1)e^{\beta_1-r_1}\frac{1-\gamma_1^t}{1-\gamma_1}\beta_1e^{r_1-\beta_1} \\ &= \beta_1,\ t = 1,2...,\tau+1 \ \text{ and } \ \varphi \in \mathcal{D}([-\tau,0],[\beta_1,\beta_2]), \end{aligned}$$

where f is from our assumption (H2). In general, by the mathematical induction, we can infer that

$$x(t;\varphi,u) \ge \beta_1 \ \text{ for all } \ t \ge 0 \ \text{ and } \ \varphi \in \mathcal{D}([-\tau,0],[\beta_1,\beta_2]).$$

The proof is complete. □

The following is concerned with the controllable permanence of (1.1).

Theorem 2.2. *Under the assumptions (H1)–(H3), (1.1) is controllably permanent with respect to* $\{T,\beta;\alpha,\mu\}$, *here*

$$\mu < \beta(1-\gamma_2)e^{1-r_2} \quad \text{and} \quad \alpha = \left(\beta - \frac{e^{r_2-1}\mu}{1-\gamma_2}\right)\gamma_2^{-T}.$$

Proof. Similar to (2.3) in the proof of Theorem 2.1, we have

$$\begin{aligned} |x(t;\varphi,u)| &\le \gamma_2^t|\varphi(0)| + e^{r_2-1}||u||_T\sum_{k=0}^{t-1}\gamma_2^{t-1-k} \\ &\le \gamma_2^T||\varphi|| + \frac{e^{r_2-1}||u||_T}{1-\gamma_2},\ t \ge T, \end{aligned}$$

which yields

$$|x(t;\varphi,u)| \le \beta \ \text{ for } \ \varphi \in \mathcal{D}([-\tau,0],(0,\alpha]),\ ||u||_T \le \mu \ \text{ and } \ t \ge T.$$

The proof is complete. □

We now give an example to demonstrate the results above.

Example 1. *Let*

$$\gamma(t) = 0.225 + 0.025\sin\frac{\pi t}{4},\ r(t) = 1.1 + 0.1\sin t.$$

Then, with the symbols in (H1) we have

$$\gamma_1 = 0.2,\ \gamma_2 = 0.25,\ r_1 = 1 \ \text{ and } r_2 = 1.2.$$

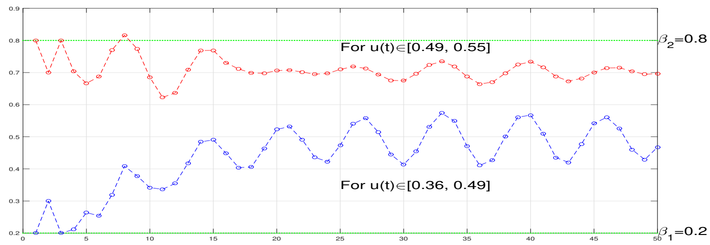
Now consider

$$x(t+1) = \gamma(t)x(t) + u(t)x(t-2)e^{r(t)-x(t-2)},\ t \in \mathbb{Z}_0. \tag{2.4}$$

Then, Theorem 2.1 implies that (2.4) is controllably stable with respect to $([0.2, 0.8]; [0.36, 0.49])$. *That is, the solution* $x(t;\varphi,u)$ *of (2.4) satisfies*

$$0.2 \le x(t;\varphi,u) \le 0.8$$

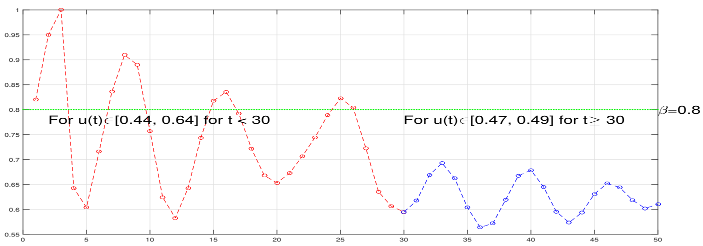
as $\varphi \in \mathcal{D}([-2,0],[0.2,0.8])$ *and* $0.36 \le u(t) \le 0.49$ *for* $t \ge 0$. *See the following graph:*



On the other hand, Theorem 2.2 implies that (2.4) is controllably permanent with respective to $\{30, 0.8; 10^{15}, 0.49\}$. *That is, the solution* $x(t;\varphi,u)$ *of (2.4) satisfies*

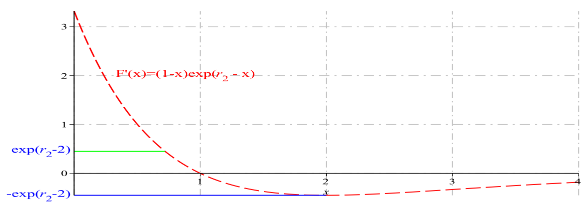
$$|x(t;\varphi,u)| \le 0.8 \quad \textit{as} \quad \varphi \in \mathcal{D}([-2,0],(0,10^{15}]) \quad \textit{and} \quad ||u||_{30} \le 0.49 \quad \textit{for} \quad t \ge 30.$$

To see this, we show the first 50*-terms of* $x(t;\varphi,u)$ *for* $\varphi(-2) = 0.82$, $\varphi(-1) = 0.95$ *and* $\varphi(0) = 1$ *as follows:*



3. Controllable Periodicity

This section is concerned with the periodicity of (1.1). We first note that the character of the function $F'(\omega)$ for $F(\omega) = \omega e^{r_2-\omega}$ (see assumption (H2)):



Then, it is easy to see that when $\delta \in (0,1)$ with $\delta + e^{\delta-2} \le 1$, $|F'(\omega)| \le (1-\delta)e^{r_2-\delta}$ for $\omega \in [\delta, \infty)$.

Lemma 3.1. *Suppose that* $\delta \in (0,1)$ *with* $\delta + e^{\delta-2} \le 1$*, and* $||u|| < \frac{(1-\gamma_2)e^{\delta-r_2}}{1-\delta}$*. Then, any two solutions* $x(t)$ *and* $y(t)$ *of (1), with* $x(t) \ge \delta$ *and* $y(t) \ge \delta$ *for all* $t \ge -\tau$*, satisfy that*

$$\lim_{t\to\infty} |x(t) - y(t)| = 0. \tag{3.1}$$

Proof. We proceed in two steps.

(i) Assertion 1: each solution $x(t;\varphi,u)$ of (1.1) is bounded.

To see this, we note that for a given $\varphi \in D([-\tau,0],(0,\infty))$ and a positive control $u \in l^\infty$, there exists a $\lambda \ge 1$ such that

$$||\varphi|| \le \lambda \frac{||u||e^{r_2-1}}{1-\gamma_2}. \tag{3.2}$$

Then, from (1.2) it follows that

$$\begin{aligned} |x(t;\varphi,u)| &\le \gamma_2^t|\varphi(0)| + ||u|| \sum_{k=0}^{t-1} \left[\prod_{s=k}^{t-2} \gamma(s+1)\right] |f(k,x(k-\tau))| \\ &\le \gamma_2^t||\varphi|| + ||u|| \sum_{k=0}^{t-1} \gamma_2^{t-k-1}|F(x(k-\tau))|, \ t \in \mathbb{Z}_0. \end{aligned} \tag{3.3}$$

Likewise, by $|F(\omega)| \le e^{r_2-1}$ for $\omega \ge 0$, it follows from (3.2)–(3.3) that

$$|x(t;\varphi,u)| \le \gamma_2^t \cdot \lambda \frac{||u||e^{r_2-1}}{1-\gamma_2} + ||u|| \frac{1-\gamma_2^t}{1-\gamma_2} e^{r_2-1} \le \lambda \frac{||u||e^{r_2-1}}{1-\gamma_2},$$

and this shows the solution $x(t;\varphi,u)$ of (1.1) is bounded.

(ii) Assertion 2: under the hypotheses in Lemma 3.1, the conclusion (3.1) is true.

Indeed, by assertion 1, we can assume that

$$\limsup_{t\to\infty} |x(t) - y(t)| = w.$$

In other words, for any $\varepsilon > 0$, there exists an integer $N > 0$ such that

$$|x(t) - y(t)| \le w + \varepsilon \quad \text{for} \quad t \ge N. \tag{3.4}$$

Now, from (1.2) we have

$$\begin{aligned} |x(t) - y(t)| &\le \gamma_2^{t-N}|x(N) - y(N)| + ||u|| \sum_{k=N}^{t-1} \gamma_2^{t-1-k}|x(k)e^{r(k)-x(k)} - y(k)e^{r(k)-y(k)}| \\ &= \gamma_2^{t-N}|x(N) - y(N)| + ||u|| \sum_{k=N}^{t-1} \gamma_2^{t-1-k}|(1-\xi(k))e^{r(k)-\xi(k)}(x(k) - y(k))| \\ &\le \gamma_2^{t-N}|x(N) - y(N)| + ||u|| \sum_{k=N}^{t-1} \gamma_2^{t-1-k}|F'(\xi(k))(x(k) - y(k))|, \ t \ge N, \end{aligned} \tag{3.5}$$

where $\xi(k)$ is between $x(k)$ and $y(k)$. Consequently, $|F'(\xi(k))| \le (1-\delta)e^{r_2-\delta}$ (see the diagram $F'(\omega)$ above). Therefore, in the light of (3.4)–(3.5), it holds that

$$|x(t) - y(t)| \le \gamma_2^{t-N}|x(N) - y(N)| + ||u|| \frac{1-\gamma_2^{t-N}}{1-\gamma_2}(1-\delta)e^{r_2-\delta}(w+\varepsilon), \ t \ge N,$$

which results in

$$w \leq \frac{||u||(1-\delta)e^{r_2-\delta}}{1-\gamma_2}(w+\varepsilon),$$

and this, coupled with the hypothesis $||u|| < \frac{(1-\gamma_2)e^{\delta-r_2}}{1-\delta}$, yields $w=0$. The proof is complete. □

Lemma 3.2. [6, Krasnoselskii's Fixed Point Theorem] *Let $\mathcal{S}$ be a closed, convex, and bounded subset of a Banach space $\mathcal{X}$ and $T, U : \mathcal{S} \to \mathcal{X}$ be two operators such that*
(i) T is a contraction;
(ii) U is completely continuous, and
(iii) $Tx + Uy \in \mathcal{S}$ for all $x, y \in \mathcal{S}$.
Then $T+U$ has a fixed point in $\mathcal{S}$.

Theorem 3.3. *Suppose that $\gamma(t)$ and $r(t)$ are periodic with a common σ-period. Suppose further that $\eta > 0$ is an integer with $\sigma|\eta$ (η divides exactly by σ), and*

$$\beta_1 e^{r_1-\beta_1} \leq \beta_2 e^{r_1-\beta_2} \ \textit{for}\ 0<\beta_1<\beta_2 \ \textit{with}\ \beta_1 \in (0,1) \ \ \textit{and}\ \beta_1 + e^{\beta_1-2} \leq 1,$$

Then, the following is true:

(i) (1.1) is controllably periodic with respect to $\{\eta, [\beta_1,\beta_2]; l_\eta^\infty[\mu_1,\mu_2]\}$, here

$$\mu_1 = (1-\gamma_1)e^{\beta_1-r_1} \leq \mu_2 < \min\left\{\beta_2(1-\gamma_2)e^{1-r_2}, \frac{(1-\gamma_2)e^{\beta_1-r_2}}{1-\beta_1}\right\}; \tag{3.6}$$

(ii) for each $u \in l_\eta^\infty[\mu_1,\mu_2]$, (1.1) has a unique η-periodic solution $p(t)$ satisfying

$$\beta_1 \leq p(t) \leq \beta_2 \ \ \textit{for all}\ \ t \geq -\tau.$$

Proof. We proceed in several steps.

(i) Assertion 1: for each $u \in l_\eta^\infty[\mu_1,\mu_2]$, (1.1) admits an asymptotically η-periodic solution.

To see this, we take

$$l_{ap}^\infty(\eta) = \{\{x(t)\}_{t\geq-\tau} \subset \mathbb{R} : x \text{ is asymptotically } \eta\text{-periodic}\}.$$

Then, $l_{ap}^\infty(\eta)$ is a Banach space with the norm $||x|| = \sup_{-\tau\leq t<\infty}|x|$. Let

$$\mathcal{S}_{ap} = \left\{\{x(t)\}_{t\geq-\tau} \in l_{ap}^\infty(\eta) : \beta_1 \leq x(t) \leq \beta_2 \ \text{ for all } t \geq -\tau\right\}.$$

Then, $\mathcal{S}_{ap}$ is bounded, convex and closed.

Now, for a fixed $u \in l_\eta^\infty[\mu_1,\mu_2]$, we define two operators $T, U : \mathcal{S}_{ap} \to l_{ap}^\infty(\eta)$ as follows:

$$(Tx)(t) = \begin{cases} 0, \ t=-\tau, \ -\tau+1, \ldots, 0, \\ \gamma(t-1)x(t-1) + f(t-1, x(t-1-\tau))u(t-1), \ t \geq 1 \end{cases}$$

and

$$(Ux)(t) = \begin{cases} x(t), & t=-\tau, \ -\tau+1, \ldots, 0, \\ 0, & t \geq 1, \end{cases}$$

where $f(t-1,\omega) = \omega e^{r(t-1)-\omega}$ as in the assumption (H2). Then, by the straightforward verification, we have for any $x, y \in \mathcal{S}_{ap}$,

$$\beta_1 \leq (Tx)(t) + (Uy)(t) \leq \beta_2 \ \text{ for } \ t \geq 1,$$

and consequently,

$$Tx + Uy \in \mathcal{S}_{ap} \ \text{ for all } \ x, y \in \mathcal{S}_{ap}.$$

Furthermore, since $U\mathcal{S}_{ap}$ is bounded and $U\mathcal{S}_{ap} \subset \mathbb{R}^{\tau+1}$, it becomes a relatively compact set. Hence, U is completely continuous.

To invoke Lemma 3.2, we show that T is contractive. Indeed, by (3.6) we can choose an $\varepsilon_0 \in (0,1)$ such that

$$\mu_2(1-\beta_1)e^{r_2-\beta_1} \leq 1-\gamma_2-\varepsilon_0. \tag{3.7}$$

Then, for any $x, y \in \mathcal{S}_{ap}$, it follows that

$$\begin{aligned}(Tx)(t)-(Ty)(t) \;&=\; \gamma(t-1)(x(t-1)-y(t-1))+\\ &\quad\; u(t-1)(f(t-1,x(t-1-\tau))-f(t-1,y(t-1-\tau))),\;\; t\geq 1,\end{aligned}$$

which, with the help of

$$\begin{aligned}&|f(t-1,x(t-1-\tau))-f(t-1,y(t-1-\tau))|\\ \leq\;& |F'(\xi((t-1-\tau)))|\cdot|x(t-1-\tau)-y(t-1-\tau)|\\ \leq\;& (1-\beta_1)e^{r_2-\beta_1}|x(t-1-\tau)-y(t-1-\tau)|,\end{aligned}$$

amounts to

$$\begin{aligned}|(Tx)(t)-(Ty)(t)| \;&=\; \gamma_2||x-y||+||u||(1-\beta_1)e^{r_2-\beta_1}|x(t-1-\tau)-y(t-1-\tau)|\\ &\leq\; \gamma_2||x-y||+\mu_2(1-\beta_1)e^{r_2-\beta_1}||x-y||\\ &\leq\; (1-\varepsilon_0)||x-y||,\;\; t\geq 1,\end{aligned}$$

where we have imposed $||u|| \leq \mu_2$ for the second step, and (3.7) for the last step.

Now we see that T verifies the contractive condition. Thus, Lemma 3.2 implies that there exists $\widetilde{x} \in \mathcal{S}_{ap}$ such that

$$\widetilde{x}(t) = (T\widetilde{x})(t) + (U\widetilde{x})(t),\;\; t \geq -\tau,$$

and this yields

$$\widetilde{x}(t+1) = \gamma(t)\widetilde{x}(t) + u(t)\widetilde{x}(t)e^{r(t)-\widetilde{x}(t)} \quad \text{for} \quad t \geq 0. \tag{3.8}$$

That is, (1.1) has an asymptotically η-periodic solution.

(ii) Assertion 2: for each $u \in l^\infty_\eta[\mu_1,\mu_2]$, the solution $x(t;\varphi,u)$ of (1.1) is asymptotically η-periodic for any $\varphi \in \mathcal{D}([-\tau,0],[\beta_1,\beta_2])$.

In reality, for each $u \in l^\infty_\eta[\mu_1,\mu_2]$, by assertion 1 we can set $\widetilde{x}(t)$ is an asymptotically η-periodic solution of (1.1) and let

$$\widetilde{x}(t) = p(t) + q(t), \tag{3.9}$$

where $p(t)$ is η-periodic and $\lim_{t\to\infty} q(t) = 0$. In addition, by Theorem 2.1 we have $\beta_1 \leq x(t;\varphi,u) \leq \beta_2$ for any $\varphi \in \mathcal{D}([-\tau,0],[\beta_1,\beta_2])$. Now by Lemma 3.1 we have

$$\lim_{t\to\infty} |x(t;\varphi,u) - \widetilde{x}(t)| = 0,$$

which means that

$$x(t;\varphi,u) = p(t) + c(t),$$

where $\lim_{t\to\infty} c(t) = 0$. Hence, $x(t;\varphi,u)$ is asymptotically η-periodic.

As thus, we have shown that (1.1) is controllably periodic with respect to $\{\eta, [\beta_1,\beta_2]; l^\infty_\eta[\mu_1,\mu_2]\}$.

(iii) Assertion 3: for each $u \in l^\infty_\eta[\mu_1,\mu_2]$, the function $p(t)$ as in (3.9) is a unique η-periodic solution of (1.1), and satisfies that $\beta_1 \leq p(t) \leq \beta_2$ for all $t \geq -\tau$.

As a matter of fact, from (3.8)–(3.9) it follows that

$$p(t+1) \to \gamma(t)p(t) + u(t)p(t-\tau)e^{r(t)-p(t-\tau)} \quad \text{as} \quad t \to \infty.$$

Since $\gamma(t)p(t)+u(t)p(t-\tau)e^{r(t)-p(t-\tau)}$ is η-periodic, the relation above implies that

$$p(t+1) = \gamma(t)p(t) + u(t)p(t-\tau)e^{r(t)-p(t-\tau)} \quad \text{for} \quad t \geq 0.$$

That is, $p(t)$ is an η-periodic solution of (1.1).

Besides, by (3.9) we have

$$\beta_1 \leq p(t) + q(t) \leq \beta_2 \quad \text{for} \quad t \geq -\tau. \tag{3.10}$$

Since $p(t)$ is periodic and $q(t) \to 0$ as $t \to \infty$, by (3.10) it follows that

$$\beta_1 \leq p(t) \leq \beta_2 \quad \text{for} \quad t \geq -\tau.$$

The uniqueness of $p(t)$ is due to Lemma 3.1. The proof is complete. □

We finally remark that, under hypotheses of Theorem 3.3, (1.1) is globally asymptotically η-periodic, since we can choose $\beta_1 \to 0$ and $\beta_2 \to \infty$ such that

$$\beta_1 e^{r_1-\beta_1} \leq \beta_2 e^{r_1-\beta_2} \quad \text{for} \quad 0 < \beta_1 < \beta_2.$$

Example 2. *Let*

$$\gamma(t) = 0.225 + 0.025 \sin\frac{\pi t}{2}, \;\; r(t) = 1.1 + 0.1 \sin\frac{\pi t}{2}.$$

Then, both $\gamma(t)$ and $r(t)$ are 4-periodic, and

$$\gamma_1 = 0.2, \;\; \gamma_2 = 0.25, \;\; r_1 = 1 \quad \text{and } r_2 = 1.2.$$

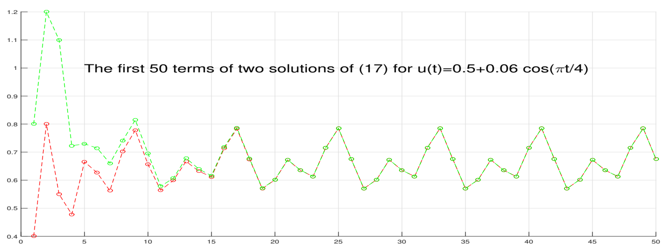
Consider

$$x(t+1) = \gamma(t)x(t) + u(t)x(t-2)e^{r(t)-x(t-2)}, \;\; t \in \mathbb{Z}_0 \tag{3.11}$$

and take $\beta_1 = 0.4$ and $\beta_2 = 2$. Then,

$$\min\left\{\beta_2(1-\gamma_2)e^{1-r_2}, \frac{(1-\gamma_2)e^{\beta_1-r_2}}{1-\beta_1}\right\} = 0.5616.$$

Now, for any positive integer η with $4|\eta$, Theorem 3.3 implies that (3.11) is controllably periodic with respect to $\{\eta, [0.4, 2]; l_\eta^\infty[0.44, 0.56]\}$. Furthermore, (3.11) has a unique η-periodic solution for each $u \in l_\eta^\infty[0.44, 0.56]$. To confirm our belief, we choose an 8-periodic control function $u(t) = 0.5 + 0.06\cos\frac{\pi t}{4} \in l_8^\infty[0.44, 0.56]$ and show the first 50 terms of two solutions $x(t; \varphi, u)$ of (3.11) as follows:



4. An Open Problem

In this paper, we employ the general analysis methods to study the stability for a class of population models. It is worth mentioning that there are other methods for the stability analysis, see, for example, [2, 3, 13, 14]. To be specific, Burton et al. contributed their pioneering works[2, 3] in where the authors used the fixed point theorems to analyze the stability for differential equations, while Cheng, Wang and the present author [13, 14] studied the similar problems by making use of the frequency analysis. A problem now emerges that whether we can invoke the method of fixed point theorems (or frequency analysis) to consider the present model. This is an interesting direction, and we leave it for our future endeavor.

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