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SOFT ω-CLOSED SET IN SOFT $\mathcal{N}$-TOPOLOGICAL SPACE

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Abstract. In this article, we extend the notion of soft topological space to soft $\mathcal{N}$-topological space using a real-world example. Furthermore, in soft $\mathcal{N}$-topological spaces, $\mathcal{N} \geq 2$, we define soft ω-closed set, soft ω-open set, soft ω-closure and soft ω-interior. In addition, we examine the properties of these soft sets using relevant instances.

1. Introduction

There were various theories like "fuzzy set theory"[13], "theory of interval mathematics" [12] etc. to deal with uncertain problems. But there were some limitations to all these theories. Molodtsov [6] used an adequate parameterization to avoid these limitations. He presented the first result of the 'soft set theory' in 1999. Many authors were fascinated by this theory. In 2003, Maji et al. [5] gave "application of soft sets in a decision making problem" .

Shabir and Naz [9] defined "soft topological spaces". This concept attracted researchers to work on it. Then, "properties of soft topological spaces" were studied by Hussain and Ahmad [2]. A new concept of closed set was introduced by Sundaram and John [10]. They gave the idea of "ω-closed set in topology". This idea motivated Paul [7] to define "soft ω-closed sets in soft topological spaces" in which "properties of soft semi-open sets and soft semi-closed sets" were used which was discussed by Chen [1]. Kelly [4] defined bitopological spaces and corresponding to his idea, Ittanagi [3] introduced "soft bitopological spaces". In 2017, Thivagar [11] defined a "new structure of N-topology". Inspiring by all these ideas, we extend "soft bitopological spaces" to "soft $\mathcal{N}$-topological spaces" and define "soft ω-closed set" and "soft ω-open set" in the newly defined spaces and discuss their characteristics.

2. Preliminaries

This section contains some important definitions which will help in our main results. Throughout this article, we will refer to I_U as the universal set, Δ as the parameter set and $(I_U, \tilde{\tau}, \Delta)$ as the soft topological space.

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Definition 2.1. [9] "Suppose I_U be an initial universal set, Δ be a parameter set and $\mathcal{P}(I_U)$ denote the power set of I_U. A pair (M_Δ, Δ) is called a soft set over I_U, if M_Δ is a mapping given by $M_\Delta : \Delta \to \mathcal{P}(I_U)$."

Definition 2.2. [6] "Let (M_Δ, Δ) be a soft set over I_U, then (M_Δ, Δ) is
(1) null soft set denoted by $\tilde{\phi}$ if for all $\delta_1 \in \Delta$, $M_\Delta(\delta_1) = \phi$.
(2) absolute soft set denoted by $\tilde{I_U}$ if for all $\delta_1 \in \Delta, M_\Delta(\delta_1) = I_U$."

Definition 2.3. [9] "The union of two soft sets (M_{Δ_a}, Δ_a) and (N_{Δ_b}, Δ_b) over the common universe I_U is the soft set (H_{Δ_c}, Δ_c), where $\Delta_c = \Delta_a \cup \Delta_b$ and for all $\delta \in \Delta_c$,

$$H_{\Delta_c}(\delta) = \begin{cases} M_{\Delta_a}(\delta) & \text{if} \quad \delta \in \Delta_a - \Delta_b \\ N_{\Delta_b}(\delta) & \text{if} \quad \delta \in \Delta_b - \Delta_a \\ M_{\Delta_a}(\delta) \cup N_{\Delta_b}(\delta) & \text{if} \quad \delta \in \Delta_a \cap \Delta_b \end{cases}$$

We write $(M_{\Delta_a}, \Delta_a)\tilde{\cup}(N_{\Delta_b}, \Delta_b) = (H_{\Delta_c}, \Delta_c)$."

Definition 2.4. [9] "The intersection (H_{Δ_c}, Δ_c) of two soft sets (M_{Δ_a}, Δ_a) and (N_{Δ_b}, Δ_b) over a common universe I_U, denoted by $(M_{\Delta_a}, \Delta_a) \tilde{\cap} (N_{\Delta_b}, \Delta_b)$, is defined as $\Delta_c = \Delta_a \cap \Delta_b$ and $H_{\Delta_c}(\delta) = M_{\Delta_a}(\delta) \cap N_{\Delta_b}(\delta)$ for all $\delta \in \Delta_c$."

Definition 2.5. [9] "The relative complement of a soft set (M_Δ, Δ) denoted by $(M_\Delta, \Delta)^c$ and is defined by $(M_\Delta, \Delta)^c = (M^c_\Delta, \Delta)$ where $M^c_\Delta : \Delta \to \mathcal{P}(I_U)$ is a mapping defined by $M^c_\Delta(\delta) = I_U - M_\Delta(\delta)$ for all $\delta \in \Delta$."

Definition 2.6. [9] "Let I_U be an initial universal set, Δ be the non-empty set of parameters and $\tilde{\tau}$ be the collection of soft sets over I_U, then $\tilde{\tau}$ is a soft topology on I_U, if
(1) $\tilde{\phi}$, $\tilde{I_U} \tilde{\in} \tilde{\tau}$,
(2) union of any number of soft sets in $\tilde{\tau}$ belongs to $\tilde{\tau}$,
(3) intersection of any two soft sets in $\tilde{\tau}$ belongs to $\tilde{\tau}$.
Then, the triplet $(I_U, \tilde{\tau}, \Delta)$ is called a soft topological space over I_U. The members of $\tilde{\tau}$ are called soft open sets and complements of them are called soft closed sets in I_U."

Definition 2.7. [9](i) "Let $(I_U, \tilde{\tau}, \Delta)$ be a soft topological space and (M_Δ, Δ) be a soft set over I_U, then the soft closure of (M_Δ, Δ), denoted by $\overline{(M_\Delta, \Delta)}$ is defined as the intersection of all soft closed supersets of (M_Δ, Δ)."
(ii) "Let $(I_U, \tilde{\tau}, \Delta)$ be a soft topological space and (M_Δ, Δ) be a soft set over I_U, then the soft interior of (M_Δ, Δ), denoted by $(M_\Delta, \Delta)^\circ$ is defined as the union of all soft open subsets of (M_Δ, Δ)."

Remark. [9] (i) "If $(M_\Delta, \Delta) \tilde{\in} \tilde{\tau}^c$, where $\tilde{\tau}^c$ is the collection of all soft closed sets in a soft topological space, then $\overline{(M_\Delta, \Delta)} = (M_\Delta, \Delta)$."
(ii) "If $(M_\Delta, \Delta) \tilde{\in} \tilde{\tau}$, then $(M_\Delta, \Delta)^\circ = (M_\Delta, \Delta)$."

Definition 2.8. [8] (i) "Let (W^0_Δ, Δ) be a soft set over I_U. Then, (W^0_Δ, Δ) is **soft ω-open set** if for any soft semi-closed set (C_Δ, Δ) contained in (W^0_Δ, Δ), we have $(C_\Delta, \Delta) \tilde{\subseteq} (W^0_\Delta, \Delta)^\circ$. The set of all soft ω-open sets is denoted by $G_{s\omega}(\tilde{I_U})$."
(ii) "Consider a soft set (W_Δ, Δ) over I_U. Then, (W_Δ, Δ) is **soft ω-closed set** if for any soft semi-open set (O_Δ, Δ) containing (W_Δ, Δ), we have $\overline{(W_\Delta, \Delta)} \tilde{\subseteq} (O_\Delta, \Delta)$. The collection of all soft ω-closed sets is denoted by $F_{s\omega}(\tilde{I_U})$."

Proposition 2.9. [8](i) "If $(W^0_\Delta, \Delta) \in \tilde{\tau}$, then $(W^0_\Delta, \Delta) \in G_{s\omega}(\tilde{I_U})$."
(ii) "If $(W_\Delta, \Delta) \tilde{\in} \tilde{\tau}^c$, then $(W_\Delta, \Delta) \tilde{\in} F_{s\omega}(\tilde{I_U})$."

Definition 2.10. [8] (i) "Consider a soft set (W_Δ, Δ) over I_U. Then, the soft ω-interior of (W_Δ, Δ), denoted by $\{(W_\Delta, \Delta)\}^\circ_\omega$, is defined as the union of all soft ω-open subsets of (W_Δ, Δ) i.e., $\{(W_\Delta, \Delta)\}^\circ_\omega = \tilde{\cup}\ \{(F_\Delta, \Delta) : (W_\Delta, \Delta) \tilde{\supseteq} (F_\Delta, \Delta), (W_\Delta, \Delta) \tilde{\in} G_{s\omega}(\tilde{I_U})\}$."
(ii) "Consider a soft set (W_Δ, Δ) over I_U. Then, the soft ω-closure of (W_Δ, Δ), denoted by $\{\overline{(W_\Delta, \Delta)}\}_\omega$, is defined as the intersection of all soft ω-closed supersets of (W_Δ, Δ) i.e.,
$\{\overline{(W_\Delta, \Delta)}\}_\omega = \tilde{\cap}\ \{(F_\Delta, \Delta) : (W_\Delta, \Delta) \tilde{\subseteq} (F_\Delta, \Delta), (W_\Delta, \Delta) \tilde{\in} F_{s\omega}(\tilde{I_U})\}$."

Remark. [8] (i) "$(W_\Delta, \Delta) \tilde{\supseteq} \{(W_\Delta, \Delta)\}^\circ_\omega$."
(ii) "$\{(W_\Delta, \Delta)\}^\circ_\omega \tilde{\in} G_{s\omega}(\tilde{I_U})$."
(iii) "$(W_\Delta, \Delta) \tilde{\subseteq} \{\overline{(W_\Delta, \Delta)}\}_\omega$."
(iv) "$\{\overline{(W_\Delta, \Delta)}\}_\omega \tilde{\in} F_{s\omega}(\tilde{I_U})$."

Lemma 2.11. [8](i) "$\{(W_\Delta, \Delta)\}^\circ_\omega$ is the largest soft ω-open set contained in (W_Δ, Δ)."
(ii) " $(W_\Delta, \Delta) \tilde{\in} G_{s\omega}(\tilde{I_U})$ if and only if $\{(W_\Delta, \Delta)\}^\circ_\omega = (W_\Delta, \Delta)$."
(iii) "$\{\overline{(W_\Delta, \Delta)}\}_\omega$ is the smallest soft ω-open set containing (W_Δ, Δ)."
(iv) " $(W_\Delta, \Delta) \tilde{\in} F_{s\omega}(\tilde{I_U})$ if and only if $\{\overline{(W_\Delta, \Delta)}\}_\omega = (W_\Delta, \Delta)$."

3. Soft $\mathcal{N}$-Topological Space

This section generalize the idea of "soft topological space" by defining soft $\mathcal{N}$-topological space with the help of real life example. We further discuss some properties of the newly defined space. Throughout this section, we use $(I_U, \tilde{\tau}_1, \tilde{\tau}_2, .., \tilde{\tau_\mathcal{N}}, \Delta)$ as soft $\mathcal{N}$-topological space.

Definition 3.1. Consider $\mathcal{N}$-soft topologies over I_U as $\tilde{\tau}_1, \tilde{\tau}_2, .., \tilde{\tau_\mathcal{N}}$, $\mathcal{N} \geq 3$. Then, the space $(I_U, \tilde{\tau}_1, \tilde{\tau}_2, ..., \tilde{\tau_\mathcal{N}}, \Delta)$ is called **soft $\mathcal{N}$-topological space**.

Example 3.1. If $I_U = \{\eta_1, \eta_2, \eta_3\}$, $\Delta = \{\delta_1, \delta_2\}$ and
$\tilde{\tau}_1 = \{\tilde{\phi},\ \tilde{I_U},\ \{(\delta_1, \{\eta_1\}), (\delta_2, \{\eta_2\})\}\}$, $\tilde{\tau}_2 = \{\tilde{\phi},\ \tilde{I_U}\}$, $\tilde{\tau}_3 = \{\tilde{\phi},\ \tilde{I_U},\ \{(\delta_1, \{\eta_2\}), (\delta_2, \{\eta_3\})\}\}$ and $\tilde{\tau}_4 = \{\tilde{\phi},\ \tilde{I_U},\ \{(\delta_1, \{\eta_1, \eta_2\}), (\delta_2, \{\eta_3\})\}\}$ are soft topologies over I_U and the space $(I_U, \tilde{\tau}_1, \tilde{\tau}_2, \tilde{\tau}_3, \tilde{\tau}_4, \Delta)$ is soft quad topological space.

Definition 3.2. The soft set (M_Δ, Δ) is **soft $\tilde{\tau}_1, \tilde{\tau}_2, ..., \tilde{\tau_\mathcal{N}}$-open set** if

$$(M_\Delta, \Delta) = \{ \bigcup_{i=1}^{\mathcal{N}} (M_{i_\Delta}, \Delta)\} \tilde{\cup} \{ \bigcap_{i=1}^{\mathcal{N}} (N_{i_\Delta}, \Delta)\}$$

where $(M_{i_\Delta}, \Delta), (N_{i_\Delta}, \Delta) \tilde{\in} \tilde{\tau}_i$, $i = 1, 2, ..., \mathcal{N}$, $\mathcal{N} \geq 3$.
The complements of soft $\tilde{\tau}_1, \tilde{\tau}_2, .., \tilde{\tau_\mathcal{N}}$-open sets are called **soft $\tilde{\tau}_1, \tilde{\tau}_2, .., \tilde{\tau_\mathcal{N}}$-closed sets**. The collection of soft $\tilde{\tau}_1, \tilde{\tau}_2, .., \tilde{\tau_\mathcal{N}}$-open sets is denoted by $\tilde{\tau}_1, \tilde{\tau}_2, .., \tilde{\tau_\mathcal{N}}$-$\breve{\mathcal{O}}(I_U)$ and the collection of soft $\tilde{\tau}_1, \tilde{\tau}_2, .., \tilde{\tau_\mathcal{N}}$-closed sets is denoted by $\tilde{\tau}_1, \tilde{\tau}_2, .., \tilde{\tau_\mathcal{N}}$-$\breve{\mathcal{C}}(I_U)$.

Example 3.2. In example 3.1, the soft $\tilde{\tau}_1, \tilde{\tau}_2, .., \tilde{\tau_\mathcal{N}}$-open sets are

$$\tilde{\tau}_1, \tilde{\tau}_2, .., \tilde{\tau_\mathcal{N}}\text{-}\breve{\mathcal{O}}(I_U) = \{\tilde{\phi}\ , \tilde{I_U}\ , \{(\delta_1, \{\eta_1\}), (\delta_2, \{\eta_2\})\},\ \{(\delta_1, \{\eta_2\}), (\delta_2, \{\eta_3\})\}, \{(\delta_1, \{\eta_1, \eta_2\}), (\delta_2, \{\eta_3\})\},\ \{(\delta_1, \{\eta_1, \eta_2\}), (\delta_2, \{\eta_2, \eta_3\})\}, \{(\delta_1, \{\eta_1\}), (\delta_2, \phi)\}\}$$

and the collection of all soft $\tilde{\tau_1}, \tilde{\tau_2}, .., \tilde{\tau_\mathcal{N}}$-closed sets are

$\tilde{\tau_1}, \tilde{\tau_2}, ..., \tilde{\tau_\mathcal{N}}$-$\breve{\mathcal{C}}(I_U)$ =$\{\tilde{\phi}$, $\tilde{I_U}$, $\{(\delta_1, \{\eta_2, \eta_3\}), (\delta_2, \{\eta_1, \eta_3\})\}$, $\{(\delta_1, \{\eta_1, \eta_3\}), (\delta_2, \{\eta_1, \eta_2\})\}$,
$\{(\delta_1, \{\eta_3\}), (\delta_2, \{\eta_1, \eta_2\})\}$, $\{(\delta_1, \{\eta_3\}), (\delta_2, \{\eta_1\})\}$, $\{(\delta_1, \{\eta_2, \eta_3\}), (\delta_2, \tilde{I_U})\}\}$.

Example 3.3. In medical diagnosis, we want to know the side affects after a patient is diagnosed with COVID-19.
Let our universal set contains 4 persons who are diagnosed with COVID-19 i.e., $I_U = \{\eta_1, \eta_2, \eta_3, \eta_4\}$.
Suppose we are dealing with 3 symptoms as parameter set, say $\Delta = \{\delta_1, \delta_2, \delta_3\}$, where δ_1, δ_2 and δ_3 show the symptoms fatigueness, joint pain and lung infection respectively.
After the experiment was conducted, the experiment tested for four time periods, the first period is after 10 days, the second period is after 20 days, the third period is after 30 days and the fourth period is after 40 days. The four time periods represent the four soft topological spaces $\tilde{\tau_1}, \tilde{\tau_2}, \tilde{\tau_3}$ and $\tilde{\tau_4}$.
When testing the experiment after 10 days, we obtain the soft open set (M_Δ, Δ) over I_U such that $M_\Delta(\delta_1) = I_U$, $M_\Delta(\delta_2) = \{\eta_1, \eta_2, \eta_4\}$, $M_\Delta(\delta_3) = \{\eta_2, \eta_3, \eta_4\}$, i.e., fatigueness is among all the 4 patients who are diagnosed with COVID-19, joint pain occurs in 3 patients namely, η_1, η_2 and η_4 while lung infection is in 3 patients namely, η_2, η_3 and η_4.
When testing the experiment after 20 days, we obtain the soft open set (N_Δ, Δ) over I_U such that $N_\Delta(\delta_1) = \{\eta_1, \eta_3, \eta_4\}$, $N_\Delta(\delta_2) = \{\eta_1, \eta_2\}$, $N_\Delta(\delta_3) = \{\eta_2, \eta_4\}$, i.e., fatigueness is among the 3 patients namely, η_1, η_3 and η_4 who are diagnosed with COVID-19, joint pain occurs in 2 patients namely, η_1 and η_2 while lung infection is in 2 patients namely, η_2 and η_4.
When testing the experiment after 30 days, we obtain soft open set (F_Δ, Δ) over I_U such that $F_\Delta(\delta_1) = \{\eta_1, \eta_3, \eta_4\}$, $F_\Delta(\delta_2) = \{\eta_2\}$, $F_\Delta(\delta_3) = \{\eta_4\}$, i.e., fatigueness is among the 3 patients namely, η_1, η_3 and η_4 who are diagnosed with COVID-19, joint pain occurs in 1 patient η_2 while lung infection is in 1 patient namely, η_4.
When testing the experiment after 40 days, we obtain the soft open set (G_Δ, Δ) over I_U such that $G_\Delta(\delta_1) = \{\eta_4\}$, $F_\Delta(\delta_2) = \phi$, $F_\Delta(\delta_3) = \phi$, i.e., fatigueness is among one patient only η_4 while there is no symptom of joint pain and lung infection in any patient.
Thus, $(I_U, \tilde{\tau_1}, \tilde{\tau_2}, \tilde{\tau_3}, \tilde{\tau_4}, \Delta)$ is soft quad topological space.

Definition 3.3. Let (M_Δ, Δ) be a soft set over I_U, then
(i) soft $\tilde{\tau_1}, \tilde{\tau_2}, ..., \tilde{\tau_\mathcal{N}}$-closure of (M_Δ, Δ), denoted by $\tilde{\tau_1}, \tilde{\tau_2}, ..., \tilde{\tau_\mathcal{N}}$-$\mathcal{CL}\{(M_\Delta, \Delta)\}$, is defined by

$$\tilde{\tau_1}, \tilde{\tau_2}, ..., \tilde{\tau_\mathcal{N}}\text{-}\mathcal{CL}\{(M_\Delta, \Delta)\} = \tilde{\cap}\ \{(F_\Delta, \Delta) : (M_\Delta, \Delta)\tilde{\subseteq}\ (F_\Delta, \Delta) \text{ and } (F_\Delta, \Delta)\tilde{\in}\ \tilde{\tau_1}, \tilde{\tau_2}, ..., \tilde{\tau_\mathcal{N}}\text{-}\breve{\mathcal{C}}(I_U)\}.$$

(ii) soft $\tilde{\tau_1}, \tilde{\tau_2}, ..., \tilde{\tau_\mathcal{N}}$-interior of (M_Δ, Δ), denoted by $\tilde{\tau_1}, \tilde{\tau_2}, ..., \tilde{\tau_\mathcal{N}}$-$\mathcal{INT}\{(M_\Delta, \Delta)\}$, is defined by

$$\tilde{\tau_1}, \tilde{\tau_2}, ..., \tilde{\tau_\mathcal{N}}\text{-}\mathcal{INT}\{(M_\Delta, \Delta)\} = \tilde{\cup}\{(F_\Delta, \Delta) : (F_\Delta, \Delta)\tilde{\subseteq}(M_\Delta, \Delta) \text{ and } (F_\Delta, \Delta)\tilde{\in}\tilde{\tau_1}, \tilde{\tau_2}, ..., \tilde{\tau_\mathcal{N}}\text{-}\breve{\mathcal{O}}(I_U)\}.$$

Remark. (i) $(M_\Delta, \Delta)\ \tilde{\in}\ \tilde{\tau_1}\tilde{\tau_2}...\tilde{\tau_\mathcal{N}}$-$\breve{\mathcal{O}}(I_U)$ iff $\tilde{\tau_1}, \tilde{\tau_2}, ...\tilde{\tau_\mathcal{N}}$-$\mathcal{INT}\{(M_\Delta, \Delta)\} = (M_\Delta, \Delta)$.
(ii) $(M_\Delta, \Delta)\ \tilde{\in}\ \tilde{\tau_1}\tilde{\tau_2}...\tilde{\tau_\mathcal{N}}$-$\breve{\mathcal{C}}(I_U)$ iff $\tilde{\tau_1}\tilde{\tau_2}...\tilde{\tau_\mathcal{N}}$-$\mathcal{CL}\{(M_\Delta, \Delta)\} = (M_\Delta, \Delta)$.

Definition 3.4. Let (M_Δ, Δ) be a soft set over I_U, then (M_Δ, Δ) is soft $\tilde{\tau_1}, \tilde{\tau_2}, ..., \tilde{\tau_\mathcal{N}}$-semi-open set if

$$(M_\Delta, \Delta) \tilde{\subseteq} \tilde{\tau_1}, \tilde{\tau_2}, ..., \tilde{\tau_\mathcal{N}}\text{-}\mathcal{CL}\{\tilde{\tau_1}, \tilde{\tau_2}, ..., \tilde{\tau_\mathcal{N}}\text{-}\mathcal{INT}(M_\Delta, \Delta)\}.$$

Example 3.4. In example 3.1, the soft set $(M_\Delta, \Delta) = \{(\delta_1, \{\eta_1\}), (\delta_2, \{\eta_1, \eta_2\})\}$ is soft $\tilde{\tau_1}, \tilde{\tau_2}, ..., \tilde{\tau_\mathcal{N}}$-semi-open set.

Remark. Any soft topological space is a soft $\mathcal{N}$-topological space. Let $(I_U, \tilde{\tau}, \Delta)$ be a soft topological space, then $(I_U, \underbrace{\tilde{\tau}, \tilde{\tau}, ..., \tilde{\tau}}_{\mathcal{N}\text{-times}}, \Delta)$ is soft $\mathcal{N}$-topological space.

Remark. Any soft $\mathcal{N}$-topological space need not be a "soft topological space" but any soft $\mathcal{N}$-topological space induces a soft topological space in various ways.
If we take the intersection of all soft topologies, then we get a soft topological space.

Example 3.5. In example 3.1, if we take the intersection of all the four soft topologies, then we get a soft topology $\tilde{\tau}$ as

$$\begin{aligned}\tilde{\tau} &= \tilde{\tau_1} \tilde{\cap} \tilde{\tau_2} \tilde{\cap} \tilde{\tau_3} \tilde{\cap} \tilde{\tau_4} \\ &= \{\tilde{\phi},\ \tilde{I_U}\}\end{aligned}$$

Clearly, $(I_U, \tilde{\tau}, \Delta)$ is a soft topological space.

4. Soft $\tilde{\tau_1}\tilde{\tau_2}$-$\omega$-closed set in soft bitopological space

In 2014, B.M. Ittanagi[3] has defined soft bitopological space and we extend this concept to define "soft ω-closed set" in "soft bitopological spaces" with some suitable examples. Further, we discuss some characteristics of these spaces. Throughout this section, we denote soft bitopological space by $(I_U, \tilde{\tau_1}, \tilde{\tau_2}, \Delta)$.

Definition 4.1. [3] "Let I_U be an initial universal set, Δ be the non-empty parameter set and $\tilde{\tau_1}, \tilde{\tau_2}$ be two soft topologies over I_U. Then, the space $(I_U, \tilde{\tau_1}, \tilde{\tau_2}, \Delta)$ is called **soft bitopological space**."

Definition 4.2. The soft set (W_Δ, Δ) is soft $\tilde{\tau_1}\tilde{\tau_2}$-$\omega$-closed set if

$$\tilde{\tau_2}\text{-}\mathcal{CL}\{(W_\Delta, \Delta)\} \tilde{\subseteq} (C_\Delta, \Delta)$$

whenever $(W_\Delta, \Delta) \tilde{\subseteq} (C_\Delta, \Delta)$, where (C_Δ, Δ) is soft $\tilde{\tau_1}$-semi-open set over I_U.
The family of all soft $\tilde{\tau_1}\tilde{\tau_2}$-$\omega$-closed set is denoted by $F_{s\omega}(\tilde{\tau_1}, \tilde{\tau_2})$.

Example 4.1. Let $I_U = \{\eta_1, \eta_2, \eta_3\}$, $\Delta = \{\delta_1, \delta_2\}$ and $\tilde{\tau_1} = \{\tilde{\phi},\ \tilde{I_U},\ \{(\delta_1, \{\eta_1, \eta_3\}), (\delta_2, \{\eta_2, \eta_3\})\}\}$, $\tilde{\tau_2} = \{\tilde{\phi},\ \tilde{I_U},\ \{(\delta_1, \{\eta_3\}), (\delta_2, \{\eta_3\})\}\}$ are soft topologies over I_U and the space $(I_U, \tilde{\tau_1}, \tilde{\tau_2}, \Delta)$ is a soft bitopological space. Let $(W_\Delta, \Delta) = \{(\delta_1, \{\eta_1, \eta_2\}), (\delta_2, I_U)\}$ be a soft set over I_U, then

$$\tilde{\tau_2}\text{-}\mathcal{CL}\{(W_\Delta, \Delta)\} \tilde{\subseteq} (C_\Delta, \Delta),$$

where (C_Δ, Δ) is soft $\tilde{\tau_1}$-semi-open set over I_U. Thus, (W_Δ, Δ) is soft $\tilde{\tau_1}\tilde{\tau_2}$-$\omega$-closed set over I_U.

Remark. If $\tilde{\tau_1} = \tilde{\tau_2}$, then soft $\tilde{\tau_1}\tilde{\tau_2}$-$\omega$-closed set is simply a soft ω-closed set.

Remark. Soft $\tilde{\tau_1}$-ω-closed set and soft $\tilde{\tau_1}\tilde{\tau_2}$-$\omega$-closed sets are independent, in general which can be seen from the following examples.

Example 4.2. Let $I_U = \{\eta_1, \eta_2, \eta_3\}$, $\Delta = \{\delta_1, \delta_2\}$, $\tilde{\tau_1} = \{\tilde{\phi},\ \tilde{I_U},\ \{(\delta_1, \{\eta_1, \eta_2\}), (\delta_2, \{\eta_1, \eta_2\})\}\}$ and $\tilde{\tau_2} = \{\tilde{\phi},\ \tilde{I_U},\ \{(\delta_1, \{\eta_1\}), (\delta_2, \{\eta_1\})\}, \{(\delta_1, \{\eta_3\}), (\delta_2, \{\eta_3\})\}, \{(\delta_1, \{\eta_1, \eta_3\}), (\delta_2, \{\eta_1, \eta_3\})\}\}$. Then, $(I_U, \tilde{\tau_1}, \tilde{\tau_2}, \Delta)$ is a "soft bitopological space" and the soft set $(W_\Delta, \Delta) = \{(\delta_1, \{\eta_1, \eta_2\}), (\delta_2, \{\eta_1, \eta_2\})\} \tilde{\in} F_{s\omega}(\tilde{\tau_1}, \tilde{\tau_2})$ but it is not soft $\tilde{\tau_1}$-ω closed set.

Example 4.3. Let $I_U = \{\eta_1, \eta_2, \eta_3\}$, $\Delta = \{\delta_1, \delta_2\}$,
$\tilde{\tau_1} = \{\tilde{\phi},\ \tilde{I_U},\ \{(\delta_1, \{\eta_1\}), (\delta_2, \{\eta_1\})\}, \{(\delta_1, \{\eta_1, \eta_2\}), (\delta_2, \{\eta_1, \eta_2\})\}\}$ and
$\tilde{\tau_2} = \{\tilde{\phi},\ \tilde{I_U},\ \{(\delta_1, \{\eta_1\}), (\delta_2, \{\eta_1\})\}, \{(\delta_1, \{\eta_3\}), (\delta_2, \{\eta_3\})\}, \{(\delta_1, \{\eta_1, \eta_3\}), (\delta_2, \{\eta_1, \eta_3\})\}\}$.
Then, $(I_U, \tilde{\tau_1}, \tilde{\tau_2}, \Delta)$ is a "soft bitopological space" and the soft set $(W_\Delta, \Delta) = \{(\delta_1, \{\eta_3\}), (\delta_2, \{\eta_3\})\}$ is soft $\tilde{\tau_1}$-ω closed set but $(W_\Delta, \Delta) \tilde{\notin} F_{s\omega}(\tilde{\tau_1}, \tilde{\tau_2})$.

Proposition 4.3. Let $(W_\Delta, \Delta) \tilde{\in} \tilde{\tau_2^c}$, then $(W_\Delta, \Delta) \tilde{\in} F_{s\omega}(\tilde{\tau_1}, \tilde{\tau_2})$.
Proof: Let (C_Δ, Δ) be a soft $\tilde{\tau_1}$-semi-open set such that $(W_\Delta, \Delta) \tilde{\subseteq} (C_\Delta, \Delta)$. But as
$(W_\Delta, \Delta) \tilde{\in} \tilde{\tau_2^c}$, thus $\tilde{\tau_2}$-$\mathcal{CL}\{(W_\Delta, \Delta)\} = (W_\Delta, \Delta)$ and hence $(W_\Delta, \Delta) \tilde{\in} F_{s\omega}(\tilde{\tau_1}, \tilde{\tau_2})$.

Remark. The converse of prop 4.3 is not true, in general by the example 4.1.

Remark. If $(W_\Delta, \Delta) \tilde{\in} \tilde{\tau_1^c}$, then in general $(W_\Delta, \Delta) \tilde{\notin} F_{s\omega}(\tilde{\tau_1}, \tilde{\tau_2})$, which can be seen from
the following example.

Example 4.4. Let $I_U = \{\eta_1, \eta_2, \eta_3\}$, $\Delta = \{\delta_1, \delta_2\}$,
$\tilde{\tau_1} = \{\tilde{\phi},\ \tilde{I_U},\ \{(\delta_1, \{\eta_1\}), (\delta_2, \{\eta_1\})\}, \{(\delta_1, \{\eta_3\}), (\delta_2, \{\eta_3\})\}, \{(\delta_1, \{\eta_1, \eta_3\}), (\delta_2, \{\eta_1, \eta_3\})\}\}$ and
$\tilde{\tau_2} = \{\tilde{\phi},\ \tilde{I_U},\ , \{(\delta_1, \{\eta_2, \eta_3\}), (\delta_2, \{\eta_2, \eta_3\})\}\}$.
Then, $(W_\Delta, \Delta) = \{(\delta_1, \{\eta_1, \eta_2\}), (\delta_2, \{\eta_1, \eta_2\})\} \tilde{\in} \tilde{\tau_1^c}$ but $(W_\Delta, \Delta) \tilde{\notin} F_{s\omega}(\tilde{\tau_1}, \tilde{\tau_2})$.

Proposition 4.4. If $(W_{1_\Delta}, \Delta) \tilde{\in} F_{s\omega}(\tilde{\tau_1}, \tilde{\tau_2})$ and $(W_{2_\Delta}, \Delta) \tilde{\in} F_{s\omega}(\tilde{\tau_1}, \tilde{\tau_2})$, then $(W_{1_\Delta}, \Delta) \tilde{\cup} (W_{2_\Delta}, \Delta) \tilde{\in} F_{s\omega}(\tilde{\tau_1}, \tilde{\tau_2})$.
Proof: Let (C_Δ, Δ) be a soft $\tilde{\tau_1}$-semi-open set such that

$$(W_{1_\Delta}, \Delta) \tilde{\cup} (W_{2_\Delta}, \Delta) \tilde{\subseteq} (C_\Delta, \Delta)$$

$$\Rightarrow \quad (W_{1_\Delta}, \Delta) \tilde{\subseteq} (C_\Delta, \Delta) \text{ and } (W_{2_\Delta}, \Delta) \tilde{\subseteq} (C_\Delta, \Delta)$$

As (W_{1_Δ}, Δ) and (W_{2_Δ}, Δ) are soft $\tilde{\tau_1}\tilde{\tau_2}$-$\omega$-closed sets, thus

$$\begin{aligned} & \tilde{\tau_2}\text{-}\mathcal{CL}\{(W_{1_\Delta}, \Delta)\} \tilde{\subseteq} (C_\Delta, \Delta) \text{ and } \tilde{\tau_2}\text{-}\mathcal{CL}\{(W_{2_\Delta}, \Delta)\} \tilde{\subseteq} (C_\Delta, \Delta) \\ \Rightarrow \quad & \tilde{\tau_2}\text{-}\mathcal{CL}\{(W_{1_\Delta}, \Delta)\} \tilde{\cup} \tilde{\tau_2}\text{-}\mathcal{CL}\{(W_{2_\Delta}, \Delta)\} \tilde{\subseteq} (C_\Delta, \Delta) \\ \Rightarrow \quad & \tilde{\tau_2}\text{-}\mathcal{CL}\{(W_{1_\Delta}, \Delta) \tilde{\cup} (W_{2_\Delta}, \Delta)\} \tilde{\subseteq} (C_\Delta, \Delta). \end{aligned}$$

Hence, $(W_{1_\Delta}, \Delta) \tilde{\cup} (W_{2_\Delta}, \Delta) \tilde{\in} F_{s\omega}(\tilde{\tau_1}, \tilde{\tau_2})$.

Remark. If $(W_{1_\Delta}, \Delta) \tilde{\in} F_{s\omega}(\tilde{\tau_1}, \tilde{\tau_2})$ and $(W_{2_\Delta}, \Delta) \tilde{\in} F_{s\omega}(\tilde{\tau_1}, \tilde{\tau_2})$, then it is not necessary that $(W_{1_\Delta}, \Delta) \tilde{\cap} (W_{2_\Delta}, \Delta) \tilde{\in} F_{s\omega}(\tilde{\tau_1}, \tilde{\tau_2})$, which can be seen from the following example.

Example 4.5. Let $I_U = \{\eta_1, \eta_2, \eta_3\}$, $\Delta = \{\delta_1, \delta_2\}$ and

$$\begin{aligned} \tilde{\tau_1} =& \{\tilde{\phi},\ \tilde{I_U},\ \{(\delta_1, \{\eta_1, \eta_3\}), (\delta_2, \{\eta_1, \eta_3\})\}, \{(\delta_1, \{\eta_2, \eta_3\}), (\delta_2, \{\eta_2, \eta_3\})\}, \{(\delta_1, \{\eta_2\}), (\delta_2, \{\eta_2\})\}, \\ & \{(\delta_1, \{\eta_3\}), (\delta_2, \{\eta_3\})\}\}, \\ \tilde{\tau_2} =& \{\tilde{\phi},\ \tilde{I_U},\ \{(\delta_1, \{\eta_1\}), (\delta_2, \{\eta_1\})\}, \{(\delta_1, \{\eta_2, \eta_3\}), (\delta_2, \{\eta_2, \eta_3\})\}\} \end{aligned}$$

are soft topologies over I_U. Then,

$$\begin{aligned}&(W_{1_\Delta}, \Delta) = \{(\delta_1, \{\eta_1, \eta_2\}), (\delta_2, \{\eta_1, \eta_2\})\} \tilde{\in} F_{s\omega}(\tilde{\tau_1}, \tilde{\tau_2}) \text{ and}\\&(W_{2_\Delta}, \Delta) = \{(\delta_1, \{\eta_2, \eta_3\}), (\delta_2, \{\eta_2, \eta_3\})\} \tilde{\in} F_{s\omega}(\tilde{\tau_1}, \tilde{\tau_2}) \text{ but}\\&(W_{1_\Delta}, \Delta) \tilde{\cap} (W_{2_\Delta}, \Delta) \tilde{\notin} F_{s\omega}(\tilde{\tau_1}, \tilde{\tau_2}).\end{aligned}$$

Remark. In general, the collection $F_{s\omega}(\tilde{\tau_1}, \tilde{\tau_2}) \neq F_{s\omega}(\tilde{\tau_2}, \tilde{\tau_1})$ as in example 4.5, we have a soft set $(W_{1_\Delta}, \Delta) = \{(\delta_1, \{\eta_2\}), (\delta_2, \{\eta_2\})\} \tilde{\in} F_{s\omega}(\tilde{\tau_2}, \tilde{\tau_1})$ but $(W_{1_\Delta}, \Delta) \tilde{\notin} F_{s\omega}(\tilde{\tau_1}, \tilde{\tau_2})$. Also,
$(W_{2_\Delta}, \Delta) = \{(\delta_1, \{\eta_2, \eta_3\}), (\delta_2, \{\eta_2, \eta_3\})\} \tilde{\in} F_{s\omega}(\tilde{\tau_1}, \tilde{\tau_2})$ but $(W_{2_\Delta}, \Delta) \tilde{\notin} F_{s\omega}(\tilde{\tau_2}, \tilde{\tau_1})$.

Proposition 4.5. The soft set (W_Δ, Δ) corresponding to the soft point W_δ^η, where $\eta \in I_U$ and $\delta \in \Delta$ is soft $\tilde{\tau_1}$-semi-closed or $(W_\Delta, \Delta)^c$ is soft $\tilde{\tau_1}\tilde{\tau_2}$-$\omega$-closed.
Proof: If (W_Δ, Δ) is soft $\tilde{\tau_1}$-semi-closed, then there is nothing to prove.
Let (W_Δ, Δ) be not soft $\tilde{\tau_1}$-semi-closed set, then $(W_\Delta, \Delta)^c$ is not soft $\tilde{\tau_1}$-semi-open set. Therefore, a soft $\tilde{\tau_1}$-semi-open set containing $(W_\Delta, \Delta)^c$ is $\tilde{I_U}$ itself. Also, $\tilde{\tau_2}$-$\mathcal{CL}\{(W_\Delta, \Delta)^c\} \tilde{\subseteq} \tilde{I_U}$. Hence, $(W_\Delta, \Delta)^c$ is soft $\tilde{\tau_1}\tilde{\tau_2}$-$\omega$-closed set.

Example 4.6. (i) In example 4.5, consider a soft set $(W_\Delta, \Delta) = \{(\delta_1, \{\eta_1\}), (\delta_2, \phi)\}$ corresponding to the soft point $W_{\delta_1}^{\eta_1}$. Then, (W_Δ, Δ) is the soft $\tilde{\tau_1}$ semi-closed set and
$(W_\Delta, \Delta)^c = \{(\delta_1, \{\eta_2, \eta_3\}), (\delta_2, I_U)\} \tilde{\notin} F_{s\omega}(\tilde{\tau_1}, \tilde{\tau_2})$.
(ii)In example 4.5, if we consider a soft set $(M_\Delta, \Delta) = \{(\delta_1, \{\eta_2\}), (\delta_2, \phi)\}$ corresponding to the soft point $W_{\delta_1}^{\eta_2}$. Then, (M_Δ, Δ) is not soft $\tilde{\tau_1}$ semi-closed set while $(M_\Delta, \Delta)^c = \{(\delta_1, \{\eta_1, \eta_3\}), (\delta_2, I_U)\} \tilde{\in} F_{s\omega}(\tilde{\tau_1}, \tilde{\tau_2})$.

Proposition 4.6. Let $(W_\Delta, \Delta) \tilde{\in} F_{s\omega}(\tilde{\tau_1}, \tilde{\tau_2})$, then no non-empty soft $\tilde{\tau_1}$-semi-closed set is contained in $\tilde{\tau_2}$-$\mathcal{CL}\{(W_\Delta, \Delta)\} - (W_\Delta, \Delta)$.
Proof: Suppose $\tilde{\tau_2}$-$\mathcal{CL}\{(W_\Delta, \Delta)\} - (W_\Delta, \Delta)$ contains a non-empty soft $\tilde{\tau_1}$-semi-closed set say (F_Δ, Δ) i.e., $(F_\Delta, \Delta) \tilde{\subseteq} \tilde{\tau_2}$-$\mathcal{CL}\{(W_\Delta, \Delta)\} - (W_\Delta, \Delta)$.

$$\begin{aligned}\Rightarrow\quad & (F_\Delta, \Delta) \tilde{\subseteq} \tilde{\tau_2}\text{-}\mathcal{CL}\{(W_\Delta, \Delta)\} \tilde{\cap} (W_\Delta, \Delta)^c\\\Rightarrow\quad & (F_\Delta, \Delta) \tilde{\subseteq} \tilde{\tau_2}\text{-}\mathcal{CL}\{(W_\Delta, \Delta)\} \text{ and } (F_\Delta, \Delta) \tilde{\subseteq} (W_\Delta, \Delta)^c\\\Rightarrow\quad & (F_\Delta, \Delta) \tilde{\subseteq} \tilde{\tau_2}\text{-}\mathcal{CL}\{(W_\Delta, \Delta)\} \text{ and } (W_\Delta, \Delta) \tilde{\subseteq} (F_\Delta, \Delta)^c\end{aligned}$$

As (F_Δ, Δ) is soft $\tilde{\tau_1}$-semi-closed set, thus $(F_\Delta, \Delta)^c$ is soft $\tilde{\tau_1}$-semi-open set. But $(W_\Delta, \Delta) \tilde{\in} F_{s\omega}(\tilde{\tau_1}, \tilde{\tau_2})$, thus $\tilde{\tau_2}$-$\mathcal{CL}\{(W_\Delta, \Delta)\} \tilde{\subseteq} (F_\Delta, \Delta)^c$ i.e., we have

$$\begin{aligned}& (F_\Delta, \Delta) \tilde{\subseteq} \tilde{\tau_2}\text{-}\mathcal{CL}\{(W_\Delta, \Delta)\} \tilde{\subseteq} (F_\Delta, \Delta)^c\\\Rightarrow\quad & (F_\Delta, \Delta) = \tilde{\phi}\end{aligned}$$

Thus, $\tilde{\tau_2}$-$\mathcal{CL}\{(W_\Delta, \Delta)\} - (W_\Delta, \Delta)$ contains no non-empty soft $\tilde{\tau_1}$-semi-closed set.

Example 4.7. Consider a soft bitopological space in example 4.5, let
$(W_\Delta, \Delta) = \{(\delta_1, \{\eta_1, \eta_2\}), (\delta_2, \{\eta_1, \eta_2\})\} \tilde{\in} F_{s\omega}(\tilde{\tau_1}, \tilde{\tau_2})$, then
$\tilde{\tau_2}$-$\mathcal{CL}\{(W_\Delta, \Delta)\} = \tilde{I_U}$ and thus $\tilde{\tau_2}$-$\mathcal{CL}\{(W_\Delta, \Delta)\} - (W_\Delta, \Delta) = \{(\delta_1, \{\eta_3\}), (\delta_2, \{\eta_3\})\}$.
Clearly, the only soft $\tilde{\tau_1}$-semi-closed set contained in $\tilde{\tau_2}$-$\mathcal{CL}\{(W_\Delta, \Delta)\} - (W_\Delta, \Delta)$ is $\tilde{\phi}$.

Remark. The converse of above proposition is not true, in general which can be seen from the following example.

Example 4.8. Let $I_U = \{\eta_1, \eta_2, \eta_3\}$, $\Delta = \{\delta_1, \delta_2\}$,

$$\tilde{\tau_1} = \{\tilde{\phi},\ \tilde{I_U},\ \{(\delta_1, \{\eta_2\}), (\delta_2, \{\eta_2\})\}, \{(\delta_1, \{\eta_1, \eta_3\}), (\delta_2, \{\eta_1, \eta_3\})\}\},$$
$$\tilde{\tau_2} = \{\tilde{\phi},\ \tilde{I_U},\ \{(\delta_1, \{\eta_1\}), (\delta_2, \{\eta_1\})\}\}$$

are soft topologies over I_U.
Let $(W_\Delta, \Delta) = \{(\delta_1, \{\eta_2\}), (\delta_2, \{\eta_2\})\}$ be a soft set over I_U, then

$$\tilde{\tau_2}\text{-}\mathcal{CL}\{(W_\Delta, \Delta)\} = \{(\delta_1, \{\eta_2, \eta_3\}), (\delta_2, \{\eta_2, \eta_3\})\}$$
$$\tilde{\tau_2}\text{-}\mathcal{CL}\{(W_\Delta, \Delta)\} - (W_\Delta, \Delta) = \{(\delta_1, \{\eta_3\}), (\delta_2, \{\eta_3\})\}$$

Thus, $\tilde{\tau_2}\text{-}\mathcal{CL}\{(W_\Delta, \Delta)\} - (W_\Delta, \Delta)$ contains no non-empty soft $\tilde{\tau_1}$-semi-closed set but
$(W_\Delta, \Delta) = \{(\delta_1, \{\eta_2\}), (\delta_2, \{\eta_2\})\} \tilde{\notin} F_{s\omega}(\tilde{\tau_1}, \tilde{\tau_2})$.

Proposition 4.7. Let $(W_\Delta, \Delta) \tilde{\in} F_{s\omega}(\tilde{\tau_1}, \tilde{\tau_2})$. If (W_Δ, Δ) is soft $\tilde{\tau_1}$-semi-open, then $(W_\Delta, \Delta) \tilde{\in} \tilde{\tau_2^c}$.
Proof: Clearly, $(W_\Delta, \Delta) \tilde{\subseteq} (W_\Delta, \Delta)$ and thus (W_Δ, Δ) is soft $\tilde{\tau_1}$-semi-open set containing itself. Since $(W_\Delta, \Delta) \tilde{\in} F_{s\omega}(\tau_1 \tilde{,} \tilde{\tau_2})$, therefore

$$\tilde{\tau_2}\text{-}\mathcal{CL}\{(W_\Delta, \Delta)\} \tilde{\subseteq} (W_\Delta, \Delta) \qquad ...(1)$$

But we always have

$$(W_\Delta, \Delta) \tilde{\subseteq} \tilde{\tau_2}\text{-}\mathcal{CL}\{(W_\Delta, \Delta)\} \qquad ...(2)$$

Thus, from (1) and (2), we have

$$(W_\Delta, \Delta) = \tilde{\tau_2}\text{-}\mathcal{CL}\{(W_\Delta, \Delta)\}$$

Hence, $(W_\Delta, \Delta) \tilde{\in} \tilde{\tau_2^c}$.

Remark. The converse of above proposition is not true i.e., if $(W_\Delta, \Delta)\tilde{\in}\tilde{\tau_2}^c$, then (W_Δ, Δ) need not be soft $\tilde{\tau_1}$- semi-open set, which can be seen from the following example.

Example 4.9. Consider a soft bitopological space from example 4.8. Let $(W_\Delta, \Delta) = \{(\delta_1, \{\eta_2, \eta_3\}), (\delta_2, \{\eta_2, \eta_3\})\}$ be a soft set over I_U such that $(W_\Delta, \Delta)\tilde{\in}\tilde{\tau_2}^c$ but (W_Δ, Δ) is not soft $\tilde{\tau_1}$-semi-open set.

We now introduce soft $\tilde{\tau_1}\tilde{\tau_2}$-$\omega$-open set in soft bitopological space $(I_U, \tilde{\tau_1}, \tilde{\tau_2}, \Delta)$.

Definition 4.8. A soft set (M_Δ, Δ) in "soft bitopological space" is soft $\tilde{\tau_1}\tilde{\tau_2}$-$\omega$-open set if $(M_\Delta, \Delta)^c$ is soft $\tilde{\tau_1}\tilde{\tau_2}$-$\omega$-closed set. The family of all soft $\tilde{\tau_1}\tilde{\tau_2}$-$\omega$-open set is denoted by $G_{s\omega}(\tilde{\tau_1}, \tilde{\tau_2})$.

Example 4.10. In example 4.1, the soft set $(M_\Delta, \Delta) = \{(\delta_1, \{\eta_3\}), (\delta_2, \phi)\}$ is soft $\tilde{\tau_1}\tilde{\tau_2}$-$\omega$-open set.

Theorem 4.9. A soft set $(M_\Delta, \Delta) \tilde{\in} G_{s\omega}(\tilde{\tau_1}, \tilde{\tau_2})$ if and only if $(F_\Delta, \Delta) \tilde{\subseteq} \tilde{\tau_2}$-$\mathcal{INT}\{(M_\Delta, \Delta)\}$ whenever $(F_\Delta, \Delta) \tilde{\subseteq} (M_\Delta, \Delta)$ where (F_Δ, Δ) is soft $\tilde{\tau_1}$-semi-closed set.
Proof: Suppose (F_Δ, Δ) is soft $\tilde{\tau_1}$-semi-closed set such that $(F_\Delta, \Delta) \tilde{\subseteq} \{(M_\Delta, \Delta)\}$. Then,

$$(F_\Delta, \Delta) \tilde{\subseteq} \tilde{\tau_2}\text{-}\mathcal{INT}\{(M_\Delta, \Delta)\}$$

Let $(C_\Delta,\Delta)=(F_\Delta,\Delta)^c$ be a soft $\tilde{\tau}_1$ -semi-open set such that $(M_\Delta,\Delta)^c \tilde{\subseteq} (C_\Delta,\Delta)$. Then, $(C_\Delta,\Delta)^c \tilde{\subseteq} (M_\Delta,\Delta)$ and $(C_\Delta,\Delta)^c$ is soft $\tilde{\tau}_1$-semi-closed set. Thus,

$$\begin{aligned} & (C_\Delta,\Delta)^c \tilde{\subseteq} \tilde{\tau}_2\text{-}\mathcal{INT}(M_\Delta,\Delta) \\ \Rightarrow \quad & \{\tilde{\tau}_2\text{-}\mathcal{INT}(M_\Delta,\Delta)\}^c \tilde{\subseteq}(C_\Delta,\Delta) \\ \Rightarrow \quad & \tilde{\tau}_2\text{-}\mathcal{CL}\{(M_\Delta,\Delta)^c\} \tilde{\subseteq}(C_\Delta,\Delta) \end{aligned}$$

and hence $(M_\Delta,\Delta)^c \tilde{\in} F_{s\omega}(\tilde{\tau}_1,\tilde{\tau}_2)$. Thus, $(M_\Delta,\Delta) \tilde{\in} G_{s\omega}(\tilde{\tau}_1,\tilde{\tau}_2)$.
Conversely, suppose $(M_\Delta,\Delta) \tilde{\in} G_{s\omega}(\tilde{\tau}_1,\tilde{\tau}_2)$ such that $(F_\Delta,\Delta) \tilde{\subseteq} (M_\Delta,\Delta)$ where (F_Δ,Δ) is soft $\tilde{\tau}_1$-semi-closed set. Then, $(F_\Delta,\Delta)^c$ is soft $\tilde{\tau}_1$-semi-open set such that $(M_\Delta,\Delta)^c \tilde{\subseteq} (F_\Delta,\Delta)^c$. Thus,

$$\begin{aligned} & \tilde{\tau}_2\text{-}\mathcal{CL}\{(M_\Delta,\Delta)\}^c \tilde{\subseteq}(F_\Delta,\Delta)^c \\ \Rightarrow \quad & \{\tilde{\tau}_2\text{-}\mathcal{INT}(M_\Delta,\Delta)\}^c \tilde{\subseteq}(F_\Delta,\Delta)^c \\ \Rightarrow \quad & (F_\Delta,\Delta) \tilde{\subseteq} \tilde{\tau}_2\text{-}\mathcal{INT}\{(M_\Delta,\Delta)\}. \end{aligned}$$

Proposition 4.10. If $(W_\Delta,\Delta) \tilde{\in} \tilde{\tau}_2$, then $(W_\Delta,\Delta) \tilde{\in} G_{s\omega}(\tilde{\tau}_1,\tilde{\tau}_2)$.
Proof: The proof follows from the prop 4.3.

Proposition 4.11. If $(M_{1_\Delta},\Delta) \tilde{\in} G_{s\omega}(\tilde{\tau}_1,\tilde{\tau}_2)$ and $(M_{2_\Delta},\Delta) \tilde{\in} G_{s\omega}(\tilde{\tau}_1,\tilde{\tau}_2)$, then $(M_{1_\Delta},\Delta) \tilde{\cap} (M_{2_\Delta},\Delta) \tilde{\in} G_{s\omega}(\tilde{\tau}_1,\tilde{\tau}_2)$.
Proof: The proof follows from the prop 4.4.

Theorem 4.12. Let (M_Δ,Δ) and (N_Δ,Δ) be two soft sets in "soft bitopological space". If $(M_\Delta,\Delta) \tilde{\in} G_{s\omega}(\tilde{\tau}_1,\tilde{\tau}_2)$, then $(N_\Delta,\Delta) = \tilde{I_U}$ whenever (N_Δ,Δ) is soft $\tilde{\tau}_1$-semi-open set and $\tilde{\tau}_2\text{-}\mathcal{INT}(M_\Delta,\Delta) \tilde{\cup} (M_\Delta,\Delta)^c \tilde{\subseteq} (N_\Delta,\Delta)$.
Proof: Let $(M_\Delta,\Delta) \tilde{\in} G_{s\omega}(\tilde{\tau}_1,\tilde{\tau}_2)$ and (N_Δ,Δ) be soft $\tilde{\tau}_1$-semi-open set such that

$$\begin{aligned} & \{\tilde{\tau}_2\text{-}\mathcal{INT}(M_\Delta,\Delta)\} \tilde{\cup} (M_\Delta,\Delta)^c \tilde{\subseteq}(N_\Delta,\Delta) \\ \Rightarrow \quad & (N_\Delta,\Delta)^c \tilde{\subseteq} \{\tilde{\tau}_2\text{-}\mathcal{INT}(M_\Delta,\Delta)\}^c - (M_\Delta,\Delta)^c \\ & = \tilde{\tau}_2\text{-}\mathcal{CL}\{(M_\Delta,\Delta)^c\} - (M_\Delta,\Delta)^c \end{aligned}$$

Since $(M_\Delta,\Delta)^c \tilde{\in} F_{s\omega}(\tilde{\tau}_1,\tilde{\tau}_2)$ and $(N_\Delta,\Delta)^c$ is soft $\tilde{\tau}_1$-semi-closed set, then by proposition 4.6, we have $(N_\Delta,\Delta)^c = \tilde{\phi}$ i.e., $(N_\Delta,\Delta) = \tilde{I_U}$.

Remark. The converse of theorem 4.12 is not true, in general by the following example.

Example 4.11. Let us consider the "soft bitopological space" in example 4.5. Suppose
$(M_\Delta,\Delta) = \{(\delta_1,\{\eta_1,\eta_3\}),(\delta_2,\{\eta_1,\eta_3\})\}$ is a soft set over I_U, then the only soft $\tilde{\tau}_1$-semi-open set containing $\tilde{\tau}_2\text{-}\mathcal{INT}(M_\Delta,\Delta) \tilde{\cup} (M_\Delta,\Delta)^c$ is $\tilde{I_U}$ itself. But $(M_\Delta,\Delta) \tilde{\notin} G_{s\omega}(\tilde{\tau}_1,\tilde{\tau}_2)$.

Proposition 4.13. If $(M_\Delta,\Delta) \tilde{\in} F_{s\omega}(\tilde{\tau}_1,\tilde{\tau}_2)$, then $\{\tilde{\tau}_2\text{-}\mathcal{CL}(M_\Delta,\Delta) - (M_\Delta,\Delta)\} \tilde{\in} G_{s\omega}(\tilde{\tau}_1,\tilde{\tau}_2)$.
Proof: Let (F_Δ,Δ) be a soft $\tilde{\tau}_1$-semi-closed set such that

$$(F_\Delta,\Delta) \tilde{\subseteq} \{\tilde{\tau}_2\text{-}\mathcal{CL}\{(M_\Delta,\Delta)\} - (M_\Delta,\Delta)\}$$

By proposition 4.6, we have $(F_\Delta,\Delta) = \tilde{\phi}$.

$$\Rightarrow \quad (F_\Delta,\Delta) \tilde{\subseteq} \tilde{\tau}_2\text{-}\mathcal{INT}[\tilde{\tau}_2\text{-}\mathcal{CL}\{(M_\Delta,\Delta)\} - (M_\Delta,\Delta)]$$

By theorem 4.9, we have

$$\{\tilde{\tau_2}\text{-}\mathcal{CL}(M_\Delta,\Delta) - (M_\Delta,\Delta)\} \tilde{\in} G_{s\omega}(\tilde{\tau_1},\tilde{\tau_2}).$$

Remark. The converse of proposition 4.13 is not true by the following example.

Example 4.12. Consider the soft bitopological space in example 4.5, let $(M_\Delta,\Delta) = \{(\delta_1,\{\eta_2\}),(\delta_2,\{\eta_2\})\}$, then $\tilde{\tau_2}\text{-}\mathcal{CL}(M_\Delta,\Delta) - (M_\Delta,\Delta) = \{(\delta_1,\{\eta_3\}),(\delta_2,\{\eta_3\})\} \tilde{\in} G_{s\omega}(\tilde{\tau_1},\tilde{\tau_2})$ but $(M_\Delta,\Delta) \tilde{\notin} F_{s\omega}(\tilde{\tau_1}\tilde{\tau_2})$.

5. Soft $\tilde{\tau_1},\tilde{\tau_2},...,\tilde{\tau_\mathcal{N}}$-$\omega$-closed set in soft $\mathcal{N}$-topological space

In this section, we define soft $\tilde{\tau_1},\tilde{\tau_2},...,\tilde{\tau_\mathcal{N}}$-$\omega$-closed set in soft $\mathcal{N}$-topological space and study some characteristics of these spaces. Throughout this section, we use $(I_U,\tilde{\tau_1},\tilde{\tau_2},...,\tilde{\tau_\mathcal{N}},\Delta)$ as soft $\mathcal{N}$-topological space.

Definition 5.1. The soft set (W_Δ,Δ) is soft $\tilde{\tau_1},\tilde{\tau_2},...,\tilde{\tau_\mathcal{N}}$-$\omega$-closed set, $\mathcal{N} \geq 3$ if for any soft $\tilde{\tau_1},\tilde{\tau_2},...,\tilde{\tau_\mathcal{N}}$-semi-open set (C_Δ,Δ) containing (W_Δ,Δ), we have

$$\tilde{\tau_1},\tilde{\tau_2},...,\tilde{\tau_\mathcal{N}}\text{-}\mathcal{CL}\{(W_\Delta,\Delta)\} \tilde{\subseteq} (C_\Delta,\Delta).$$

The collection of all soft $\tilde{\tau_1},\tilde{\tau_2},...,\tilde{\tau_\mathcal{N}}$-$\omega$-closed sets is denoted by $F_{s\omega}(\tilde{\tau_1},\tilde{\tau_2},...,\tilde{\tau_\mathcal{N}})$.

Example 5.1. In example 3.2, the soft set $(W_\Delta,\Delta) = \{(\delta_1,I_U),(\delta_2,\{\eta_1,\eta_2\})\} \tilde{\in} F_{s\omega}(\tilde{\tau_1},\tilde{\tau_2},...,\tilde{\tau_\mathcal{N}})$.

Proposition 5.2. If $(W_\Delta,\Delta) \tilde{\in} \tilde{\tau_1},\tilde{\tau_2},...,\tilde{\tau_\mathcal{N}}\text{-}\breve{\mathcal{C}}(I_U)$, then $(W_\Delta,\Delta) \tilde{\in} F_{s\omega}(\tilde{\tau_1},\tilde{\tau_2},...,\tilde{\tau_\mathcal{N}})$.
Proof: The proof is obvious using remark 4.3.

Remark. The converse of proposition 5.2 is not true, in general by example 5.1.

Proposition 5.3. If $(W_{1_\Delta},\Delta) \tilde{\in} F_{s\omega}(\tilde{\tau_1},\tilde{\tau_2},...,\tilde{\tau_\mathcal{N}})$ and $(W_{2_\Delta},\Delta) \tilde{\in} F_{s\omega}(\tilde{\tau_1},\tilde{\tau_2},...,\tilde{\tau_\mathcal{N}})$, then
$(W_{1_\Delta},\Delta) \tilde{\cup} (W_{2_\Delta},\Delta) \tilde{\in} F_{s\omega}(\tilde{\tau_1},\tilde{\tau_2},...,\tilde{\tau_\mathcal{N}})$.
Proof: The proof is similar to the proof of proposition 4.4.

Proposition 5.4. If $(W_{1_\Delta},\Delta) \tilde{\in} F_{s\omega}(\tilde{\tau_1},\tilde{\tau_2},...,\tilde{\tau_\mathcal{N}})$ and $(W_{2_\Delta},\Delta) \tilde{\in} F_{s\omega}(\tilde{\tau_1},\tilde{\tau_2},...,\tilde{\tau_\mathcal{N}})$, then
$(W_{1_\Delta},\Delta) \tilde{\cap} (W_{2_\Delta},\Delta) \tilde{\in} F_{s\omega}(\tilde{\tau_1},\tilde{\tau_2},...,\tilde{\tau_\mathcal{N}})$.
Proof: The proof is similar to theorem 4.15[8].

We now introduce soft $\tilde{\tau_1}\tilde{\tau_2},...,\tilde{\tau_\mathcal{N}}$-$\omega$-open set in soft $\mathcal{N}$-topological space $(I_U,\tilde{\tau_1},\tilde{\tau_2},...,\tilde{\tau_\mathcal{N}},\Delta)$.

Definition 5.5. A soft set (M_Δ,Δ) is soft $\tilde{\tau_1}\tilde{\tau_2},...,\tilde{\tau_\mathcal{N}}$-$\omega$-open set if $(M_\Delta,\Delta)^c$ is soft $\tilde{\tau_1}\tilde{\tau_2},...,\tilde{\tau_\mathcal{N}}$-$\omega$-closed set. The family of all soft $\tilde{\tau_1}\tilde{\tau_2},...,\tilde{\tau_\mathcal{N}}$-$\omega$-open set is denoted by $G_{s\omega}(\tilde{\tau_1},\tilde{\tau_2},...,\tilde{\tau_\mathcal{N}})$.

Example 5.2. In example 3.2, the soft set $(M_\Delta,\Delta) = \{(\delta_1,\phi),(\delta_2,\{\eta_3\})\}$ is soft $\tilde{\tau_1}\tilde{\tau_2},...,\tilde{\tau_\mathcal{N}}$-$\omega$-open set.

Proposition 5.6. In a soft $\mathcal{N}$-topological space $(I_U,\tilde{\tau_1},\tilde{\tau_2},,...,\tilde{\tau_\mathcal{N}},\Delta)$, every soft $\tilde{\tau_1},\tilde{\tau_2},...,\tilde{\tau_\mathcal{N}}$-open set is soft $\tilde{\tau_1}\tilde{\tau_2},...,\tilde{\tau_\mathcal{N}}$-$\omega$-open set.
Proof: The proof follows from the prop 4.10.

Proposition 5.7. If $(M_{1_\Delta},\Delta) \tilde{\in} G_{s\omega}(\tilde{\tau_1}\tilde{\tau_2},...,\tilde{\tau_\mathcal{N}})$ and $(M_{2_\Delta},\Delta) \tilde{\in} G_{s\omega}(\tilde{\tau_1}\tilde{\tau_2},...,\tilde{\tau_\mathcal{N}})$, then
$(M_{1_\Delta},\Delta) \tilde{\cap} (M_{2_\Delta},\Delta) \tilde{\in} G_{s\omega}(\tilde{\tau_1}\tilde{\tau_2},...,\tilde{\tau_\mathcal{N}})$.
Proof: The proof follows from the prop 4.11.

Theorem 5.8. A soft set (M_Δ,Δ) is soft $\tilde{\tau_1}\tilde{\tau_2},...,\tilde{\tau_\mathcal{N}}$-$\omega$-open set if and only if $(F_\Delta,\Delta)\tilde{\subseteq}\tilde{\tau_1}\tilde{\tau_2}...\tilde{\tau_\mathcal{N}}$-$\mathcal{INT}\{(M_\Delta,\Delta)\}$ whenever $(F_\Delta,\Delta)\tilde{\subseteq}\{(M_\Delta,\Delta)\}$ where (F_Δ,Δ) is soft $\tilde{\tau_1},\tilde{\tau_2},...,\tilde{\tau_\mathcal{N}}$-semi-closed set.
Proof: Suppose (F_Δ,Δ) is soft $\tilde{\tau_1},\tilde{\tau_2},...,\tilde{\tau_\mathcal{N}}$-semi-closed set such that

$$(F_\Delta,\Delta)\tilde{\subseteq}\{(M_\Delta,\Delta)\}\text{ and }(F_\Delta,\Delta)\tilde{\subseteq}\tilde{\tau_1}\tilde{\tau_2}...\tilde{\tau_\mathcal{N}}\text{-}\mathcal{INT}\{(M_\Delta,\Delta)\}$$

Let (C_Δ,Δ) be a soft $\tilde{\tau_1}\tilde{\tau_2},...,\tilde{\tau_\mathcal{N}}$ -semi-open set such that $(M_\Delta,\Delta)^c\tilde{\subseteq}(C_\Delta,\Delta)$. Then,
$(C_\Delta,\Delta)^c\tilde{\subseteq}(M_\Delta,\Delta)$ and $(C_\Delta,\Delta)^c$ is soft $\tilde{\tau_1}\tilde{\tau_2},...,\tilde{\tau_\mathcal{N}}$-semi-closed set. Thus,

$$\begin{aligned} & (C_\Delta,\Delta)^c\tilde{\subseteq}\tilde{\tau_1}\tilde{\tau_2},...,\tilde{\tau_\mathcal{N}}\text{-}\mathcal{INT}(M_\Delta,\Delta)\\ \Rightarrow\quad & \{\tilde{\tau_1}\tilde{\tau_2},...,\tilde{\tau_\mathcal{N}}\text{-}\mathcal{INT}(M_\Delta,\Delta)\}^c\tilde{\subseteq}(C_\Delta,\Delta)\\ \Rightarrow\quad & \tilde{\tau_1}\tilde{\tau_2},...,\tilde{\tau_\mathcal{N}}\text{-}\mathcal{CL}\{(M_\Delta,\Delta)^c\}\tilde{\subseteq}(C_\Delta,\Delta)\end{aligned}$$

and hence $(M_\Delta,\Delta)^c$ is soft $\tilde{\tau_1}\tilde{\tau_2},...,\tilde{\tau_\mathcal{N}}$-$\omega$-closed set. Hence, $(M_\Delta,\Delta)\tilde{\in}G_{s\omega}(\tilde{\tau_1}\tilde{\tau_2},...,\tilde{\tau_\mathcal{N}})$.
Conversely, suppose (M_Δ,Δ) is soft $\tilde{\tau_1}\tilde{\tau_2},...,\tilde{\tau_\mathcal{N}}$-$\omega$-open set such that $(F_\Delta,\Delta)\tilde{\subseteq}\{(M_\Delta,\Delta)\}$ where (F_Δ,Δ) is soft $\tilde{\tau_1},...,\tilde{\tau_\mathcal{N}}$-semi-closed set. Then, $(F_\Delta,\Delta)^c$ is soft $\tilde{\tau_1},...,\tilde{\tau_\mathcal{N}}$-semi-open set such that $(M_\Delta,\Delta)^c\tilde{\subseteq}(F_\Delta,\Delta)^c$. Thus,

$$\begin{aligned} & \tilde{\tau_2},...,\tilde{\tau_\mathcal{N}}\text{-}\mathcal{CL}\{(M_\Delta,\Delta)\}^c\tilde{\subseteq}(F_\Delta,\Delta)^c\\ \Rightarrow\quad & \{\tilde{\tau_2},...,\tilde{\tau_\mathcal{N}}\text{-}\mathcal{INT}(M_\Delta,\Delta)\}^c\tilde{\subseteq}(F_\Delta,\Delta)^c\\ \Rightarrow\quad & (F_\Delta,\Delta)\tilde{\subseteq}\tilde{\tau_2},...,\tilde{\tau_\mathcal{N}}\text{-}\mathcal{INT}\{(M_\Delta,\Delta)\}.\end{aligned}$$

Definition 5.9. Consider a soft set (W_Δ,Δ) over I_U. Then, the soft $\tilde{\tau_1}\tilde{\tau_2}...\tilde{\tau_\mathcal{N}}$-$\omega$-closure of (W_Δ,Δ), denoted by $\tilde{\tau_1}\tilde{\tau_2}...\tilde{\tau_\mathcal{N}}$-$\mathcal{CL}_\omega\{(W_\Delta,\Delta)\}$, is defined as the intersection of all soft $\tilde{\tau_1}\tilde{\tau_2}...\tilde{\tau_\mathcal{N}}$-$\omega$-closed supersets of (W_Δ,Δ), $\mathcal{N}\geq 2$ i.e.,

$$\tilde{\tau_1}\tilde{\tau_2}...\tilde{\tau_\mathcal{N}}\text{-}\mathcal{CL}_\omega\{(W_\Delta,\Delta)\}=\tilde{\cap}\{(F_\Delta,\Delta):(W_\Delta,\Delta)\tilde{\subseteq}(F_\Delta,\Delta),(W_\Delta,\Delta)\tilde{\in}F_{s\omega}(\tilde{\tau_1},\tilde{\tau_2}...,\tilde{\tau_\mathcal{N}})\}.$$

Remark. (i) $(W_\Delta,\Delta)\tilde{\subseteq}\tilde{\tau_1}\tilde{\tau_2}...\tilde{\tau_\mathcal{N}}$-$\mathcal{CL}_\omega\{(W_\Delta,\Delta)\}$, $\mathcal{N}\geq 2$
(ii) $\tilde{\tau_1}\tilde{\tau_2}...\tilde{\tau_\mathcal{N}}$-$\mathcal{CL}_\omega\{(W_\Delta,\Delta)\}\tilde{\in}F_{s\omega}(\tilde{\tau_1},\tilde{\tau_2},...,\tilde{\tau_\mathcal{N}})$, $\mathcal{N}\geq 2$.

Theorem 5.10. $\tilde{\tau_1}\tilde{\tau_2}...\tilde{\tau_\mathcal{N}}$-$\mathcal{CL}_\omega\{(W_\Delta,\Delta)\}$ is the "smallest soft ω-closed set" containing (W_Δ,Δ), $\mathcal{N}\geq 2$.
Proof: The proof is similar to theorem 4.19 [8].

Theorem 5.11. A soft set $(W_\Delta,\Delta)\tilde{\in}F_{s\omega}(\tilde{\tau_1},\tilde{\tau_2},...\tilde{\tau_\mathcal{N}})$ if and only if $\tilde{\tau_1}\tilde{\tau_2}...\tilde{\tau_\mathcal{N}}$-$\mathcal{CL}_\omega\{(W_\Delta,\Delta)\}=(W_\Delta,\Delta)$, $\mathcal{N}\geq 2$.
Proof: The proof is similar to theorem 4.20 [8].

Theorem 5.12. Consider a soft $\mathcal{N}$-topological space $(I_U,\tilde{\tau_1},\tilde{\tau_2},...,\tilde{\tau_\mathcal{N}},\Delta)$, $\mathcal{N}\geq 2$. Let (W_Δ,Δ) and (W'_Δ,Δ) be soft sets over I_U, then
(i) $\tilde{\tau_1}\tilde{\tau_2}...\tilde{\tau_\mathcal{N}}$-$\mathcal{CL}_\omega\{\tilde{\phi}\}=\tilde{\phi}$ and $\tilde{\tau_1}\tilde{\tau_2}...\tilde{\tau_\mathcal{N}}$-$\mathcal{CL}_\omega\{\tilde{I_U}\}=\tilde{I_U}$.
(ii) $\tilde{\tau_1}\tilde{\tau_2}...\tilde{\tau_\mathcal{N}}$-$\mathcal{CL}_\omega[\tilde{\tau_1}\tilde{\tau_2}...\tilde{\tau_\mathcal{N}}$-$\mathcal{CL}_\omega\{(W_\Delta,\Delta)\}]=\tilde{\tau_1}\tilde{\tau_2}...\tilde{\tau_\mathcal{N}}$-$\mathcal{CL}_\omega\{(W_\Delta,\Delta)\}$.
(iii) If $(W_\Delta,\Delta)\tilde{\subseteq}(W'_\Delta,\Delta)$, then $\tilde{\tau_1}\tilde{\tau_2}...\tilde{\tau_\mathcal{N}}$-$\mathcal{CL}_\omega\{(W_\Delta,\Delta)\}\tilde{\subseteq}\tilde{\tau_1}\tilde{\tau_2}...\tilde{\tau_\mathcal{N}}$-$\mathcal{CL}_\omega\{(W'_\Delta,\Delta)\}$.
(iv) $\tilde{\tau_1}\tilde{\tau_2}...\tilde{\tau_\mathcal{N}}$-$\mathcal{CL}_\omega\{(W_\Delta,\Delta)\cup(W'_\Delta,\Delta)\}=\tilde{\tau_1}\tilde{\tau_2}...\tilde{\tau_\mathcal{N}}$-$\mathcal{CL}_\omega\{(W_\Delta,\Delta)\}\tilde{\cup}\tilde{\tau_1}\tilde{\tau_2}...\tilde{\tau_\mathcal{N}}$-$\mathcal{CL}_\omega\{(W'_\Delta,\Delta)\}$.
(v)$\tilde{\tau_1}\tilde{\tau_2}...\tilde{\tau_\mathcal{N}}$-$\mathcal{CL}_\omega\{(W_\Delta,\Delta)\cap(W'_\Delta,\Delta)\}\tilde{\subseteq}\tilde{\tau_1}\tilde{\tau_2}...\tilde{\tau_\mathcal{N}}$-$\mathcal{CL}_\omega\{(W_\Delta,\Delta)\}\tilde{\cap}\tilde{\tau_1}\tilde{\tau_2}...\tilde{\tau_\mathcal{N}}$-$\mathcal{CL}_\omega\{(W'_\Delta,\Delta)\}$.
(vi) $\tilde{\tau_1}\tilde{\tau_2}...\tilde{\tau_\mathcal{N}}$-$\mathcal{CL}_\omega\{(W_\Delta,\Delta)\}\tilde{\subseteq}\tilde{\tau_1}\tilde{\tau_2}...\tilde{\tau_\mathcal{N}}$-$\mathcal{CL}\{(W_\Delta,\Delta)\}$.
Proof: The proof is similar to theorem 4.21 [8].

Definition 5.13. Consider a soft set (W_Δ, Δ) over I_U. Then, the soft ω-interior of (W_Δ, Δ), denoted by $\tilde{\tau_1}\tilde{\tau_2}...\tilde{\tau_\mathcal{N}}$-$\mathcal{INT}_\omega\{(W_\Delta, \Delta)\}$, is defined as the union of all soft ω-open subsets of (W_Δ, Δ) for $\mathcal{N} \geq 2$ i.e.,

$$\tilde{\tau_1}\tilde{\tau_2}...\tilde{\tau_\mathcal{N}}\text{-}\mathcal{INT}_\omega\{(W_\Delta, \Delta)\} = \tilde{\cup}\ \{(F_\Delta, \Delta) : (W_\Delta, \Delta) \tilde{\supseteq} (F_\Delta, \Delta), (W_\Delta, \Delta) \tilde{\in} G_{s\omega}(\tilde{\tau_1}, \tilde{\tau_2}, ..., \tilde{\tau_\mathcal{N}})\}.$$

Remark. (i) $(W_\Delta, \Delta) \tilde{\supseteq} \tilde{\tau_1}\tilde{\tau_2}...\tilde{\tau_\mathcal{N}}$-$\mathcal{INT}_\omega\{(W_\Delta, \Delta)\}$, $\mathcal{N} \geq 2$.
(ii)$\tilde{\tau_1}\tilde{\tau_2}...\tilde{\tau_\mathcal{N}}$-$\mathcal{INT}_\omega\{(W_\Delta, \Delta)\} \tilde{\in} G_{s\omega}(\tilde{\tau_1}, \tilde{\tau_2}, ..., \tilde{\tau_\mathcal{N}})$, $\mathcal{N} \geq 2$.

Theorem 5.14. $\tilde{\tau_1}\tilde{\tau_2}...\tilde{\tau_\mathcal{N}}$-$\mathcal{INT}_\omega\{(W_\Delta, \Delta)\}$ is the "largest soft ω-open set" contained in (W_Δ, Δ), $\mathcal{N} \geq 2$.
Proof: The proof is similar to theorem 3.13 [8].

Theorem 5.15. A soft set $(W_\Delta, \Delta) \tilde{\in} G_{s\omega}(\tilde{\tau_1}, \tilde{\tau_2}, ..., \tilde{\tau_\mathcal{N}})$ if and only if $\tilde{\tau_1}\tilde{\tau_2}$-$\mathcal{INT}_\omega\{(W_\Delta, \Delta)\} = (W_\Delta, \Delta)$, $\mathcal{N} \geq 2$.
Proof: The proof is similar to theorem 3.14 [8].

Theorem 5.16. Consider a soft $\mathcal{N}$-topological space $(I_U, \tilde{\tau_1}, \tilde{\tau_2}, ..., \tilde{\tau_\mathcal{N}}, \Delta)$,$\mathcal{N} \geq 2$. Let (W_Δ, Δ) and (W'_Δ, Δ) be soft sets over I_U, then
(i) $\tilde{\tau_1}\tilde{\tau_2}...\tilde{\tau_\mathcal{N}}$-$\mathcal{INT}_\omega\{\tilde{\phi}\} = \tilde{\phi}$ and $\tilde{\tau_1}\tilde{\tau_2}...\tilde{\tau_\mathcal{N}}$-$\mathcal{INT}_\omega\{\tilde{I_U}\} = \tilde{I_U}$.
(ii) $\tilde{\tau_1}\tilde{\tau_2}...\tilde{\tau_\mathcal{N}}$-$\mathcal{INT}_\omega[\tilde{\tau_1}\tilde{\tau_2}...\tilde{\tau_\mathcal{N}}$-$\mathcal{INT}_\omega\{(W_\Delta, \Delta)\}] = \tilde{\tau_1}\tilde{\tau_2}...\tilde{\tau_\mathcal{N}}$-$\mathcal{INT}_\omega\{(W_\Delta, \Delta)\}$.
(iii) If $(W_\Delta, \Delta) \tilde{\subseteq} (W'_\Delta, \Delta)$, then $\tilde{\tau_1}\tilde{\tau_2}...\tilde{\tau_\mathcal{N}}$-$\mathcal{INT}_\omega\{(W_\Delta, \Delta)\} \tilde{\subseteq} \tilde{\tau_1}\tilde{\tau_2}...\tilde{\tau_\mathcal{N}}$-$\mathcal{INT}_\omega\{(W'_\Delta, \Delta)\}$.
(iv)$\tilde{\tau_1}\tilde{\tau_2}...\tilde{\tau_\mathcal{N}}$-$\mathcal{INT}_\omega\{(W_\Delta, \Delta)\} \tilde{\cup} \tilde{\tau_1}\tilde{\tau_2}...\tilde{\tau_\mathcal{N}}$-$\mathcal{INT}_\omega\{(W'_\Delta, \Delta)\} \tilde{\subseteq} \tilde{\tau_1}\tilde{\tau_2}...\tilde{\tau_\mathcal{N}}$-$\mathcal{INT}_\omega\{(W_\Delta \cup W'_\Delta, \Delta)\}$.
(v)$\tilde{\tau_1}\tilde{\tau_2}...\tilde{\tau_\mathcal{N}}$-$\mathcal{INT}_\omega\{(W_\Delta \cap W'_\Delta, \Delta)\} = \tilde{\tau_1}\tilde{\tau_2}...\tilde{\tau_\mathcal{N}}$-$\mathcal{INT}_\omega\{(W_\Delta, \Delta)\} \tilde{\cap} \tilde{\tau_1}\tilde{\tau_2}...\tilde{\tau_\mathcal{N}}$-$\mathcal{INT}_\omega\{(W'_\Delta, \Delta)\}$.
(vi) $\tilde{\tau_1}\tilde{\tau_2}...\tilde{\tau_\mathcal{N}}$-$\mathcal{INT}\{(W_\Delta, \Delta)\} \tilde{\subseteq} \tilde{\tau_1}\tilde{\tau_2}...\tilde{\tau_\mathcal{N}}$-$\mathcal{INT}_\omega\{(W_\Delta, \Delta)\}$.
Proof: The proof is similar to theorem 3.15 [8].

6. Concluding Remarks

We have generalized the concept of soft topological spaces and explore the idea of soft ω-closed set and soft ω-open set soft $\mathcal{N}$-topological spaces, $\mathcal{N} \geq 2$. The properties of these spaces were studied with some examples.

References

[1] Chen, B.(2013). Soft semi-open sets and related properties in soft topological spaces. Applied Mathematics and Informatics Sciences, 1(7), 287-294.
[2] Hussain, S. and Ahmad, B.(2011). Some properties of soft topological spaces.Computers and Mathematics with Applications, 62(2011),4058-4067.
[3] Ittanagi, B.M.(2014).Soft Bitopological Spaces. International Journal of Computer Applications, 107(7),1-4.
[4] Kelly, J.C.(1963). Bitopological spaces. Proceedings of the London Mathematical Society, 13(1), 71-81.
[5] Maji, P.K. and Roy, A.R.(2002). An Application of soft sets in a decision making problem, Computers and Mathematics with Applications, 44(2002), 1077-1083.
[6] Molodtsov, D.(1999). Soft set theory-First Results. Computers and Mathematics with Applications, 37(1999), 19-31.
[7] Paul, N.R.(2015). Remarks on soft ω-closed sets in soft topological space. Bol. Soc. Paran. Mat., 33(2015), 183-192.
[8] Rathee,S., Girdhar,R. and Dhingra, K.(2020). On Soft ω-interior and soft ω-closure in soft topological spaces. Journal of Interdisciplinary Mathematics, 23(6), 1223-1239.
[9] Shabir, M. and Naz, M.(2011).On soft topological spaces.Computers and Mathematics with Applications, 61(2011), 1786-1799.
[10] Sundaram, P. and John, M.S.(2000), On ω-closed set in topology. Acta Ciencia Indica, 4(2000), 389-392.

[11] Thivagar, M.L., Ramesh, V. and Dasan, M.A.(2017), On new structure of N-topology, Cogent Mathematics, 3(1), 1-10.
[12] Wolfe, M.A.(2000), Interval Mathematics, algebraic equations and optimization, Journal of Computational and Applied Mathematics, 124(2000), 263-280.
[13] Zadeh, L.A.(1965), Fuzzy sets, Inf. Control, 8(1965), 338-353.

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