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ON THE SOLVABILITY OF A SEMILINEAR SECOND ORDER PARABOLIC EQUATION WITH INTEGRAL CONDITION

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Abstract. In this paper we study a semilinear second order parabolic equation with mixed nonlocal boundary condition combined integral with another two-point boundary conditions of the Neumann type. First, we study the unique solvability of the associated linear problem by using the energy inequality method, then via an iteration process we prove the existence and uniqueness of the weak solution of the stated problem.

1. Introduction

Many phenomena of modern natural science often lead to nonlocal problems on mathematical modeling, and nonlocal models turn out to be often more precise that local conditions; see [3, 8, 9, 12]. These nonlocal boundary conditions appear when the data on the body can not be measured directly, but their average values are known. For instance, in some cases, describing the solution (pressure, temperature, etc.) pointwise is not possible, because only the average value of the solution can be estimate along the boundary or along a part of it. Nonlocal problems form a relatively new division of differential equations theory and generate a need in developing some new methods of research and the importance of these problems have been also pointed out by Samarskii [31]. The studies of nonlocal problems with integral conditions originated with the papers by Cannon [7] and Kamynin [20]. Recently various nonlocal problems for partial differential equations are actively studied and one can find a lot of papers dealing with them; see [2, 5, 4, 10, 7, 11, 19, 20, 21, 32, 36, 6, 23, 24, 34, 32] and references therein for parabolic equations, and [5, 27, 28, 29, 30, 35] for hyperbolic equations, and in [13, 14, 15, 16, 17, 22, 25] for mixed type equations. Problems for elliptic equations with operator nonlocal conditions were considered by Mikhailov and Gushchina [18], Skubachevskii [33], Paneyakh [26]. In this article, we focus our attention on nonlocal problems with integral conditions for parabolic equations. It is a continuation of previous studies of this type of problem with the difference in non-local boundary conditions and

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the equation itself. (case of the semi-linear equation). We establish the existence and uniqueness of the weak solution for the semi-linear second order parabolic equation with nonlocal boundary conditions combined integral with another two-point boundary conditions of the Neumann type. Firstly, we study the solvability of the associated linear problem by using an energy inequality method. Then we prove the unicity of the strong solution by using an a priori estimate, the existence of the strong solution is based on the density of the operator range. Using the results obtained for the associated linear problem and basing an iterative process, we prove the existence and the uniqueness of the weak solution of the stated problem. This paper is organized as follows: In Section 2, the problematic studied in this paper is formulated with the corresponding hypotheses. In Section 3, we state and pose associated the linear problem and introduce the function spaces used throughout the paper as well and present an abstract formulation of the posed linear problem. In Section 4, an a priori bound from which we deduce the uniqueness of the strong solution is then established by energy inequality technique. For the solvability of the associated linear problem, the density of the operator range of the operator generated by the considered problem is proved in Section 5 . Finally, in Section 6 , on the basis of results obtained in Sections 4 and 5, by using an iterative process, we prove the existence and uniqueness of the weak solution of the considered semi-linear problem.

2. Statement of the problem

In the rectangle $\Omega = (0,1) \times (0,T)$, with $T < +\infty$, we consider the equation

$$\frac{\partial u}{\partial t} - \frac{\partial}{\partial x}\left(a(x,t)\frac{\partial u}{\partial x}\right) = f(x,t,u), \tag{2.1}$$

with the initial condition

$$lu = u(x,0) = \varphi\left(x\right), \quad \forall x \in (0,1), \tag{2.2}$$

the boundary condition

$$\frac{\partial u}{\partial x}(0,t) = \frac{\partial u}{\partial x}(1,t), \quad \forall t \in (0,T), \tag{2.3}$$

and the integral condition

$$\int_0^1 u(x,t)dx = 0, \qquad \forall t \in (0,T). \tag{2.4}$$

In addition, we assume that the function $a(x,t)$ and its derivatives satisfies the conditions

$$\begin{cases} 0 < a_0 \le a(x,t) \le a_1 \quad \forall x,t \in \Omega, \\ c_1 \le \dfrac{\partial a}{\partial t}(x,t) \le c_2, \quad \forall x,t \in \Omega, \\ \qquad\left|\dfrac{\partial a}{\partial x}(x,t)\right| \le b, \\ \qquad a(1,t) \ne a(0,t), \end{cases} \tag{2.5}$$

Here, we assume that the known function φ satisfies the compatibility conditions given by (2.3) and (2.4), and there exists a positive constant d such that

$$\left|f(x,t,w_1) - f(x,t,w_2)\right| \le d\left|w_1 - w_2\right|. \tag{2.6}$$

3. Associated linear problem

In this section we study a linear problem related to (2.1)-(2.4) and establish the existence and uniqueness of a strong solution. Thus we consider

$$\frac{\partial u}{\partial t} - \frac{\partial}{\partial x}\left(a(x,t)\frac{\partial u}{\partial x}\right) = f(x,t), \tag{3.1}$$

with the initial conditions

$$lu = u(x,0) = \varphi\left(x\right), \qquad x \in (0,1), \tag{3.2}$$

the boundary condition

$$\frac{\partial u}{\partial x}(0,t) = \frac{\partial u}{\partial x}(1,t), \qquad \forall t \in (0,T), \tag{3.3}$$

and the integral condition

$$\int_0^1 u(x,t)dx = 0, \qquad t \in (0,T). \tag{3.4}$$

The problem (3.1)-(3.4) can be considered as a solving of the operator equation

$$Lu = (\pounds u, lu) = (f, \varphi) = \mathcal{F}, \tag{3.5}$$

where the operator L has domain of definition $D(L)$ consisting of functions $u \in L^2(\Omega)$ such that $\dfrac{\partial u}{\partial t}, \dfrac{\partial u}{\partial x}, \dfrac{\partial^2 u}{\partial x \partial t}, \dfrac{\partial^2 u}{\partial x^2} \in L^2(\Omega)$ and satisfying the conditions (3.3) and (3.4).

The operator L is an operator defined on E into F, where E is the Banach space of function $u \in L^2(\Omega)$, with the finite norm

$$\|u\|_E^2 = \int_\Omega |u(x,t)|^2 \, dxdt + \sup_t \int_0^1 |u(x,t)|^2 \, dx. \tag{3.6}$$

F is the Hilbert space of functions $\mathcal{F} =(f, \varphi)$, $f \in L^2(\Omega)$, $\varphi \in H^1\left(0,1\right)$ with the finite norm

$$\|\mathcal{F}\|_F^2 = \int_\Omega |f(x,t)|^2 \, dxdt + \int_0^1 |\varphi|^2 \, dx. \tag{3.7}$$

4. An energy inequality and their results

The following a priori estimate gives the uniqueness of the solution of the posed problem (3.1)-(3.4).

Theorem 4.1. *There exists a positive constant K, such that for each function $u \in D(L)$ we have*

$$\|u\|_E \le K \|Lu\|_F \,. \tag{4.1}$$

Proof 1. *Taking the scalar product in $L^2\left(\Omega^s\right)$ of equation (3.1) and the integro-differential operator*

$$Qu = e^{-ct}h\left(t\right)\left[a\left(0,t\right)\int_0^x d\zeta \int_0^\zeta \frac{\partial u}{\partial t}d\eta + a\left(1,t\right)\int_x^1 d\zeta \int_0^\zeta \frac{\partial u}{\partial t}d\eta\right],$$

where

$$h\left(t\right) = a(1,t) - a(0,t),$$

Integrating over $\Omega^s = [0,1] \times [0,s]$ with $0 \leq s \leq T$, and taking the real part

$$\Phi(u,u) = \mathrm{Re} \int_{\Omega^s} e^{-ct} f(x,t)\overline{Qu}dxdt$$
$$= \mathrm{Re} \int_{\Omega^s} e^{-ct} \frac{\partial u}{\partial t}\overline{Qu}dxdt - \mathrm{Re} \int_{\Omega^s} e^{-ct} \frac{\partial}{\partial x}\left(a(x,t)\frac{\partial u}{\partial x}\right)\overline{Qu}dxdt. \quad (4.2)$$

From conditions (3.2), (3.3) and the integral condition (3.4) and integrating by parts with respect to x, and t, we can evaluate each integral term on the right-hand side of (4.2), as follows

$$\mathrm{Re} \int_{\Omega^s} e^{-ct} f\overline{Qu}dxdt = \int_{\Omega^s} h^2(t)\, e^{-ct} \left|\int_0^x \frac{\partial u}{\partial t} d\zeta\right|^2 dxdt \quad (4.3)$$
$$- \int_{\Omega^s} \frac{2h(t)\frac{\partial h}{\partial t}a + h^2(t)\frac{\partial a}{\partial t} - ch^2(t)\,a}{2} e^{-ct}|u|^2 dxdt$$
$$+ \frac{1}{2}\int_0^1 h^2(t)\, a e^{-ct}|u|^2 dx\Big|_{t=s} - \int_0^1 \frac{h^2(0)\, a(x,0)}{2}|\varphi|^2 dx$$
$$+ \mathrm{Re} \int_{\Omega^s} h^2(t)\frac{\partial a}{\partial x} e^{-ct} u \overline{\int_0^x \frac{\partial u}{\partial t} d\zeta} dxdt.$$

Using Cauchy $\varepsilon-$ inequality, we obtain

$$- \mathrm{Re} \int_{\Omega^s} h^2(t)\frac{\partial a}{\partial x} e^{-ct} u \overline{\int_0^x \frac{\partial u}{\partial t} d\zeta} dxdt$$
$$\leq \int_{\Omega^s}^2 \left(\frac{\partial a}{\partial x}\right)^2 h^2(t)\, e^{-ct}|u|^2 dxdt + \frac{1}{4}\int_{\Omega^s} h^2(t)\, e^{-ct}\left|\int_0^x \frac{\partial u}{\partial t} d\zeta\right|^2 dxdt, \quad (4.4)$$

$$\mathrm{Re} \int_{\Omega^s} e^{-ct} f\overline{Qu}dxdt \leq a_1^2 \int_{\Omega^s} e^{-ct}|f|^2 dxdt + \frac{1}{4}\int_{\Omega^s} h^2(t)\, e^{-ct}\left|\int_0^x \frac{\partial u}{\partial t} d\zeta\right|^2 dxdt. \quad (4.5)$$

Combining (4.4) and (4.5) with (4.3), we get

$$\int_{\Omega^s} \frac{\left(ca - \frac{\partial a}{\partial t} - 2\left(\frac{\partial a}{\partial x}\right)^2\right)h^2(t) - 2h(t)\frac{\partial h}{\partial t}a}{2} e^{-ct}|u|^2 dxdt$$
$$+ \frac{1}{2}\int_0^1 h^2(t)\, a e^{-ct}|u|^2 dx\Big|_{t=s} + \int_{\Omega^s} \frac{h^2(t)}{2} e^{-ct}\left|\int_0^x \frac{\partial u}{\partial t} d\zeta\right|^2 dxdt$$
$$\leq 4a_1^2 \int_{\Omega^s} e^{-ct}|f|^2 dxdt + c_2 \int_0^1 \frac{h^2(0)\, a(x,0)}{2}|\varphi|^2 dx. \quad (4.6)$$

We suppose that

$$\delta \leq h^2(t),$$

and

$$\left|\frac{d}{dt}h^2(t)\right| = \left|2h(t)\frac{d}{dt}h(t)\right| = 2|h(t)|\left|\frac{d}{dt}h(t)\right| \leq 4a_1 \max\{|c_1|, |c_2|\}$$

.

Choosing c such that

$$c \geq \frac{8a_1^2 \max\{|c_1|, |c_2|\}}{\delta a_0} + \frac{c_2 + 2b^2}{a_0}, \quad (4.7)$$

and then last two terms in the left-hand side in (4.6) are non negatives, hence

$$\int_\Omega |u|^2\,dxdt + \int_0^1 |u|^2\,dx\bigg|_{t=s} \le m\left(\int_\Omega |f|^2\,dxdt + \int_0^1 |\varphi|^2\,dx\right),$$

where

$$m = \frac{\max\left(4a_1^2,\ 2a_1^3\right)}{\min(\frac{1}{2}\delta, \frac{1}{2}\delta a_0, \frac{(ca_0 - c_2 - 2b^2)\delta - 4a_1\max\{|c_1|,|c_2|\})}{2}}.$$

By taking the least upper bound of the left side with respect to s from 0 to T, we get the desired estimate (4.1) with $K = \sqrt{m}$.

Remark. *It can be proved in a standard way that the operator $L\colon E \to F$ is closable. Let $\overline{L}$ be the closure of this operator, with the domain of definition $D\left(\overline{L}\right)$.*

Definition 4.2. *The solution of the operator equation $\overline{L}u = \mathcal{F}$ is called a strong solution of problem (3.1)-(3.4).*

Remark. *The a priori estimate (4.1) can be extended to strong solutions after passing to limit, that is we have the inequality*

$$\|u\|_E \le c\left\|\overline{L}u\right\|_F, \qquad \forall u \in D\left(\overline{L}\right).$$

This last inequality implies the following corollaries.

Corollary 4.3. *If a strong solution of (3.1)-(3.4) exists, it is unique and depends continuously on $\mathcal{F} = (f, \varphi)$.*

Corollary 4.4. *The range $R\left(\overline{L}\right)$ of $\overline{L}$ is closed in F and $\overline{R\left(L\right)} = R\left(\overline{L}\right)$.*

5. Solvability of problem (2.1)-(2.4)

Corollary 4.4 shows that, to prove that problem (3.1)-(3.4) has a strong solution for arbitrary $\mathcal{F}$, it sufficient to prove that the set $R\left(L\right)$ is dense in F. The proof is based on the following lemma.

Lemma 5.1. *Suppose that $a(x,\,t)$ and its derivative $\dfrac{\partial^2 a}{\partial t \partial x}$ are bounded.*
Let $D_0(L) = \{u \in D(L), u(x,0) = 0\}$. If, for $u \in D_0(L)$ and for some function $v \in L^2\left(\Omega\right)$, we have

$$\int_\Omega \left(\frac{\partial u}{\partial t} - \frac{\partial}{\partial x}\left(a(x,t)\frac{\partial u}{\partial x}\right)\right)\overline{v}dxdt = 0. \tag{5.1}$$

Then v vanishes almost everywhere in Ω; that is $v = 0$.

Proof 2. *From (5.1) we have*

$$\int_\Omega \frac{\partial u}{\partial t}\overline{v}dxdt = \int_\Omega \frac{\partial}{\partial x}\left(a(x,t)\frac{\partial u}{\partial x}\right)\overline{v}dxdt, \tag{5.2}$$

We introduce the smoothing operators $J_\varepsilon^{-1} = \left(I - \varepsilon\dfrac{\partial}{\partial t}\right)^{-1}$ and $\left(J_\varepsilon^{-1}\right)^* = \left(I + \varepsilon\dfrac{\partial}{\partial t}\right)^{-1}$, with respect to t, then, these operators provide the solution of the problems:

$$\begin{cases} u_\varepsilon\left(t\right) - \varepsilon\dfrac{\partial u_\varepsilon}{\partial t} = u\left(t\right), & u_\varepsilon\left(0\right) = 0, \\ v_\varepsilon^*\left(t\right) + \varepsilon\dfrac{\partial v_\varepsilon^*}{\partial t} = v\left(t\right), & v_\varepsilon^*\left(T\right) = 0. \end{cases} \tag{5.3}$$

We also have the following properties: for any $g \in L^2(0,T)$, *the functions* $J_\epsilon^{-1} g$, $\left(J_\epsilon^{-1}\right)^* g \in W_2^1(0,T)$. *If* $g \in D(L)$, *then* $J_\epsilon^{-1} g \in D(L)$ *and we have*

$$\begin{cases} \lim \left\| J_\epsilon^{-1} g - g \right\|_{L^2(0,T)} = 0, & \text{for } \varepsilon \to 0, \\ \lim \left\| \left(J_\epsilon^{-1}\right)^* g - g \right\|_{L^2(0,T)} = 0, & \text{for } \varepsilon \to 0. \end{cases} \tag{5.4}$$

Substituting the function u *in (5.2) by the smoothing function* u_ε *and using the relation*

$$A(t)\, u_\varepsilon = J_\varepsilon^{-1} A(t)\, u - \varepsilon J_\varepsilon^{-1} B_\epsilon(t) u_\varepsilon,$$

where $B_\epsilon(t) = \dfrac{\partial A(t)}{\partial t} u_\epsilon$. *We obtain*

$$-\int_\Omega u \overline{\frac{\partial v_\epsilon^*}{\partial t}} dx dt = \int_\Omega A(t) u \overline{v_\epsilon^*} dx dt - \epsilon \int_\Omega B_\epsilon(t) u \overline{v_\epsilon^*} dx dt. \tag{5.5}$$

The operator $A(t)$ *has a continuous inverse in* $L^2(0,1)$ *defined by*

$$A^{-1}(t) g = \int_0^x \frac{d\zeta}{a(\zeta,t)} \int_0^\zeta g(\eta) d\eta + C_1(t) \int_0^x \frac{d\zeta}{a(\zeta,t)} + C_2(t), \tag{5.6}$$

where

$$\begin{cases} C_1(t) = \dfrac{a(0,t)}{a(1,t) - a(0,t)} \int_0^1 g(\eta) d\eta, \\ C_2(t) = -C_1(t) \int_0^1 \dfrac{1-x}{a(x,t)} dx - \int_0^1 \dfrac{1-x}{a(x,t)} dx \int_0^x g(\eta) d\eta. \end{cases} \tag{5.7}$$

Then we have $\int_0^1 A^{-1}(t) u = 0$, *hence, the function* $J_\epsilon^{-1} u = u_\varepsilon$ *can be represented in the form*

$$u_\varepsilon = J_\epsilon^{-1} A^{-1}(t) A(t) u.$$

The adjoint of $B_\epsilon(t)$ *has the form*

$$B_\epsilon^*(t) g = \frac{1}{a} \left(J_\epsilon^{-1}\right)^* \frac{\partial a}{\partial t} g + G_\epsilon(g)(x) + C_1(t) G_\epsilon(g)(1), \tag{5.8}$$

where

$$G_\epsilon(g)(x) = \int_x^1 \left[\frac{1}{a} \left(J_\epsilon^{-1}\right)^* \frac{\partial^2 a}{\partial t \partial \zeta} g - \frac{1}{a^2} \frac{\partial a}{\partial \zeta} \left(J_\epsilon^{-1}\right)^* \left(\frac{\partial a}{\partial t} g \right) \right] d\zeta.$$

Consequently, equality (5.5), becomes

$$-\int_\Omega u \overline{\frac{\partial v_\epsilon^*}{\partial t}} dx dt = \int_\Omega A(t) u \overline{h_\epsilon} dx dt, \tag{5.9}$$

where $h_\epsilon = v_\epsilon^* - \epsilon B_\epsilon^*(t) v_\epsilon^*$.
The left hand side of (5.9) is a continuous linear fonctionel of u, *hence the function* h_ϵ *has the derivatives* $\dfrac{\partial h_\epsilon}{\partial x}, \dfrac{\partial^2 h_\epsilon}{\partial x^2} \in L^2(\Omega)$ *and the following condition are satisfied*

$$\begin{cases} a(1,t)\, h_\epsilon(1,t) = a(0,t)\, h_\epsilon(0,t), \\ \dfrac{\partial h_\epsilon}{\partial x}(0,t) = \dfrac{\partial h_\epsilon}{\partial x}(1,t) = 0. \end{cases}$$

From the equality

$$\frac{\partial h_\epsilon}{\partial x} = \left[I - \epsilon \frac{1}{a} \left(J_\epsilon^{-1}\right)^* \left(\frac{\partial a}{\partial t} \right) \right] \frac{\partial v_\epsilon^*}{\partial x},$$

and since the operator $\left(J_{\epsilon}^{-1}\right)^{*}$ *is bounded in* $L^{2}(\Omega)$, *for sufficiently small* ϵ, *we have*

$$\left\|\epsilon\frac{1}{a}\left(J_{\epsilon}^{-1}\right)^{*}\left(\frac{\partial a}{\partial t}\right)\right\|_{L^{2}(\Omega)}<1.$$

Hence, the operator $I-\epsilon\frac{1}{a}\left(J_{\epsilon}^{-1}\right)^{*}\left(\frac{\partial a}{\partial t}\right)$ *has a bounded inverse in* $L^{2}(\Omega)$. *We conclude that* $\frac{\partial v_{\epsilon}^{*}}{\partial x},\ \frac{\partial^{2} v_{\epsilon}^{*}}{\partial x^{2}}\in L^{2}(\Omega)$, *and the following condition are satisfied*

$$\left\{\begin{array}{c} a\left(1,t\right)v_{\epsilon}^{*}(1,t)=a\left(0,t\right)v_{\epsilon}^{*}(0,t),\\ \frac{\partial v_{\epsilon}^{*}}{\partial x}\left(0,t\right)=\frac{\partial v_{\epsilon}^{*}}{\partial x}\left(1,t\right)=0. \end{array}\right. \tag{5.10}$$

Putting

$$u=\int_{0}^{t}e^{ct}d\tau\left(\lambda_{1}\int_{0}^{x}av_{\epsilon}^{*}\left(\eta,t\right)d\eta+\lambda_{2}v_{\epsilon}^{*}\left(x,t\right)+k\left(t\right)\right), \tag{5.11}$$

in (5.2), we obtain

$$\begin{aligned}&\int_{\Omega}e^{ct}\left(\lambda_{1}\int_{0}^{x}av_{\epsilon}^{*}\left(\eta,t\right)d\eta+\lambda_{2}v_{\epsilon}^{*}\left(x,t\right)+k\left(t\right)\right)\overline{v}dxdt\\ &=\int_{\Omega}A(t)u\overline{v_{\varepsilon}^{*}}dxdt+\varepsilon\int_{\Omega}A(t)u\overline{\frac{\partial v_{\epsilon}^{*}}{\partial t}}dxdt,\end{aligned} \tag{5.12}$$

where

$$\left\{\begin{array}{c} k\left(t\right)=-\int_{0}^{1}\left(\lambda_{1}\left(1-x\right)a+\lambda_{2}\right)v_{\epsilon}^{*}\left(x,t\right)dx,\\ \lambda_{2}>8\left|\lambda_{1}\right|a_{1}+2. \end{array}\right.$$

By integrating with respect to x *and using condition (5.10), we get*

$$\operatorname{Re}\int_{\Omega}A(t)u\overline{v_{\varepsilon}^{*}}dxdt=-\operatorname{Re}\int_{\Omega}a\frac{\partial u}{\partial x}\overline{\frac{\partial v_{\varepsilon}^{*}}{\partial x}}dxdt.$$

Using (5.11), the last equality can be represented as

$$\operatorname{Re}\int_{\Omega}A(t)u\overline{v_{\varepsilon}^{*}}dxdt=-\operatorname{Re}\int_{\Omega}\frac{a}{\lambda_{2}}e^{-ct}\frac{\partial u}{\partial x}\overline{\frac{\partial^{2}u}{\partial x\partial t}}dxdt+\operatorname{Re}\int_{\Omega}\frac{\lambda_{1}}{\lambda_{2}}a^{2}\frac{\partial u}{\partial x}\overline{v_{\varepsilon}^{*}}dxdt.$$

Integrating with respect to t, *we get*

$$\begin{aligned}\operatorname{Re}\int_{\Omega}A(t)u\overline{v_{\varepsilon}^{*}}dxdt=&-\int_{0}^{1}\frac{a\left(x,T\right)}{2\lambda_{2}}e^{-cT}\left|\frac{\partial u}{\partial x}\left(x,T\right)\right|^{2}dx\\ &+\int_{\Omega}\frac{a_{t}-ca}{2\lambda_{2}}e^{-ct}\left|\frac{\partial u}{\partial x}\right|^{2}dxdt+\operatorname{Re}\int_{\Omega}\frac{\lambda_{1}}{\lambda_{2}}a^{2}\frac{\partial u}{\partial x}\overline{v_{\varepsilon}^{*}}dxdt.\end{aligned}$$

Using $\varepsilon-$ *inequalities, we have*

$$\begin{aligned}&\operatorname{Re}\int_{\Omega}A(t)u\overline{v_{\varepsilon}^{*}}dxdt\\ &\leq-\int_{0}^{1}\frac{a\left(x,T\right)}{2\lambda_{2}}e^{-cT}\left|\frac{\partial u}{\partial x}\left(x,T\right)\right|^{2}dx+\int_{\Omega}\left(\frac{a_{t}-ca}{2\lambda_{2}}+2\left(\frac{\left|\lambda_{2}\right|}{\lambda_{1}}\right)^{2}a_{1}^{2}\right)e^{-ct}\left|\frac{\partial u}{\partial x}\right|^{2}dxdt\\ &+\frac{a_{1}^{2}}{4}\left(\frac{\left|\lambda_{1}\right|}{\lambda_{2}}\right)^{2}\int_{\Omega}e^{ct}\left|\overline{v_{\varepsilon}^{*}}-v\right|^{2}dxdt+\int_{\Omega}e^{ct}\left|v\right|^{2}dxdt\end{aligned} \tag{5.13}$$

By integrating with respect to x, t and using the condition (5.10), we get

$$\varepsilon \operatorname{Re} \int_{\Omega} A(t) u \overline{\frac{\partial v_\epsilon^*}{\partial t}} dxdt = \varepsilon \operatorname{Re} \int_{\Omega} a \frac{\partial^2 u}{\partial x \partial t} \overline{\frac{\partial v_\epsilon^*}{\partial x}} dxdt + \varepsilon \operatorname{Re} \int_{\Omega} \frac{\partial a}{\partial t} \frac{\partial u}{\partial x} \overline{\frac{\partial v_\epsilon^*}{\partial x}} dxdt.$$

Hence

$$\begin{aligned} &\varepsilon \operatorname{Re} \int_{\Omega} A(t) u \overline{\frac{\partial v_\epsilon^*}{\partial t}} dxdt \\ &\leq \varepsilon \left(a_1 + \max\left(|c_1|, \ |c_2|\right)\right) \int_{\Omega} \left(\left| \frac{\partial^2 u}{\partial x \partial t} \right|^2 + \left| \frac{\partial u}{\partial x} \right|^2 + \left| \frac{\partial v_\epsilon^*}{\partial x} \right|^2 \right) dxdt, \end{aligned} \tag{5.14}$$

Now from (5.13) and (5.14), we deduce that

$$\begin{aligned} &\operatorname{Re} \int_{\Omega} A(t) u \overline{v_\varepsilon^*} dxdt + \varepsilon \operatorname{Re} \int_{\Omega} A(t) u \overline{\frac{\partial v_\epsilon^*}{\partial t}} dxdt \\ &\leq -\int_0^1 \frac{a_0}{2\lambda_2} e^{-cT} \left| \frac{\partial u}{\partial x}(x,T) \right|^2 dx + \int_{\Omega} \left(\frac{c_2 - c a_1}{2\lambda_2} + 2 \left(\frac{|\lambda_1|}{\lambda_2} \right)^2 a_1^2 \right) e^{-ct} \left| \frac{\partial u}{\partial x} \right|^2 dxdt \\ &+ \int_{\Omega} e^{ct} |v|^2 dxdt + \frac{a_1^2}{4} \left(\frac{|\lambda_1|}{\lambda_2} \right)^2 \int_{\Omega} \left| \overline{v_\varepsilon^*} - v \right|^2 dxdt \\ &+ \varepsilon \left(a_1 + \max\left(|c_1|, |c_2|\right)\right) \int_{\Omega} \left(\left| \frac{\partial^2 u}{\partial x \partial t} \right|^2 + \left| \frac{\partial u}{\partial x} \right|^2 + \left| \frac{\partial v_\epsilon^*}{\partial x} \right|^2 \right) dxdt. \end{aligned} \tag{5.15}$$

From the left-hand side in (5.12), we have

$$\begin{aligned} &\operatorname{Re} \int_{\Omega} \frac{\partial u}{\partial t} \overline{v} dxdt \\ &= \int_{\Omega} \lambda_2 e^{ct} |v|^2 + \operatorname{Re} \int_{\Omega} \lambda_1 e^{ct} \overline{v} dxdt \int_0^x a v(\eta, t)\, d\eta \\ &- \lambda_1 \operatorname{Re} \int_0^T e^{ct} dt \int_0^1 \overline{v} dx \int_0^1 (1-x)\, a v(x,t)\, dx - \lambda_2 \operatorname{Re} \int_0^T e^{ct} dt \int_0^1 \overline{v} dx \int_0^1 v_\epsilon^*(x,t)\, dx \\ &- \operatorname{Re} \int_{\Omega} \lambda_2 e^{ct} v \left(v_\epsilon^* - v\right) dxdt + \operatorname{Re} \int_{\Omega} \lambda_1 e^{ct} \overline{v} dxdt \int_0^x a \left(v_\epsilon^*(\eta,t) - v(\eta,t)\right) d\eta \\ &- \lambda_1 \operatorname{Re} \int_0^T e^{ct} dt \int_0^1 \overline{v} dx \int_0^1 (1-x)\, a \left(v_\epsilon^*(\eta,t) - v(\eta,t)\right) dx. \end{aligned} \tag{5.16}$$

It is easy to show that

$$\operatorname{Re} \int_{\Omega} \lambda_1 e^{ct} \overline{v} dxdt \int_0^x a v(\eta,t)\, d\eta \leq |\lambda_1|\, a_1 \int_{\Omega} e^{ct} |v|^2 dxdt,$$

$$-\operatorname{Re} \int_{\Omega} \lambda_1 e^{ct} dt \int_0^1 \overline{v} dx \int_0^1 a v(\eta,t)\, d\eta \leq |\lambda_1|\, a_1 \int_{\Omega} e^{ct} |v|^2 dxdt,$$

$$-\operatorname{Re} \int_{\Omega} \lambda_2 e^{ct} \left(v_\epsilon^* - v\right) \overline{v} dxdt \leq \frac{\lambda_2}{2} \int_{\Omega} e^{ct} |v|^2 dxdt + \frac{\lambda_2}{2} \int_{\Omega} e^{ct} \left|v_\epsilon^* - v\right|^2 dxdt,$$

$$-\operatorname{Re} \int_{\Omega} \lambda_1 e^{ct} dt \int_0^1 \overline{v} dx \int_0^1 a \left(v_\epsilon^* - v\right) d\eta \leq |\lambda_1|\, a_1 \int_{\Omega} e^{ct} |v|^2 dxdt + \frac{\lambda_1 a_1}{4} \int_{\Omega} e^{ct} \left|v_\epsilon^* - v\right|^2 dxdt,$$

$$\operatorname{Re} \int_{\Omega} \lambda_1 e^{ct} \overline{v} dxdt \int_0^x a \left(v_\epsilon^* - v\right) d\eta \leq |\lambda_1|\, a_1 \int_{\Omega} e^{ct} |v|^2 dxdt + \frac{\lambda_1 a_1}{4} \int_{\Omega} e^{ct} \left|v_\epsilon^* - v\right|^2 dxdt.$$

From (5.3) we have

$$v_{\epsilon}^{*}(x,t)=\frac{-1}{\varepsilon}e^{\frac{-t}{\varepsilon}}\int_{t}^{T}e^{\frac{\tau}{\varepsilon}}v(x,\tau)\,d\tau.$$

Then

$$\begin{aligned}&-\lambda_2\operatorname{Re}\int_0^T e^{ct}dt\int_0^1\overline{v}dx\int_0^1 v_{\epsilon}^{*}(x,t)\,dx\\&=\frac{\lambda_2}{2\varepsilon}\int_0^T e^{\frac{2t}{\varepsilon}}\left|\int_0^1 vdx\right|^2dt+\frac{\lambda_2}{2\varepsilon}\left(c-\frac{2}{\varepsilon}\right)\int_0^T e^{\left(c-\frac{2}{\varepsilon}\right)t}dt\int_t^T e^{\frac{\tau}{\varepsilon}}d\tau\left|\int_0^1 vdx\right|^2.\end{aligned}$$

By integrating with respect to t, we have

$$\frac{\lambda_2}{2\varepsilon}\left(c-\frac{2}{\varepsilon}\right)\int_0^T e^{\left(c-\frac{2}{\varepsilon}\right)t}\int_t^T\left|\int_0^1 e^{\frac{\tau}{\varepsilon}}vd\tau dx\right|^2dt=\frac{\lambda_2}{2\varepsilon}\int_0^T\left|\int_0^1 vd\tau\right|^2\left[e^{ct}-e^{\frac{2t}{\varepsilon}}\right]dxdt.$$

Therefore,

$$-\lambda_2\operatorname{Re}\int_0^T e^{ct}dt\int_0^1\overline{v}dx\int_0^1 v_{\epsilon}^{*}(x,t)\,dx=\frac{\lambda_2}{2\varepsilon}\int_0^T e^{ct}\left|\int_0^1 vd\tau\right|^2.$$

If, we choose c, such that

$$\left[\frac{c_2}{a_0}+\frac{a_1^2}{a_0}\left(\frac{|\lambda_1|}{\lambda_2}\right)^2\right]\leq c.$$

Then from the previous inequalities and (5.15), we have

$$\begin{aligned}&\int_{\Omega}\left(\frac{\lambda_2}{2}-4|\lambda_1|a_1+1\right)e^{ct}|v|^2+\frac{\lambda_2}{2\varepsilon}\int_0^T e^{ct}\left|\int_0^1 vd\tau\right|^2\\&\leq\left(\frac{\lambda_2}{2}+\frac{\lambda_1a_1}{2}+\frac{a_1^2}{4}\left(\frac{|\lambda_1|}{\lambda_2}\right)^2\right)\int_{\Omega}e^{ct}|v_{\epsilon}^{*}-v|^2dxdt\\&+\varepsilon\left(a_1+\max\left(|c_1|,\ |c_2|\right)\right)\int_{\Omega}\left(\left|\frac{\partial^2u}{\partial x\partial t}\right|^2+\left|\frac{\partial u}{\partial x}\right|^2+\left|\frac{\partial v_{\epsilon}^{*}}{\partial x}\right|^2\right)dxdt.\end{aligned}$$

By passing to the limit as $\varepsilon\to 0$, we conclude that $v=0$. The proof of lemma is now completed.

Now, we give the main result in this section.

Theorem 5.2. *The range $R\left(\overline{L}\right)$ of the operator $\overline{L}$ coincides with F.*

Proof 3. *Since F is a Hilbert space, we have $R\left(\overline{L}\right)=F$ if and only if the relation*

$$\int_Q f\overline{g}dxdt+\int_0^1 lu\overline{\varphi_1}dx=0, \tag{5.17}$$

for arbitrary $u\in D(L)$ and $(g,\,\varphi_1)\in F$, implies that $g=0$ and $\varphi_1=0$.
Putting $u\in D_0(L)$ in (5.17), we conclude from the Lemma 5.1 that $g=v=0$, a.e.
Taking $u\in D(L)$ in (5.17) yields

$$\int_0^1 lu\overline{\varphi_1}dx=0.$$

Since the range of the trace operator l is everywhere dense in Hilbert space with the norm $\int_0^1 |lu|^2 dx$, see ([1]). It follows that $\varphi_1 = 0$. The proof of theorem is completed.

6. Study of the semilinear problem

In this section, we prove the existence, uniqueness and continuous dependance of the solution on the data of the problem (2.1)-(2.4).
If the solution of problem (2.1)-(2.4) exists, it can be expressed in the form $u = w + U$, where
U is a solution of the homogeneous problem

$$\pounds U = \frac{\partial U}{\partial t} - \frac{\partial}{\partial x}\left(a\frac{\partial U}{\partial x}\right) = 0, \tag{6.1}$$

$$U(x,0) = \varphi(x), \tag{6.2}$$

$$\frac{\partial U}{\partial x}(0,t) = \frac{\partial U}{\partial x}(1,t), \tag{6.3}$$

$$\int_0^1 U(x,t)\,dx = 0. \tag{6.4}$$

And w is a solution of the problem

$$\pounds w = \frac{\partial w}{\partial t} - \frac{\partial}{\partial x}\left(a\frac{\partial w}{\partial x}\right) = F(x,t,w)\ , \tag{6.5}$$

$$w(x,0) = 0, \tag{6.6}$$

$$\frac{\partial w}{\partial x}(0,t) = \frac{\partial w}{\partial x}(1,t), \tag{6.7}$$

$$\int_0^1 w(x,t)\,dx = 0, \tag{6.8}$$

where $F(x,t,w) = f(x,t,w)$ and satisfied the condition

$$|F(x,t,u_1) - F(x,t,u_2)| \leq d\,|u_1 - u_2| \quad \text{for all } x,\ t \in \Omega. \tag{6.9}$$

According to Theorem 4.1 and Lemma 5.1, the problem (6.1)-(6.4) has a unique solution that depend continuously on $\varphi \in L^2(0,1)$ where $L^2(0,1)$ is a Hilbert space with the scalar product

$$(u,v)_{L^2(0,1)} = \int_0^1 u\bar{v}dx,$$

and with associated norm

$$\|u\|^2_{L^2(0,1)} = \int_0^1 |u|^2\,dx.$$

We shall prove that the problem (6.5)-(6.8) has a weak solution by using an approximation process and passage to the limit.
Assume that v and $w \in C^1(\Omega)$, and the following conditions are satisfied

$$\begin{cases} v(x,T) = 0,\ \int_0^1 v(x,t)\,dx = 0, \\ w(x,0) = 0,\ \dfrac{\partial w}{\partial x}(0,t) = \dfrac{\partial w}{\partial x}(1,t). \end{cases} \tag{6.10}$$

Taking the scalar product in $L^2(\Omega)$ of equation (6.5) and the integro-differential operator

$$Mv = h(t)\left(a(0,t)\int_0^x d\zeta\int_0^\zeta v d\eta + a(1,t)\int_x^1 d\zeta\int_0^\zeta v d\eta\right),$$

by taking the real part, we obtain

$$\begin{aligned} H(w,v) &= \operatorname{Re}\int_\Omega F(x,t,w)\,\overline{Mv}dxdt \\ &= \operatorname{Re}\int_\Omega \frac{\partial w}{\partial t}\overline{Mv}dxdt - \operatorname{Re}\int_\Omega \frac{\partial}{\partial x}\left(a\frac{\partial w}{\partial x}\right)\overline{Mv}dxdt. \end{aligned} \tag{6.11}$$

Substituting the expression of Mv in the first integral of the right-hand side of (6.11), integrating with respect to t, using the condition (6.10), we get

$$\begin{aligned} &\operatorname{Re}\int_\Omega \frac{\partial w}{\partial t}\overline{Mv} \\ &= -\operatorname{Re}\int_\Omega w\left(a(0,t)\,h(t)\int_0^x d\zeta\int_0^\zeta \overline{\frac{\partial v}{\partial t}}d\eta + a(1,t)h(t)\int_x^1 d\zeta\int_0^\zeta \overline{\frac{\partial v}{\partial t}}d\eta\right)dxdt \\ &- \operatorname{Re}\int_\Omega w\left[\left(h(t)\frac{\partial a}{\partial t}(0,t) + a(0,t)\frac{\partial h}{\partial t}\right)\int_0^x d\zeta\int_0^\zeta \overline{v}d\eta + \left(h(t)\frac{\partial a}{\partial t}(1,t) + a(1,t)\frac{\partial h}{\partial t}\right)\int_x^1 d\zeta\int_0^\zeta \overline{v}d\eta\right]. \end{aligned} \tag{6.12}$$

Substituting the expression of Mv in the second integral of the right-hand side of (6.11), integrating with respect to x, using the condition (6.10), we get

$$-\operatorname{Re}\int_\Omega \frac{\partial}{\partial x}\left(a\frac{\partial w}{\partial x}\right)\overline{Mv}dxdt = \operatorname{Re}\int_\Omega h^2(t)w\left(\frac{\partial a}{\partial x}\int_0^x \overline{v}d\eta + a\overline{v}\right)dxdt. \tag{6.13}$$

Insertion of (6.12), (6.13) into (6.11) yields

$$H(w,v) = \operatorname{Re}\int_\Omega h(t)\overline{v}\left(a(1,t)\int_x^1 d\zeta\int_0^\zeta F(\eta,t,w)\,d\eta - a(0,t)\int_0^x d\zeta\int_\zeta^1 F(\eta,t,w)\,d\eta\right), \tag{6.14}$$

where

$$\begin{aligned} &H(w,v) \\ &= \operatorname{Re}\int_\Omega h^2(t)w\left(\frac{\partial a}{\partial x}\int_0^x \overline{v}d\eta + a\overline{v}\right)dxdt \\ &- \operatorname{Re}\int_Q h(t)w\left(a(0,t)\int_0^x d\zeta\int_0^\zeta \overline{\frac{\partial v}{\partial t}}d\eta + a(1,t)\int_x^1 d\zeta\int_0^\zeta \overline{\frac{\partial v}{\partial t}}d\eta\right)dxdt \\ &- \operatorname{Re}\int_\Omega w\left[\left(h(t)\frac{\partial a}{\partial t}(0,t) + a(0,t)\frac{\partial h}{\partial t}\right)\int_0^x d\zeta\int_0^\zeta \overline{v}d\eta + \left(h(t)\frac{\partial a}{\partial t}(1,t) + a(1,t)\frac{\partial h}{\partial t}\right)\int_x^1 d\zeta\int_0^\zeta \overline{v}d\eta\right]. \end{aligned} \tag{6.15}$$

Definition 6.1. *By a weak solution of problem (6.5)-(6.8) we mean a function $w \in L^2\left(0,\, T : L^2(0,\,1)\right)$ satisfying the identity (6.14) and the integral condition (6.8).*

We will construct an iteration sequence in the following way.
Starting with $w_0 = 0$, the sequence $(w_n)_{n\in\mathbb{N}}$ is defined as follows: given w_{n-1}, then for $n \geq 1$, we solve the problem

$$\pounds w_n = \frac{\partial w_n}{\partial t} - \frac{\partial}{\partial x}\left(a\frac{\partial w_n}{\partial x}\right) = F\left(x,t,w_{n-1}\right), \tag{6.16}$$

$$w_n\left(x,0\right) = 0, \tag{6.17}$$

$$\frac{\partial w_n}{\partial x}\left(0,t\right) = \frac{\partial w_n}{\partial x}\left(1,t\right), \tag{6.18}$$

$$\int_0^1 w_n\left(x,t\right)dx = 0. \tag{6.19}$$

From Theorem 4.1 and Lemma 5.2, we deduce that for fixed n, each problem (6.16)-(6.19) has a unique solution $w_n\left(x,t\right)$. If we set $V_n\left(x,t\right) = w_{n+1}\left(x,t\right) - w_n\left(x,t\right)$, we obtain the new problem

$$\pounds V_n = \frac{\partial V_n}{\partial t} - \frac{\partial}{\partial x}\left(a\frac{\partial V_n}{\partial x}\right) = \sigma_{n-1}, \tag{6.20}$$

$$V_n\left(x,0\right) = 0, \tag{6.21}$$

$$\frac{\partial V_n}{\partial x}\left(0,t\right) = \frac{\partial V_n}{\partial x}\left(1,t\right), \tag{6.22}$$

$$\int_0^1 V_n\left(x,t\right)dx = 0, \tag{6.23}$$

where

$$\sigma_{n-1} = F\left(x,t,w_n\right) - F\left(x,t,w_{n-1}\right). \tag{6.24}$$

Lemma 6.2. *Assume that the condition (6.9) holds, for the linearized problem (6.20)-(6.23), there exists a positive constant k, such that*

$$\left\|V_n\right\|_{L^2(0,\ T:\ L^2(0,1))} \leq k\left\|V_{n-1}\right\|_{L^2(0,T:\ L^2(0,1))}. \tag{6.25}$$

Proof 4. *We denote by*

$$QV_n = a\left(0,t\right)h\left(t\right)\int_0^x d\zeta\int_0^\zeta \frac{\partial V_n}{\partial t}d\eta + a\left(1,t\right)h\left(t\right)\int_x^1 d\zeta\int_0^\zeta \frac{\partial V_n}{\partial t}d\eta.$$

We consider the quadratic form obtained by multiplying equation (6.20) by $e^{-ct}\overline{QV_n}$, with the constant c satisfying (4.7), integrating over $\Omega_s = [0,1]\times[0,s]$, with $0 \leq s \leq T$, taking the real part, we obtain

$$\begin{aligned}\Phi\left(V_n,V_n\right) &= \mathrm{Re}\int_{\Omega_s}\sigma_{n-1}\overline{QV_n}dxdt\\ &= \mathrm{Re}\int_{\Omega_s}e^{-ct}\frac{\partial V_n}{\partial t}\overline{QV_n}dxdt - RE\int_{\Omega_s}e^{-ct}\frac{\partial}{\partial x}\left(a\frac{\partial V_n}{\partial x}\right)\overline{QV_n}dxdt.\end{aligned} \tag{6.26}$$

By integrating with respect to x, t and using the conditions (6.21), (6.22) and (6.23) we get

$$
\begin{aligned}
&-\operatorname{Re}\int_{\Omega_s} e^{-ct}\frac{\partial}{\partial x}\left(a\frac{\partial V_n}{\partial x}\right)\overline{QV_n}dxdt \\
&=\frac{1}{2}\int_{\Omega_s}\left(-\frac{\partial\left(ah^2(t)\right)}{\partial t}+cah^2(t)\right)e^{-ct}\left|V_n\right|^2dxdt+\frac{1}{2}\int_0^1 ah^2(t)e^{-ct}\left|V_n\right|^2dx\Big|_{t=s} \\
&+\operatorname{Re}\int_{\Omega_s} e^{-ct}h^2(t)\frac{\partial a}{\partial x}V_n\int_0^x\overline{\frac{\partial V_n}{\partial t}}d\zeta dxdt. \qquad (6.27)
\end{aligned}
$$

By integrating with respect to x and using the condition (6.23), we get

$$
\operatorname{Re}\int_{\Omega_s} e^{-ct}\frac{\partial V_n}{\partial t}\overline{QV_n}dxdt=\int_{\Omega_s} h^2(t)e^{-ct}\left|\frac{\partial V_n}{\partial t}\right|^2dxdt. \qquad (6.28)
$$

Combined (6.27) and (6.28) with (6.26), we obtain

$$
\begin{aligned}
&\frac{1}{2}\int_{\Omega_s}\left(-2ah(t)\frac{\partial h}{\partial t}+\left(ca-\left(\frac{\partial a}{\partial x}\right)^2-\frac{\partial a}{\partial t}\right)h^2(t)\right)e^{-ct}\left|V_n\right|^2dxdt \\
&+\frac{1}{2}\int_{\Omega_s} h^2(t)e^{-ct}\left|\frac{\partial V_n}{\partial t}\right|^2dxdt+\int_0^1\frac{ah^2(t)}{2}e^{-ct}\left|V_n\right|^2dx\Big|_{t=s} \\
&\leq\operatorname{Re}\int_{\Omega^s} e^{-ct}\sigma_{n-1}\overline{QV_n}dxdt. \qquad (6.29)
\end{aligned}
$$

Following the same procedure done in establishing the proof of Theorem 4.1, we have

$$
\begin{aligned}
&\operatorname{Re}\int_{\Omega^s} e^{-ct}\sigma_{n-1}\overline{QV_n}dxdt \\
&\leq 8a_1^2\int_{\Omega^s} exp(-ct)\left|\sigma_{n-1}\right|^2dxdt+\frac{1}{2}\int_{\Omega^s} h^2(t)\,exp(-ct)\left|\int_0^x\frac{\partial V_n}{\partial t}d\zeta\right|^2dxdt. \qquad (6.30)
\end{aligned}
$$

Combining the previous inequalities with (6.29), using (6.9) we have

$$
\left\|V_n\right\|^2_{L^2(0,T:L^2(0,1))}\leq k^2\left\|V_{n-1}\right\|^2_{L^2(0,T:L^2(0,1))}, \qquad (6.31)
$$

where

$$
k^2=\frac{16d^2a_1^2}{\min(\delta a_0,(ca_0-c_2-2b^2)\,\delta-4a_1\max\{|c_1|,|c_2|\})}e^{cT}.
$$

Since $V_n(x,t)=w_{n+1}(x,t)-w_n(x,t)$, then the sequence $w_n(x,t)$ can be written as follows

$$
w_n(x,t)=\sum_{k=1}^{k=n-1}V_k+w_0(x,t).
$$

The sequence $w_n(x,t)$ converge to an element $w\in L^2\left(0,\,T:L^2(0,\,1)\right)$ if

$$
d^2<\frac{\min(\delta a_0,\left(ca_0-c_2-2b^2\right)-4a_1\max\{|c_1|,|c_2|\})}{16a_1^2}e^{-cT}.
$$

Now to prove that this limit function w is a solution of the problem under consideration (6.20)-(6.23), we should show that w satisfies (6.8) and (6.14).
For problem (6.16)-(6.19), we have

$$\begin{aligned}&H(w_n-w,v)+H(w,v)\\&=\operatorname{Re}\int_\Omega a(1,t)h(t)\overline{v}\int_x^1 d\zeta\int_0^\zeta\left(F(\eta,t,w_{n-1})-F(\eta,t,w)\,d\eta\right)dxdt\\&-\operatorname{Re}\int_Q a(0,t)h(t)\int_0^x d\zeta\int_\zeta^1\left(F(\eta,t,w_{n-1})\,d\eta-F(\eta,t,w)\,d\eta\right)dxdt\\&+\int_\Omega a(1,t)h(t)\overline{v}\int_x^1 d\zeta\int_0^\zeta F(\eta,t,w)\,d\eta-a(0,t)h(t)\int_0^x d\zeta\int_\zeta^1 F(\eta,t,w)\,d\eta.\end{aligned}\tag{6.32}$$

From the equation (6.16), we have

$$\begin{aligned}&H(w_n-w,v)\\&=\operatorname{Re}\int_\Omega\frac{\partial(w_n-w)}{\partial t}\left(a(0,t)h(t)\int_0^x d\zeta\int_0^\zeta V_n d\eta+a(1,t)h(t)\int_x^1 d\zeta\int_0^\zeta V_n d\eta\right)dxdt\\&-\operatorname{Re}\int_\Omega\frac{\partial}{\partial x}\left(a\frac{\partial(w_n-w)}{\partial x}\right)\left(a(0,t)h(t)\int_0^x d\zeta\int_0^\zeta V_n d\eta+a(1,t)h(t)\int_x^1 d\zeta\int_0^\zeta V_n d\eta\right)dxdt.\end{aligned}$$

Integrating with respect to t and x, using the conditions (6.10), we obtain

$$\begin{aligned}&H(w_n-w,v)\\&=\operatorname{Re}\int_\Omega\overline{\frac{\partial v}{\partial t}}\left(a(0,t)h(t)\int_0^x d\zeta\int_\zeta^1(w_n-w)(\eta,t)\,d\eta+a(1,t)h(t)\int_0^x d\zeta\int_0^\xi(w_n-w)(\eta,t)\,d\eta\right)dxdt\\&+\operatorname{Re}\int_\Omega\overline{v}\left(\frac{d\left(a(0,t)h(t)\right)}{dt}\int_0^x d\zeta\int_\zeta^1(w_n-w)(\eta,t)\,d\eta+\frac{d\left(a(1,\,t)h(t)\right)}{dt}\int_0^x d\zeta\int_0^\xi(w_n-w)(\eta,t)\,d\eta\right)dxdt\\&+\operatorname{Re}\int_\Omega h^2(t)a(w_n-w)\,\overline{v}dxdt+\operatorname{Re}\int_\Omega h^2(t)\frac{\partial a}{\partial x}(w_n-w)\int_0^x v d\zeta dxdt.\end{aligned}\tag{6.33}$$

Each terms of the left-hand side of (6.33) is controlled by

$$\begin{aligned}&\operatorname{Re}\int_\Omega\overline{\frac{\partial v}{\partial t}}\left(a(0,t)h(t)\int_0^x d\zeta\int_\zeta^1(w_n-w)(\eta,t)\,d\eta+a(1,t)h(t)\int_0^x d\zeta\int_0^\xi(w_n-w)(\eta,t)\,d\eta\right)\\&\le k_1\left(\int_\Omega|w_n-w|^2\,dxdt\right)^{\frac12}\left(\int_\Omega\left|\frac{\partial v}{\partial t}\right|^2dxdt\right)^{\frac12},\end{aligned}\tag{6.34}$$

where

$$k_1=\sup_t\left\{a(0,t)h(t),a(1,t)h(t)\right\}.$$

And

$$\operatorname{Re}\int_{\Omega}\overline{v}\left(\frac{d\left(a(0,t)h(t)\right)}{dt}\int_{0}^{x}d\zeta\int_{\zeta}^{1}\left(w_{n}-w\right)\left(\eta,t\right)d\eta+\frac{d\left(a(1,t)h(t)\right)}{dt}\int_{0}^{x}d\zeta\int_{0}^{\xi}\left(w_{n}-w\right)\left(\eta,t\right)d\eta\right)dxdt$$

$$\leq k_{2}\left(\int_{\Omega}\left|w_{n}-w\right|^{2}dxdt\right)^{\frac{1}{2}}\left(\int_{Q}\left|v\right|^{2}dxdt\right)^{\frac{1}{2}}, \tag{6.35}$$

where

$$k_{2}=\sup_{t}\left\{\frac{d}{dt}a(0,t)h(t),\frac{d}{dt}a(1,t)h(t)\right\},$$

And the last term by

$$\operatorname{Re}\int_{\Omega}h^{2}(t)a\left(w_{n}-w\right)\overline{v}dxdt+\operatorname{Re}\int_{\Omega}h^{2}(t)\frac{\partial a}{\partial x}\left(w_{n}-w\right)\int_{0}^{x}vd\zeta dxdt$$

$$\leq\sup_{t}\left(a_{1}h^{2}(t),bh^{2}(t)\right)\left(\int_{\Omega}\left|\left(w_{n}-w\right)\right|^{2}dxdt\right)^{\frac{1}{2}}\left(\int_{Q}\left|v\right|^{2}dxdt\right)^{\frac{1}{2}}. \tag{6.36}$$

From (6.34)-(6.36), we deduce that

$$\left|H\left(w_{n}-w,v\right)\right|\leq C\left\|w_{n}-w\right\|_{L^{2}(0,T:L^{2}(0,1))}\left(\int_{\Omega}\left|\frac{\partial v}{\partial t}\right|^{2}+\left|v\right|^{2}dxdt\right)^{\frac{1}{2}}, \tag{6.37}$$

where

$$C=3\max\left(k_{1},k_{2},\sup_{t}\left(a_{1}h^{2}(t),bh^{2}(t)\right)\right).$$

Integrating with respect to x, using (6.10), we obtain

$$\begin{aligned}
&\operatorname{Re}\int_{\Omega}a(1,t)h(t)\overline{v}\int_{x}^{1}d\zeta\int_{0}^{\zeta}\left(F\left(\eta,t,w_{n-1}\right)-F\left(\eta,t,w\right)d\eta\right)dxdt\\
&-\operatorname{Re}\int_{\Omega}a(0,t)h(t)\overline{v}\int_{0}^{x}d\zeta\int_{\zeta}^{1}\left(F\left(\eta,t,w_{n-1}\right)-F\left(\eta,t,w\right)d\eta\right)dxdt\\
&=\operatorname{Re}\int_{\Omega}a(1,t)h(t)\left[F\left(x,t,w_{n-1}\right)-F\left(x,t,w\right)\right]\int_{x}^{1}d\eta\int_{0}^{\eta}vd\zeta dxdt\\
&+\operatorname{Re}\int_{\Omega}a(0,t)h(t)\left[F\left(x,t,w_{n-1}\right)-F\left(x,t,w\right)\right]\int_{0}^{x}d\eta\int_{0}^{\eta}vd\zeta dxdt.
\end{aligned}$$

Using the condition (6.9) and the Cauchy-Schwartz inequality, we obtain

$$\begin{aligned}
&\operatorname{Re}\int_{\Omega}a(1,t)h(t)\overline{v}\int_{x}^{1}d\zeta\int_{0}^{\zeta}\left(F\left(\eta,t,w_{n-1}\right)-F\left(\eta,t,w\right)d\eta\right)dxdt\\
&+\operatorname{Re}\int_{\Omega}a(1,t)h(t)\overline{v}\int_{x}^{1}d\zeta\int_{0}^{\zeta}\left(F\left(\eta,t,w_{n-1}\right)-F\left(\eta,t,w\right)d\eta\right)dxdt\\
&\leq 2k_{1}d\left\|w_{n}-w\right\|_{L^{2}(0,T:L^{2}(0,1))}\left(\int_{\Omega}\left|v\right|^{2}dxdt\right)^{\frac{1}{2}}.
\end{aligned} \tag{6.38}$$

From (6.37), (6.38) and passing to the limit in (6.32) as $n \to +\infty$, we deduce that

$$H(w,v) = \int_Q a(1,t)h(t)\overline{v}\int_x^1 d\zeta \int_0^\zeta F(\eta,t,w)\,d\eta - a(0,t)h(t)\int_0^x d\zeta \int_\zeta^1 F(\eta,t,w)\,d\eta.$$

Now we show that (6.8) holds. Since $\lim_{n\to+\infty}\|w_n - w\|_{L^2(0,T:L^2(0,1))} = 0$, *then*

$$\lim_{n\to+\infty}\left|\int_0^1 (w_n - w)\,dx\right| \le \lim_{n\to+\infty}\int_0^1 |w_n - w|^2\,dx \to 0. \tag{6.39}$$

From (6.39) we conclude that $\int_0^1 w dx = 0$.

Thus, we have proved the following

Theorem 6.3. *If condition (6.9) is satisfied, then the solution of problem (6.5)-(6.8) is unique.*

Proof 5. *Suppose that* w_1, $w_2 \in L^2\left(0,T : L^2(0,1)\right)$ *are two solution of (6.5)-(6.8), the function* $v = w_1 - w_2$ *is in* $L^2\left(0,T : L^2(0,1)\right)$ *and satisfies*

$$\frac{\partial v}{\partial t} - \frac{\partial}{\partial x}\left(a\frac{\partial v}{\partial x}\right) = G(x,t), \tag{6.40}$$

$$v(x,0) = 0, \tag{6.41}$$

$$\frac{\partial v}{\partial x}(0,t) = \frac{\partial v}{\partial x}(1,t), \tag{6.42}$$

$$\int_0^1 v dx = 0, \tag{6.43}$$

where $G(x,t) = F(x,t,w_1) - F(x,t,w_2)$.
Taking the inner product in $L^2(\Omega)$ *of equation (6.40) and the integro-differential operator*

$$Qv = a(0,t)\,h(t)\int_0^x d\zeta \int_0^\zeta \frac{\partial v}{\partial t}d\eta + a(1,t)\,h(t)\int_x^1 d\zeta \int_0^\zeta \frac{\partial v}{\partial t}d\eta$$

where $\lambda > 2a_1$ *and following the same procedure done in establishing the proof of Lemma 6.2, we get*

$$\|v\|^2_{L^2(0,T:L^2(0,1))} \le k^2 \|v\|^2_{L^2(0,T:L^2(0,1))},$$

where

$$k^2 = \frac{16a_1^2}{\min(\delta a_0, (ca_0 - c_2 - 2b^2)\,\delta - 4a_1 \max\{|c_1|, |c_2|\})}e^{cT}.$$

Since $k^2 < 1$, *then* $v = 0$, *which implies that* $w_1 = w_2 \in L^2\left(0,T : L^2(0,1)\right)$.

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References

[1] R. A. Adams, *Sobolev spaces*, Academic Press, New York, 1975.
[2] G. W. J. Batten, Second-order correct boundary conditions for the numerical solution of mixed boundary problem for parabolic equations, *Math. Comp.* **1963**, No. 17, 405–413.
[3] Z. P. Bazant, M. Jirasek, Nonlocal integral formulations of plasticity and damage: survey of progress, *Journal of engineering mechanics* **2002**, No. 128 (11), 1119–1149.
[4] N. E. Benouar, N. I. Yurchuk, A mixed problem with an integral condition for parabolic equations with a Bessel operator, *Differ. Equ.* **1991**, No. 27, 1482–1487.

[5] S. A. Beilin, Existence of solutions for one-dimensional wave equation with nonlocal conditions, *Electron. J. Diff. Equ.* **2001**, No. 76, 1–8.
[6] I. Belmouloud, A. Memou, On the solvability of a class of nonlinear singular parabolic equation with integral boundary condition, *Applied Mathematics and Computation* **2020**, No. 373, 124999, 12 pp.
[7] J. R. Cannon, The solution of the heat equation subject to the specification of energy, *Quart. Appl. Math.* **1963**, No. 21, 155–160.
[8] J. R. Cannon, *The one-dimensional heat equation*, No. 23. Cambridge University Press, 1984.
[9] V. Cappasso, K. Kunisch, A reaction-diffusion system arising in modelling man-environment diseases, *Quart. Appl. Math* **1986**, No. 46, 431–450.
[10] B. Cahlon, D. M. Kulkarni and P. Shi, Stepwise stability for the heat equation with nonlocal constraint, *SIAM J. Numer. Anal.* **1995**, No. 32, 571–593.
[11] Y. S. Choi, K. Y. Chan, A parabolic equation with nonlocal boundary conditions arising from electrochemistry, *Nonlinear Analysis: Theory, Methods and Applications* **1992**, No. 18, 317–331.
[12] J. H. Cushman, B. X. Hu and F. Deng , Nonlocal reactive transport with physical and chemical heterogeneity: localization errors, *Water Resources Research* **1995**, No. 31, 2219–2237.
[13] M. Denche, A. L. Marhoune, A three-point boundary value problem with an integral condition for parabolic equations with the Bessel operator, *Applied Mathematics Letters* **2000**, No. 13, 85–89.
[14] M. Denche, A. L. Marhoune, High order mixed-type differential equations with weighted integral boundary conditions, *Electron. J. Differ. Equ.* **2000**, No. 60, 1–10.
[15] M. Denche, A. L. Marhoune, Mixed problem with nonlocal boundary conditions for a third order partial differential equation of mixed type, *Int. J. Math. Math. Sci.* **2001**, No. 26, 417–426.
[16] M. Denche, A. L. Marhoune, Mixed problem with integral boundary condition for a high order mixed type partial differential equation, *Journal of Applied Mathematics and Stochastic Analysis* **2003**, No. 16, 69–79.
[17] M. Denche, A. Memou, Boundary value problem with integral conditions for a linear third order equation, *J. Applied Math.* **2003**, No. 2003, 553–567.
[18] A. K. Gushchina, V. P. Mikhailov , On solvability of nonlocal problems for a second-order elliptic equation, *Matem. Sbornik* **1994**, No. 185, 121–160.
[19] N. I. Ionkin, The solution of a certain boundary value problem of the theory of heat conduction with a nonclassical boundary condition, *Differ. Uravn.* **1977**, No. 13, 294–304.
[20] L. I. Kamynin, A boundary value problem in the theory of heat conduction with a nonclassical boundary condition, *Th.Vychisl. Mat. Fiz.* **1964**, No. 43, 1006–1024.
[21] A. V. Kartynnik, A three-point mixed problem with an integral condition with respect to the space variable for second-order parabolic equations, *Differ. Equ.* **1990**, No. 26, 1060–1166.
[22] C. Latrous, A. Memou, A three- point Boundary Value Problem with an Integral condition for a third order partial differential equation, *Abstract and Applied Analysis* **2005**, No. 1, 33–43.
[23] A. L. Marhoune, A three-point boundary value problem with an integral two-space variables condition for parabolic equations, *Computers and Mathematics with Applications* **2007**, No. 53, 940–947.
[24] A. L. Marhoune, A. Memou, Nonlocal singular problem with integral condition for second-order parabolic equation, *Differ. Equ.* **2015**, No. 64, 1–9.
[25] A. M. Nakhushev, An approximate method for solving boundary value problems for differential equations and its application to the dynamics of ground moisture and ground water, *Differ. Uravn.* **1982**, No. 18, 72–81.
[26] B. P. Paneyakh , Certain nonlocal boundary-value problems for linear differential operators, *Mat. Zametki* **1984**, No. 35, 425–434.
[27] L. S. Pulkina, A non-local problem with integral conditions for hyperbolic equations, *Electron. J. Differ. Equ.* **1999**, No. 45, 1–6.
[28] L. S. Pulkina, A non-local problem with integral conditions for the quasilinear hyperbolic equation, *Mathematical Notes* **2001**, No. 70, 79–85.

[29] L. S. Pulkina, A Mixed Problem With Integral Condition for the Hyperbolic Equation, *Mathematical Notes* **2003**, No. 74, 411–421.
[30] L. S. Pulkina, Nonlocal problems for hyperbolic equations from the viewpoint of strongly regular boundary conditions, *Electron. J. Differ. Equ.* **2020**, No. 28, 1–20.
[31] A. A. Samarskii, Some problems of the theory of differential equations, *Differ. Uravn.* **16**(1980), No. 11, 1225–1235.
[32] P. Shi, Weak solution to evolution problem with a nonlocal constraint, *SIAM J. Math. Anal.* **1993**, No. 24, 46–58.
[33] A. L. Skubachevskii, G. M. Steblov, On spectrum of differential operators with domain non-dense in L_2, *Doklad. Akad. Nauk. USSR.* **1991**, No. 321, 1158–1163.
[34] V. A. Vogahova, A boundary value problem with A. M. Nakhushev's nonlocal condition for a pseudoparabolic equation of moisture transfer , *Differ. Uravn.* **1982**, No. 18, 280–285.
[35] V. F. Volkodavov, V. E. Zhukov., Two problems for the string vibration equation with integral conditions and special conjugation conditions on the characteristic, *Differ. Equ.* **1998**, No. 34, 501–505.
[36] N. I. Yurchuk, Mixed problem with an integral condition for certain parabolic equations, *Differ. Equ.* **1986**, No. 22, 1457–1463.

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