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MULTIDIMENSIONAL SUMMABILITY PROCESS ON BASKAKOV-TYPE APPROXIMATION

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Abstract. In 1973 Bell presented a notion of convergence that is weaker than the usual notion and a series of theorems and summability methods using his notion. Following Bell's work Aslan and Duman extended this notion to approximation theory and likewise presented a series of natural results. In this paper we extended Aslan and Duman results to double sequences via four dimensional summability matrices. To accomplish this extension we begin with the notion of Pringsheim convergence. Using Pringsheim convergence the authors presented a series of multidimensional approximation theorems. These approximation results are natural extends and improvements of both Baskakov and Korovkin notions of approximation.

1. Preliminaries

Throughout this article, we will use the following notations and definitions. Let $x = [x_{k,l}]$ be a complex double sequence and let A denote an infinite four dimensional matrix with complex entries. Then Ax is the transformed double sequence whose mn-th term is $(Ax)_{m,n} = \sum_{k,l=1,1}^{\infty,\infty} a_{m,n,k,l}x_{k,l}$. Additionally, we let $\widetilde{x}$ and $\widetilde{A}$ denote single dimensional sequences and ordinary Summability matrix, respectively, with the following transformation $(\widetilde{A}\widetilde{x})_n = \sum_{k=1}^{\infty} \widetilde{a}_{n,k}\widetilde{x}_k$.

Definition 1.1 (Pringsheim [11]). *A double sequence $x = [x_{k,l}]$ has* Pringsheim limit *L, denoted by $P\text{-}\lim x = L$, provided that given $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that $|x_{k,l} - L| < \epsilon$ whenever $k, l > N$. We shall describe such an x more briefly as* "P-convergent."

Definition 1.2 (Patterson [10]). *The double sequence y is a* double subsequence *of the sequence x provided that there exist two increasing double index sequences*

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$\{n_j\}$ and $\{k_j\}$ such that if $z_j = x_{n_j,k_j}$, then y is formed by

$$\begin{array}{cccc} z_1 & z_2 & z_5 & z_{10} \\ z_4 & z_3 & z_6 & - \\ z_9 & z_8 & z_7 & - \\ - & - & - & - \end{array}$$

Definition 1.3 (Patterson [10]). *A number β is called a* Pringsheim limit point *of the double sequence $x = [x_{n,k}]$ provided that there exists a subsequence $y = [y_{n,k}]$ of $[x_{n,k}]$ that has Pringsheim limit β, that is, $P\text{-}\lim y_{n,k} = \beta$.*

Remark. *The definition of a Pringsheim limit point can also be stated as follows: β is a Pringsheim limit point of x provided that there exist two increasing index sequences $\{n_i\}$ and $\{k_i\}$ such that $\lim_i x_{n_i,k_i} = \beta$.*

Definition 1.4 (Patterson [10]). *A double sequence x is* divergent in the Pringsheim sense *(P-divergent) provided that x does not* converge in the Pringsheim sense *(P-convergent).*

The following is an example of an unbounded double sequence with three Pringsheim limit points, namely -1, 0, 1:

$$x_{k,l} := \begin{cases} k, & \text{if} \quad l = 1, \\ (-1)^k, & \text{if} \quad l > 1, \\ 0, & \text{otherwise.} \end{cases}$$

In 1926 Robison presented a four dimensional analog of the Silverman-Toeplitz theorem [8] for the regularity of double sequences in which he added an additional assumption of boundedness. This assumption was made because a double sequence which is P-convergent is not necessarily bounded. The definition of the regularity for four dimensional matrices will be stated next, followed by the Robison-Hamilton characterization of the regularity of four dimensional matrices. The reader can refer to e-book [2] which contains the chapter entitled *Double Sequences*, and the recent monograph [3] devoted to the spaces of double sequences and four dimensional matrices.

Definition 1.5 (Patterson [10]). *The four dimensional matrix A is said to be* RH-regular *if it maps every bounded P-convergent sequence into a P-convergent sequence with the same P-limit.*

Theorem 1.6 (Hamilton [7], Robison [12]). *The four dimensional matrix A is RH-regular if and only if*

- RH_1**:** $P\text{-}\lim_{m,n} a_{m,n,k,l} = 0$ *for each k and l;*
- RH_2**:** $P\text{-}\lim_{m,n} \sum_{k,l=1,1}^{\infty,\infty} a_{m,n,k,l} = 1$*;*
- RH_3**:** $P\text{-}\lim_{m,n} \sum_{k=1}^{\infty} |a_{m,n,k,l}| = 0$ *for each l;*
- RH_4**:** $P\text{-}\lim_{m,n} \sum_{l=1}^{\infty} |a_{m,n,k,l}| = 0$ *for each k;*
- RH_5**:** $\sum_{k,l=1,1}^{\infty,\infty} |a_{m,n,k,l}|$ *is P-convergent; and*
- RH_6**:** *there exist finite positive integers A and B such that*

$$\sum_{k,l>B} |a_{m,n,k,l}| < A.$$

Definition 1.7 (Patterson [9]). *A double sequence x is of bounded variation provided that there exists a positive integer B such that*

$$\sum_{k,l=1,1}^{\infty,\infty} |x_{m,n} - x_{m-r,n-s}| < B,$$

where r and $s = 0$ and/or 1.

Using the following notions presented by Bell in [4] we will present a series of multidimensional analog of Aslan and Duman results.

Definition 1.8 (Bell [4]). *For $u = 1, 2, \ldots,$ let $\widetilde{A}^u = [a^u_{n,k}]$ be an infinite complex matrix and let $\widetilde{\mathcal{A}}$ denote the sequence of matrices $\{\widetilde{A}^u\}$. Let $\widetilde{x} = \{\widetilde{x}_k\}$ be a sequence of real numbers. Define the double sequence $\widetilde{A}\widetilde{x} = \{(\widetilde{A}\widetilde{x})^u_n\}$ by*

$$(\widetilde{A}\widetilde{x})^u_n := \sum_{k=1}^{\infty} \widetilde{a}^u_{n,k}\widetilde{x}_k.$$

Then $\widetilde{A}\widetilde{x}$ is called $\widetilde{\mathcal{A}}$-transform of x whenever the series converges for all n and u. We say the sequence $\widetilde{x}$ is $\widetilde{\mathcal{A}}$-summable to L if $\lim_{n\to\infty}(\widetilde{A}\widetilde{x})^u_n = L$ uniformly in $u \in \mathbb{N}$.

In [4] Bell established the following Silverman-Toeplitz type characterization of two dimensional summability matrix.

Theorem 1.9. *The transformation $\widetilde{\mathcal{A}} = \{\widetilde{A}^u\} = \{[\widetilde{a}^u_{n,k}]\}$ $(k, n, u \in \mathbb{N})$ is regular if and only if*

1. *$\lim_{n\to\infty} \widetilde{a}^u_{n,k} = 0$ uniformly in u;*
2. *$\lim_{n\to\infty} \sum_{k=1}^{\infty} \widetilde{a}^u_{n,k} = 1$ uniformly in u; and*
3. *for $n, u \in \mathbb{N}$, $\sum_{k=1}^{\infty} |\widetilde{a}^u_{n,k}| < \infty$ and there exist integers N, M such that $\sum_{k=1}^{\infty} |\widetilde{a}^u_{n,k}| < M$ for $n \geq N$ and for all $u \in \mathbb{N}$.*

In a similar manner to Definition 1.8, let us consider the following notion

Definition 1.10. *Let us consider the following double sequence of four-dimensional infinite matrices $\mathcal{A} = \{A^{u,v}\} = \{[a^{u,v}_{m,n,k,l}]\}$, where (m, n), (k, l), $(u, v) \in \mathbb{N} \times \mathbb{N}$, and the real number double sequence $x = \{x_{k,l}\}$. Then the transformation*

$$\mathcal{A}x = \{(Ax)^{u,v}_{m,n}\}$$

define by

$$(Ax)^{u,v}_{m,n} := \left\{ \sum_{k,l=1,1}^{\infty,\infty} a^{u,v}_{m,n,k,l} x_{k,l} \right\},$$

where (m, n), $(u, v) \in \mathbb{N} \times \mathbb{N}$, is called the $\mathcal{A}$-transform of the double sequence x whenever the series is P-convergent for every (m, n), $(u, v) \in \mathbb{N} \times \mathbb{N}$. We say the double sequence x is $\mathcal{A}$-summable to L if $P\text{-}\lim_{m,n\to\infty,\infty}(Ax)^{u,v}_{m,n} = L$ uniformly in $(u, v) \in \mathbb{N} \times \mathbb{N}$.

The following is a Silverman-Toeplitz type characterization of four dimensional summability matrix, whose proof is a multidimensional analog of Robison-Hamilton as of such the proof is omitted.

Theorem 1.11. *The transformation* $\mathcal{A} = \{A^{u,v}\} = \{[a^{u,v}_{m,n,k,l}]\}$, (m,n), (k,l), $(u,v) \in \mathbb{N}\times\mathbb{N}$, *is regular provided that*

(1) $P\text{-}\lim_{m,n\to\infty,\infty} a^{u,v}_{m,n,k,l} = 0$, *uniformly in* (u,v);
(2) $P\text{-}\lim_{m,n\to\infty,\infty} \sum_{k,l=1,1}^{\infty,\infty} a^{u,v}_{m,n,k,l} = 1$, *uniformly in* (u,v); *and*
(3) *For* $(u,v),(m,n)\in\mathbb{N}\times\mathbb{N}$, $\sum_{l=1}^{\infty} |a^{u,v}_{m,n,k,l}| < \infty$ *for each* k
(4) *For* $(u,v),(m,n)\in\mathbb{N}\times\mathbb{N}$, $\sum_{k=1}^{\infty} |a^{u,v}_{m,n,k,l}| < \infty$ *for each* l
(5) *For* $(u,v),(m,n)\in\mathbb{N}\times\mathbb{N}$, $\sum_{k,l=1,1}^{\infty,\infty} |a^{u,v}_{m,n,k,l}| < \infty$ *and there exist integers* N, M *such that* $\sum_{k,l=1,1}^{\infty,\infty} |a^{u,v}_{m,n,k,l}| < M$ *for* $m,n\geq N$ *and for all* $(u,v)\in\mathbb{N}\times\mathbb{N}$.

2. Approximation to Function in the four dimensional Summability Methods

Let us consider the following multi-dimensional linear operators:

$$L_{k,l}(f;(\alpha,\beta)) = \int_a^b\int_a^b f(x,y)d\phi_{k,l}(\alpha,\beta,x,y), \tag{2.1}$$

where

$$f\in C[a,b]\times C[a,b], \quad (\alpha,\beta)\in[a,b]\times[a,b], \quad (k,l)\in\mathbb{N}\times\mathbb{N}.$$

Similar to Baskakov assumption, we will assume that for every $(k,l)\in\mathbb{N}\times\mathbb{N}$ and for every $(\alpha,\beta)\in[a,b]\times[a,b]$, $\phi_{k,l}(\alpha,\beta,x,y)$ is a double bounded variation with respect to $(x,y)\in[a,b]\times[a,b]$. The following are necessary conditions that for a given non-negative RH-regular summability method A, the double sequence $\{L_{k,l}(f)\}$ is uniformly A-summable to f on $[a,b]\times[a,b]$, that is

$$\text{P-}\lim_{m,n\to\infty,\infty}\left\|\sum_{k,l=1,1}^{\infty,\infty} a^{u,v}_{m,n,k,l}L_{k,l}(f) - L\right\| = 0, \text{ uniformly in } (u,v)\in\mathbb{N}\times\mathbb{N},$$

where $\|\cdot\|$ denote the sup-norm on $C[a,b]\times C[a,b]$. We are assuming that

$$\sum_{k,l=1,1}^{\infty,\infty} a^{u,v}_{m,n,k,l}L_{k,l}(f)$$

is bounded for each $m,n,u,v\in\mathbb{N}$ and $f\in C[a,b]\times C[a,b]$.

Definition 2.1. *Let* $\{[a^{u,v}_{m,n,k,l}]\}$, $(m,n),(k,l),(u,v)\in\mathbb{N}\times\mathbb{N}$, *be a non-negative RH-regular summability methods. We say the operators* (2.1) *belong to the class* $E^A_{2s,2t}; s,t\geq 1$ *provide that, for each fix* $(\alpha,\beta)\in[a,b]\times[a,b]$ *the following integrals*

$$I^{(1)}_{2s,2t,m,n,u,v} = \sum_{k,l=1,1}^{\infty,\infty} a^{u,v}_{m,n,k,l}\int_a^x\int_a^{x_1}\cdots\int_a^{x_{2s-1}}\int_a^b d_{\phi_{k,l}(\alpha,\beta,x_{2s},y)}\cdots dx_2dx_1$$

where $a\leq x\leq\alpha$,

$$I^{(2)}_{2s,2t,m,n,u,v} = \sum_{k,l=1,1}^{\infty,\infty} a^{u,v}_{m,n,k,l}\int_x^b\int_x^b\int_{x_1}^b\cdots\int_{x_{2s-1}}^b dyd_{\phi_{k,l}(\alpha,\beta,x_{2s},y)}\cdots dx_2dx_1$$

where $\alpha\leq x\leq b$,

$$I^{(3)}_{2s,2t,m,n,u,v} = \sum_{k,l=1,1}^{\infty,\infty} a^{u,v}_{m,n,k,l}\int_a^y\int_a^{y_1}\cdots\int_a^{y_{2s-1}}\int_a^b d_{\phi_{k,l}(\alpha,\beta,x,y_{2t})}\cdots dy_2dy_1$$

where $a \leq y \leq \beta$,

$$I^{(4)}_{2s,2t,m,n,u,v} = \sum_{k,l=1,1}^{\infty,\infty} a^{u,v}_{m,n,k,l} \int_y^b \int_y^b \int_{y_1}^b \cdots \int_{y_{2s-1}}^b dx d_{\phi_{k,l}(\alpha,\beta,x,y_{2t})} \cdots dy_2 dy_1$$

where $\beta \leq y \leq b$,

$$I^{(5)}_{2s,2t,m,n,u,v} = \sum_{k,l=1,1}^{\infty,\infty} a^{u,v}_{m,n,k,l} \int_a^x \int_a^{x_1} \cdots \int_a^{x_{2s-1}} \int_a^y \int_a^{y_1} \cdots \int_a^{y_{2s-1}} d_{\phi_{k,l}(\alpha,\beta,x_{2s},y)} \cdots dx_2 dx_1 d_{\phi_{k,l}(\alpha,\beta,x,y_{2t})} \cdots dy_2 dy_1$$

where $a \leq x \leq \alpha$ *and* $a \leq y \leq \beta$,

$$I^{(6)}_{2s,2t,m,n,u,v} = \sum_{k,l=1,1}^{\infty,\infty} a^{u,v}_{m,n,k,l} \int_y^b \int_{y_1}^b \cdots \int_{y_{2s-1}}^b \int_x^b \int_{x_1}^b \cdots \int_{x_{2s-1}}^b d_{\phi_{k,l}(\alpha,\beta,x,y_{2t})} \cdots dy_2 dy_1 d_{\phi_{k,l}(\alpha,\beta,x_{2s},y)} \cdots dx_2 dx_1$$

where $\alpha \leq x \leq b$ *and* $\beta \leq y \leq b$, *for all* $(x,y) \in [a,b] \times [a,b]$ *we have a well-defined sign, which depend on* $(m,n),(u,v) \in \mathbb{N} \times \mathbb{N}$.

This definition grants us a multi-dimensional family of positive linear operators that is consistent with Baskakov [6] operators.

Theorem 2.2. *Let* $\{[a^{u,v}_{m,n,k,l}]\}$, (m,n), (k,l), $(u,v) \in \mathbb{N}\times\mathbb{N}$, *be a non-negative RH-regular summability methods, and let the operators* (2.1) *belong to the class* $E^A_{2s,2t}$, $s,t \geq 1$. *If* $\{L_{k,l}(e_{i,j})\}$ *is uniformly A-summable to* $e_{i,j}$ *for each* $i = 0,1,2,\ldots 2s$ *and* $j = 0,1,2,\ldots 2t$, *then the sequence* $\{L_{k,l}(f)\}$ *is uniformly A-summable to* f *on* $[a,b]\times[a,b]$ *for all* $f \in C^{2s}[a,b] \times C^{2t}[a,b]$, *the set of two-dimension functions having continuous partial derivative of order* $2s \cdot 2t$ *on the interval* $[a,b]\times[a,b]$.

Proof. Let us consider the case when $s = t = 1$ and $\rho(x,y) := (x-\alpha)(y-\beta)$ for fix α and β in $[a,b]$. Also note that this is similar for m or n grater than 1. Let us observation that:

$$\begin{aligned}
&\left\| \sum_{k,l=1,1}^{\infty,\infty} a^{u,v}_{m,n,k,l} L_{k,l}(\rho^2) \right\| \\
\leq \quad & \left\| \sum_{k,l=1,1}^{\infty,\infty} a^{u,v}_{m,n,k,l} L_{k,l}(e_{2,2}) - e_{2,2} \right\| \quad + \quad 2C \left\| \sum_{k,l=1,1}^{\infty,\infty} a^{u,v}_{m,n,k,l} L_{k,l}(e_{2,1}) - e_{2,1} \right\| \\
+\,2C & \left\| \sum_{k,l=1,1}^{\infty,\infty} a^{u,v}_{m,n,k,l} L_{k,l}(e_{1,2}) - e_{1,2} \right\| \quad + \quad C^2 \left\| \sum_{k,l=1,1}^{\infty,\infty} a^{u,v}_{m,n,k,l} L_{k,l}(e_{2,0}) - e_{2,0} \right\| \\
+\,C^2 & \left\| \sum_{k,l=1,1}^{\infty,\infty} a^{u,v}_{m,n,k,l} L_{k,l}(e_{0,2}) - e_{0,2} \right\| \quad + \quad 2C^3 \left\| \sum_{k,l=1,1}^{\infty,\infty} a^{u,v}_{m,n,k,l} L_{k,l}(e_{0,1}) - e_{0,1} \right\| \\
+\,2C^3 & \left\| \sum_{k,l=1,1}^{\infty,\infty} a^{u,v}_{m,n,k,l} L_{k,l}(e_{1,0}) - e_{1,0} \right\| \quad + \quad C^4 \left\| \sum_{k,l=1,1}^{\infty,\infty} a^{u,v}_{m,n,k,l} L_{k,l}(e_{0,0}) - e_{0,0} \right\|.
\end{aligned}$$

The above inequality holds for each $(m,n),(u,v)\in\mathbb{N}\times\mathbb{N}$, where $C=\max\{|a|,|b|\}$. It is clear via the Pringsheim limit and the hypothesis we are granted the following:

$$\text{P-}\lim_{m,n\to\infty}\left\|\sum_{k,l=1,1}^{\infty,\infty}a_{m,n,k,l}^{u,v}L_{k,l}(\rho^2)\right\|=0,\ \text{uniformly of }(u,v)\in\mathbb{N}\times\mathbb{N}.$$

It is also in a similar manner we are granted the following:

$$\text{P-}\lim_{m,n\to\infty}\left\|\sum_{k,l=1,1}^{\infty,\infty}a_{m,n,k,l}^{u,v}L_{k,l}(\rho)\right\|=0,\ \text{uniformly of }(u,v)\in\mathbb{N}\times\mathbb{N}.$$

The following is obtain by partitioning $[a,b]\times[a,b]$ into the following $\{(\mu,\nu):a\leq\mu\leq\alpha\text{ and }a\leq\nu\leq\beta\}$ and $\{(\mu,\nu):\alpha\leq\mu\leq b\text{ and }\beta\leq\nu\leq b\}$ and them use multidimensional integration by parts and obtain the following:

$$\begin{aligned}L_{k,l}(\rho^2;(\alpha,\beta))=2\Bigg\{&\int_a^\alpha\int_a^\beta dydx\int_a^s\int_a^t dt_1ds_1\int_a^{s_1}\int_a^{t_1}d\phi_{k,l}(\alpha,\beta,s_2,t_2)\\&+\int_\alpha^b\int_\beta^b dydx\int_s^b\int_t^b dt_1ds_1\int_{s_1}^b\int_{t_1}^b d\phi_{k,l}(\alpha,\beta,s_2,t_2)\Bigg\}\end{aligned}$$

We transform the above sequence using the above RH-regularity A to obtain the following:

$$\begin{aligned}&\left\|\sum_{k,l=1,1}^{\infty,\infty}a_{m,n,k,l}^{u,v}L_{k,l}(\rho^2)\right\|\\&=2\sup_{\substack{(\alpha,\beta)\\ \in[a,b]\times[a,b]}}\Bigg|\int_a^\alpha\int_a^\beta\left(\sum_{k,l=1,1}^{\infty,\infty}a_{m,n,k,l}^{u,v}\int_a^s\int_a^t dt_1ds_1\int_a^{s_1}\int_a^{t_1}d\phi_{k,l}(\alpha,\beta,s_2,t_2)\right)dydx\\&\qquad+\int_\alpha^b\int_\beta^b\left(\sum_{k,l=1,1}^{\infty,\infty}a_{m,n,k,l}^{u,v}\int_s^b\int_t^b dt_1ds_1\int_{s_1}^b\int_{t_1}^b d\phi_{k,l}(\alpha,\beta,s_2,t_2)\right)dydx\Bigg|.\end{aligned}$$

Clearly, it follows

$$\begin{aligned}&\frac{1}{2}\left\|\sum_{k,l=1,1}^{\infty,\infty}a_{m,n,k,l}^{u,v}L_{k,l}(\rho^2)\right\|\\&\leq\sup_{\substack{(\alpha,\beta)\\ \in[a,b]\times[a,b]}}\Bigg(\int_a^\alpha\int_a^\beta\left|\sum_{k,l=1,1}^{\infty,\infty}a_{m,n,k,l}^{u,v}\int_a^s\int_a^t dt_1ds_1\int_a^{s_1}\int_a^{t_1}d\phi_{k,l}(x,y,s_2,t_2)\right|dydx\\&\qquad+\int_\alpha^b\int_\beta^b\left|\sum_{k,l=1,1}^{\infty,\infty}a_{m,n,k,l}^{u,v}\int_s^b\int_t^b dt_1ds_1\int_{s_1}^b\int_{t_1}^b d\phi_{k,l}(x,y,s_2,t_2)\right|dydx\Bigg)\end{aligned}\tag{2.2}$$

Since the Pringshim limit of the last equation of the inequality (2.2) we are granted that

$$\text{P-}\lim_{m,n\to\infty}\left\|\sum_{k,l=1,1}^{\infty,\infty} a_{m,n,k,l}^{u,v} L_{k,l}(\rho^2)\right\| = 0, \text{ uniformly in for all } (u,v)\in\mathbb{N}\times\mathbb{N}.$$

One should also observe that for $f\in\mathcal{C}^3[a,b]\times\mathcal{C}^3[a,b]$ we are granted the following multidimensional Taylor series expansion

$$\begin{aligned}
f(x,y) = f(\alpha,\beta) &+ \frac{\partial f(\alpha,\beta)}{\partial x}(x-\alpha) + \frac{\partial f(\alpha,\beta)}{\partial x}(y-\beta)\\
&+ \frac{\partial^2 f(\alpha,\beta)}{\partial x\partial y}\frac{(x-\alpha)(y-\beta)}{2} + \frac{\partial^2 f(\alpha,\beta)}{\partial x^2}\frac{(x-\alpha)}{2}^2\\
&+ \frac{\partial^2 f(\alpha,\beta)}{\partial y^2}\frac{(y-\beta)^2}{2} + \frac{1}{6}\int_\alpha^x\int_\beta^y \frac{\partial^3 f(\eta,\xi)}{\partial x^3}(x-\eta)^3 d\eta d\xi\\
&+ 3\int_\alpha^x\int_\beta^y \frac{\partial^3 f(\eta,\xi)}{\partial x^2\partial y}(x-\eta)^2(y-\xi) d\eta d\xi\\
&+ 3\int_\alpha^x\int_\beta^y \frac{\partial^3 f(\eta,\xi)}{\partial x\partial y^2}(x-\eta)(y-\xi)^2 d\eta d\xi\\
&+ \int_\alpha^x\int_\beta^y \frac{\partial^3 f(\eta,\xi)}{\partial y^3}(y-\xi)^3 d\eta d\xi.
\end{aligned}$$

Therefore, for each $(m,n),(u,v)\in\mathbb{N}\times\mathbb{N}$ the above expansion grants us the following:

$$\begin{aligned}
&\sum_{k,l=1,1}^{\infty,\infty} a_{m,n,k,l}^{u,v} L_{k,l}(f,(\alpha,\beta)) - f(\alpha,\beta) = \ f(\alpha,\beta)\left(\sum_{k,l=1,1}^{\infty,\infty} a_{m,n,k,l}^{u,v} L_{k,l}(e_{0,0}:(\alpha,\beta)) - e_{0,0}\right)\\
&+ \frac{1}{2}\frac{\partial^2 f(\alpha,\beta)}{\partial x\partial y}\sum_{k,l=1,1}^{\infty,\infty} a_{m,n,k,l}^{u,v} L_{k,l}(\rho:(\alpha,\beta)) + \frac{\partial f(\alpha,\beta)}{\partial x}\sum_{k,l=1,1}^{\infty,\infty} a_{m,n,k,l}^{u,v} L_{k,l}(x-\alpha:(\alpha,\beta))\\
&+ \frac{\partial f(\alpha,\beta)}{\partial y}\sum_{k,l=1,1}^{\infty,\infty} a_{m,n,k,l}^{u,v} L_{k,l}(y-\alpha:(\alpha,\beta)) + \frac{\partial^2 f(\alpha,\beta)}{\partial x^2}\sum_{k,l=1,1}^{\infty,\infty} a_{m,n,k,l}^{u,v} L_{k,l}((x-\alpha)^2:(\alpha,\beta))\\
&+ \frac{\partial^2 f(\alpha,\beta)}{\partial y^2}\sum_{k,l=1,1}^{\infty,\infty} a_{m,n,k,l}^{u,v} L_{k,l}((y-\alpha)^2:(\alpha,\beta)) + \sum_{k,l=1,1}^{\infty,\infty} a_{m,n,k,l}^{u,v} R_{k,l}(\alpha,\beta),
\end{aligned}$$

where

$$\begin{aligned}
R_{k,l}(\alpha,\beta) =&\frac{1}{6}\int_a^b\int_a^b\left[\int_\alpha^x\int_\beta^y \frac{\partial^3 f(\eta,\xi)}{\partial x^3}(x-\eta)^3 d\eta d\xi\right] d\phi_{k,l}(\alpha,\beta,x,y)\\
&+ 3\int_a^b\int_a^b\left[\int_\alpha^x\int_\beta^y \frac{\partial^3 f(\eta,\xi)}{\partial x^2\partial y}(x-\eta)^2(y-\xi) d\eta d\xi\right] d\phi_{k,l}(\alpha,\beta,x,y)\\
&+ 3\int_a^b\int_a^b\left[\int_\alpha^x\int_\beta^y \frac{\partial^3 f(\eta,\xi)}{\partial x\partial y^2}(x-\eta)(y-\xi)^2 d\eta d\xi\right] d\phi_{k,l}(\alpha,\beta,x,y)\\
&+ \int_a^b\int_a^b\left[\int_\alpha^x\int_\beta^y \frac{\partial^3 f(\eta,\xi)}{\partial y^3}(y-\xi)^3 d\eta d\xi\right] d\phi_{k,l}(\alpha,\beta,x,y).
\end{aligned}$$

Similar to the partition above we can write $R_{k,l}(\alpha,\beta)$ as follows:

$$
\begin{aligned}
R_{k,l}(\alpha,\beta) = 3 \Bigg[& \int_a^\alpha \int_a^\beta \frac{\partial^3 f(s,t)}{\partial s^3} \left\{ \int_a^s \int_a^t dt_1 ds_1 \int_a^{s_1} \int_a^{t_1} d\phi_{k,l}(\alpha,\beta,s_2,t_2) \right\} dsdt \\
& + \int_\alpha^b \int_\beta^b \frac{\partial^3 f(s,t)}{\partial s^3} \left\{ \int_s^b \int_t^b dt_1 ds_1 \int_{s_1}^b \int_{t_1}^b d\phi_{k,l}(\alpha,\beta,s_2,t_2) \right\} dsdt \Bigg] \\
& + 3 \Bigg[\int_a^\alpha \int_a^\beta \frac{\partial^3 f(s,t)}{\partial s^2 \partial t} \left\{ \int_a^s \int_a^t dt_1 ds_1 \int_a^{s_1} \int_a^{t_1} d\phi_{k,l}(\alpha,\beta,s_2,t_2) \right\} dsdt \\
& + \int_\alpha^b \int_\beta^b \frac{\partial^3 f(s,t)}{\partial s^2 \partial t} \left\{ \int_s^b \int_t^b dt_1 ds_1 \int_{s_1}^b \int_{t_1}^b d\phi_{k,l}(\alpha,\beta,s_2,t_2) \right\} dsdt \Bigg] \\
& + 3 \Bigg[\int_a^\alpha \int_a^\beta \frac{\partial^3 f(s,t)}{\partial s \partial t^2} \left\{ \int_a^s \int_a^t dt_1 ds_1 \int_a^{s_1} \int_a^{t_1} d\phi_{k,l}(\alpha,\beta,s_2,t_2) \right\} dsdt \\
& + \int_\alpha^b \int_\beta^b \frac{\partial^3 f(s,t)}{\partial s \partial t^2} \left\{ \int_s^b \int_t^b dt_1 ds_1 \int_{s_1}^b \int_{t_1}^b d\phi_{k,l}(\alpha,\beta,s_2,t_2) \right\} dsdt \Bigg] \\
& + 3 \Bigg[\int_a^\alpha \int_a^\beta \frac{\partial^3 f(s,t)}{\partial t^3} \left\{ \int_a^s \int_a^t dt_1 ds_1 \int_a^{s_1} \int_a^{t_1} d\phi_{k,l}(\alpha,\beta,s_2,t_2) \right\} dsdt \\
& + \int_\alpha^b \int_\beta^b \frac{\partial^3 f(s,t)}{\partial t^3} \left\{ \int_s^b \int_t^b dt_1 ds_1 \int_{s_1}^b \int_{t_1}^b d\phi_{k,l}(\alpha,\beta,s_2,t_2) \right\} dsdt \Bigg].
\end{aligned}
$$

Thus for each $(m,n),(u,v) \in \mathbb{N} \times \mathbb{N}$ we are granted the following:

$$
\begin{aligned}
& \left\| \sum_{k,l=1,1}^{\infty,\infty} a_{m,n,k,l}^{u,v} R_{k,l} \right\| \\
& \leq M \sup_{\substack{(\alpha,\beta) \\ \in [a,b]\times[a,b]}} \Bigg\{ 3 \Bigg[\int_a^\alpha \int_a^\beta \left| \sum_{k,l=1,1}^{\infty,\infty} a_{m,n,k,l}^{u,v} \left\{ \int_a^s \int_a^t dt_1 ds_1 \int_a^{s_1} \int_a^{t_1} d\phi_{k,l}(\alpha,\beta,s_2,t_2) \right\} \right| dsdt \\
& \qquad + \int_\alpha^b \int_\beta^b \left| \sum_{k,l=1,1}^{\infty,\infty} a_{m,n,k,l}^{u,v} \left\{ \int_s^b \int_t^b dt_1 ds_1 \int_{s_1}^b \int_{t_1}^b d\phi_{k,l}(\alpha,\beta,s_2,t_2) \right\} \right| dsdt \Bigg] \\
& \qquad + 3 \Bigg[\int_a^\alpha \int_a^\beta \left| \sum_{k,l=1,1}^{\infty,\infty} a_{m,n,k,l}^{u,v} \left\{ \int_a^s \int_a^t dt_1 ds_1 \int_a^{s_1} \int_a^{t_1} d\phi_{k,l}(\alpha,\beta,s_2,t_2) \right\} \right| dsdt \\
& \qquad + \int_\alpha^b \int_\beta^b \left| \sum_{k,l=1,1}^{\infty,\infty} a_{m,n,k,l}^{u,v} \left\{ \int_s^b \int_t^b dt_1 ds_1 \int_{s_1}^b \int_{t_1}^b d\phi_{k,l}(\alpha,\beta,s_2,t_2) \right\} \right| dsdt \Bigg] \\
& \qquad + 3 \Bigg[\int_a^\alpha \int_a^\beta \left| \sum_{k,l=1,1}^{\infty,\infty} a_{m,n,k,l}^{u,v} \left\{ \int_a^s \int_a^t dt_1 ds_1 \int_a^{s_1} \int_a^{t_1} d\phi_{k,l}(\alpha,\beta,s_2,t_2) \right\} \right| dsdt \\
& \qquad + \int_\alpha^b \int_\beta^b \left| \sum_{k,l=1,1}^{\infty,\infty} a_{m,n,k,l}^{u,v} \left\{ \int_s^b \int_t^b dt_1 ds_1 \int_{s_1}^b \int_{t_1}^b d\phi_{k,l}(\alpha,\beta,s_2,t_2) \right\} \right| dsdt \Bigg]
\end{aligned}
$$

$$+ 3\left[\int_a^\alpha \int_a^\beta \left|\sum_{k,l=1,1}^{\infty,\infty} a_{m,n,k,l}^{u,v}\left\{\int_a^s \int_a^t dt_1 ds_1 \int_a^{s_1}\int_a^{t_1} d\phi_{k,l}(\alpha,\beta,s_2,t_2)\right\}\right| dsdt\right.$$

$$\left.\left. + \int_\alpha^b \int_\beta^b \left|\sum_{k,l=1,1}^{\infty,\infty} a_{m,n,k,l}^{u,v}\left\{\int_s^b \int_t^b dt_1 ds_1 \int_{s_1}^{b}\int_{t_1}^{b} d\phi_{k,l}(\alpha,\beta,s_2,t_2)\right\}\right| dsdt\right]\right\},$$

where $M = \max\left\{\left\|\frac{\partial^{i+j} f}{\partial^i \partial^j}\right\| : i+j=3\right\}$. Therefore

$$\text{P-}\lim_{m,n\to\infty}\left\|\sum_{k,l=1,1}^{\infty,\infty} a_{m,n,k,l}^{u,v} R_{k,l}\right\| = 0 \text{ uniformly in } (u,v)\in\mathbb{N}\times\mathbb{N}.$$

Additionally, observe that

$$\begin{aligned}
\left\|\sum_{k,l=1,1}^{\infty,\infty} a_{m,n,k,l}^{u,v} L_{k,l}(f) - f\right\| \;\le\; & M_1\left\|\sum_{k,l=1,1}^{\infty,\infty} a_{m,n,k,l}^{u,v} L_{k,l}(e_{0,0}:(\alpha,\beta)) - e_{0,0}\right\| \\
&+ M_2\left\|\sum_{k,l=1,1}^{\infty,\infty} a_{m,n,k,l}^{u,v} L_{k,l}(x-\alpha:(\alpha,\beta))\right\| + M_3\left\|\sum_{k,l=1,1}^{\infty,\infty} a_{m,n,k,l}^{u,v} L_{k,l}(\rho:(\alpha,\beta))\right\| \\
&+ M_4\left\|\sum_{k,l=1,1}^{\infty,\infty} a_{m,n,k,l}^{u,v} L_{k,l}(y-\alpha:(\alpha,\beta))\right\| + M_5\left\|\sum_{k,l=1,1}^{\infty,\infty} a_{m,n,k,l}^{u,v} L_{k,l}((x-\alpha)^2:(\alpha,\beta))\right\| \\
&+ M_6\left\|\sum_{k,l=1,1}^{\infty,\infty} a_{m,n,k,l}^{u,v} L_{k,l}((y-\alpha)^2:(\alpha,\beta))\right\| + \left\|\sum_{k,l=1,1}^{\infty,\infty} a_{m,n,k,l}^{u,v} R_{k,l}(\alpha,\beta)\right\|.
\end{aligned}$$

Thus

$$\text{P-}\lim_{m,n\to\infty,\infty}\left\|\sum_{k,l=1,1}^{\infty,\infty} a_{m,n,k,l}^{u,v} L_{k,l}(f) - L\right\| = 0, \text{ uniformly in } (u,v)\in\mathbb{N}\times\mathbb{N}.$$

Therefore $L_{k,l}(f)$ is uniformly A-summable to f on $[a,b]\times[a,b]$. □

Remark. *The above theorem is identical to Aslan and Duman's Theorem 2.2 in* [1] *if s or t is zero.*

Remark. *Similar to Aslan and Duman's observation, Theorem 2.2 reduces to the two dimensional classical Korovkin theorem if $A = A^{u,v} = \{I\}$, the four dimensional identical matrix with $(k,l)\in\mathbb{N}\times\mathbb{N}$, $(x,y)\in[a,b]\times[a,b]$, and $\phi_{k,l}(\alpha,\beta,x,y)$ is a non-deceasing function in both x and y.*

In addition, the following theorem is a multi-dimensional analog of Swetits' Theorem 1 in [13] and the proof follows in a manner to that of Theorem 2.1.

Theorem 2.3. *Let $\{L_{k,l}\}$ be a double sequence of positive linear operators for $\mathcal{C}^2[a,b]\times\mathcal{C}^2[a,b]$. Let $\left\{A^{\xi,\mu}\right\}$ be a double sequence of four-dimensional matrices with non-negative real entries, with $\|A_{m,n}(C_{0,0})\| < \infty$. Then for $f\in C[a,b]\times C[a,b]$*

and $m, n = 1, 2, 3, \ldots,$

$$\begin{aligned}\|f - A_{m,n}(f)\| \ &\leq \|f\| \cdot \|A_{m,n}(e_{0,0}) - I\| + \left\|\frac{\partial f}{\partial x}\right\| \cdot \|A_{m,n}(e_{1,0}) - I\| \\ &+ \left\|\frac{\partial f}{\partial y}\right\| \cdot \|A_{m,n}(e_{0,1}) - I\| + \Pi(\Xi^1_{m,n}) \cdot \|A_{m,n}(e_{0,0}) - I\| \\ &+ \Pi(\Xi^2_{m,n}) \cdot \|A_{m,n}(e_{1,0}) - I\| + \Pi(\Xi^3_{m,n}) \cdot \|A_{m,n}(e_{0,0}) - I\|,\end{aligned}$$

where

$$\begin{aligned}\|A_{m,n}(f) - f\| &= \sup_{(\xi,\mu)}\left(\sup_{\{(x,y)\in[a,b]\times[a,b]\}} \left|A^{\xi,\mu}_{m,n}(f) - f(x,y)\right|\right), \\ \Xi^1_{m,n} &= \||A_{m,n}((x-\alpha)(y-\beta))\|, \\ \Xi^2_{m,n} &= \left\||A_{m,n}((x-\alpha)^2)\right\|, \\ \Xi^1_{m,n} &= \left\||A_{m,n}((y-\beta)^2)\right\|,\end{aligned}$$

and Π *denotes the modulus of the function* f.

Corollary 2.4. *Let* $\mathcal{A} = \{[a^{u,v}_{m,n,k,l}]\}$*, where* $(m,n), (k,l), (u,v) \in \mathbb{N}\times\mathbb{N}$ *be a non-negative RH-regular summability method and let* (2.1) *belong to* $E^{A}_{2s,2t}$ *where* $s, t \geq 1$*. If for each* $i = 1, 2, 3, \ldots 2s$ *and/or* $j = 1, 2, 3, \ldots, 2t$

$$P\text{-}\lim_{m,n\to\infty,\infty} \sum_{k,l=1,1}^{\infty,\infty} a^{u,v}_{m,n,k,l} \, ||L_{k,l}(e_{i,j}) - e_{i,j}|| = 0, \ \textit{uniformly in } (u,v) \in \mathbb{N}\times\mathbb{N},$$

then $\{L_{k,l}(f)\}$ *is uniformly A-summable to* f *on* $[a,b]\times[a,b]$ *for* $f \in C^{2s}[a,b] \times C^{2t}[a,b]$.

The proof follows in a similar manner to Aslan and Duman proof. Thus the proof is omitted. The following is a multidimensional analog of [1, Theorem 2.4].

Theorem 2.5. *Let* $A = [a^{u,v}_{m,n,k,l}]$ *be a four dimensional summability method and* $L_{k,l}$ *and two-dimensional operator that satisfy*

$$\int_a^b \int_a^b |d\phi_{k,l}(\alpha,\beta,x,y)| \leq M \textit{ for all } (x,y) \in [a,b]\times[a,b] \textit{ and } (k,l) \in \mathbb{N}\times\mathbb{N}$$

where M *is an absolute positive constant. Then for* $f \in C[a,b] \times C[a,b]$ *it follows that* $\{L_{k,l}(f)\}$ *is uniformly A-summable.*

Proof. It is also well known that for $f \in C^2 \times C^2$ there exists

$$P(x,y) = \sum_{k,l=1,1}^{2s,2t} b_{k,l} x^k y^l$$

such that $\|f - P\| < \epsilon$ for all $\epsilon > 0$. Therefore

$$\begin{aligned}\left\|\sum_{k,l=1,1}^{\infty,\infty} a^{u,v}_{m,n,k,l} L_{k,l}(f-P)\right\| &\leq \|f-P\| \sum_{k,l=1,1}^{\infty,\infty} a^{u,v}_{m,n,k,l} \int_a^b \int_a^b |d\phi_{k,l}(\alpha,\beta,x,y)| \\ &\leq M\epsilon \sum_{k,l=1,1}^{\infty,\infty} a^{u,v}_{m,n,k,l}.\end{aligned}$$

Thus, form the RH-regularity of A we know that

$$\sum_{k,l=1,1}^{\infty,\infty} a_{m,n,k,l}^{u,v}$$

is finite for each (m,n) and (u,v) in $\mathbb{N}\times\mathbb{N}$. Also observe that for the linearity of $L_{k,l}$ we are granted the following:

$$\sum_{k,l=1,1}^{\infty,\infty} a_{m,n,k,l}^{u,v} L_{k,l}(P) = \sum_{i,j=0,0}^{2s,2t} b_{i,j} \sum_{k,l=1,1}^{\infty,\infty} a_{m,n,k,l}^{u,v} L_{k,l}(e_{i,j}).$$

Thus

$$\left\| \sum_{k,l=1,1}^{\infty,\infty} a_{m,n,k,l}^{u,v} L_{k,l}(P) - P \right\| \le \bar{C} \sum_{i,j=0,0}^{2s,2t} \left\| \sum_{k,l=1,1}^{\infty,\infty} a_{m,n,k,l}^{u,v} L_{k,l}(e_{i,j}) - e_{i,j} \right\|,$$

where $\bar{C} = \max\{|b_{i,j}| : i = 0,1,2,3,\ldots,2s; j = 0,1,2,3,\ldots,2t\}$. The last two inequalities grant us the following for each $(m,n),(u,v)\in\mathbb{N}\times\mathbb{N}$:

$$\begin{aligned}
&\left\| \sum_{k,l=1,1}^{\infty,\infty} a_{m,n,k,l}^{u,v} L_{k,l}(f)) - f \right\| \\
&\quad \le \left\| \sum_{k,l=1,1}^{\infty,\infty} a_{m,n,k,l}^{u,v} L_{k,l}(f-P) \right\| + \left\| \sum_{k,l=1,1}^{\infty,\infty} a_{m,n,k,l}^{u,v} L_{k,l}(P)) - P \right\| + \epsilon \\
&\quad \le M \left(\sum_{k,l=1,1}^{\infty,\infty} a_{m,n,k,l}^{u,v} - 1 \right) \epsilon + \bar{C} \sum_{i,j=0,0}^{2s,2t} \left\| \sum_{k,l=1,1}^{\infty,\infty} a_{m,n,k,l}^{u,v} L_{k,l}(e_{i,j}) - e_{i,j} \right\|.
\end{aligned}$$

The last inequality and the RH-regularity of A grants the following

$$\text{P-}\lim_{m,n\to\infty,\infty} \left\| \sum_{k,l=1,1}^{\infty,\infty} a_{m,n,k,l}^{u,v} L_{k,l}(f)) - f \right\| = 0, \text{ uniformly in } (u,v)\in\mathbb{N}\times\mathbb{N}.$$

The result follows. □

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