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ANALYSIS OF A FLUID QUEUE DRIVEN BY A QUEUE WITH MULTIPLE EXPONENTIAL VACATIONS AND IMPATIENT CUSTOMERS

S. SOPHIA, B. MUTHU DEEPIKA AND S. R. ANANATHA LAKSHMI

Abstract. This paper analyses a single buffer fluid queueing system where the fluid flow into the buffer is regulated by a queue with impatient customers and exponential vacations. The governing differential equations of the fluid queueing system are solved using generating function method and continued fraction methodology. Analytical expressions for the stationary distribution of the buffer content are obtained in terms of Bessel functions and confluent hypergeometric functions by employing Laplace transforms. The average buffer content of the system under consideration is also obtained.

1. Introduction

Queues are an essential way of managing the flow of customers when the resources are limited. In todays world, fluid queues play a significant role in analysing resource sharing in computer systems and also in the performance evaluation of computer communication networks. A fluid queue acts as an input-output system where a continuous fluid (customers) get accumulated in the buffer, wait for service and get depleted after being served. The accumulation and depletion of the buffer are regulated by the background queueing system. In many real time situations, the server cannot serve the customers continuously due to several factors such as repair time, maintenance period, secondary jobs and so on and these situations can be modeled as queueing systems with vacations. Sophia and Muthu Deepika (2020) [11] used continued fraction methodology to find the buffer content distribution of a fluid queue driven by a single server queue with catastrophes. Horvath and Telek (2014) [7] derived the phase-type representation of the sojourn time distribution of the fluid queues when the input and output processes are dependent and independent and made a comparative study of the two models.

Fluid queues driven by queueing systems with various vacation policies have been studied by many authors. Deng et al. (1999) [6] studied an $M/M/1$ queue with delayed vacation and obtained the steady-state probability distribution and

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associated performance measures. Mao et al. (2010) [9] analysed a fluid model driven by an $M/M/1$ queue with single exponential vacation and obtained the steady-state distribution of the buffer content and associated system performance measures. On the other hand, Xu et al. (2013) [12] determined the stationary distribution of the buffer content for a fluid model driven by an $M/M/c$ queue with working vacations using Laplace transforms. Ammar (2014) [3] considered a fluid queue model driven by an $M/M/1$ queue with multiple exponential vacations and derived an explicit expression for the stationary distribution of the buffer content in terms of modified Bessel function of first kind, where the author used generating function methodology. In 2017, Yu et al. [13] considered a fluid queue driven by a multi-server queue with multiple working vacations and vacation interruption and obtained the buffer content distribution by employing matrix-geometric method.

In many queueing situations, the customers leave the system or lost if service has not begun within a limited time after their arrival. For instance, in a data communication network with time-out protocol packets are lost if they experience a long waiting line. It may lead to an adverse impact on the revenue generation of a business organization. Yue et al. (2016) [14] considered an $M/M/1$ queue with vacation and impatience timer, where the timer variable depends on the state of the server. The authors analysed both multiple and single vacation scenarios and obtained various performance measures for both vacation period and busy period. For further information on vacation queueing systems, readers may refer to Zhang and Gao (2020) [15] and Perel and Yechiali (2010) [10] and the references therein.

This paper studies a fluid queue driven by a queue with multiple exponential vacations and impatient customers. This paper is sketched as follows: Section 2 provides the model description. In Section 3, the joint state probabilities are determined by means of continued fraction method and generating function method by employing Laplace transforms. Analytical results are obtained for the buffer content distribution in terms of modified Bessel function and confluent hypergeometric function. The mean buffer content is explicitly derived in Section 4. Definition of confluent hypergeometric function and its identities are given as appendix.

2. Model Description

Transient analysis of an $M/M/1$ queue with impatient customers and multiple vacations was studied by Ammar (2015) [4] and Altman and Yechiali (2006) [2] considered the same queue and analysed its steady-state behavior. Here we consider an infinite buffer fluid queue regulated by an $M/M/1$ queue subject to multiple exponential vacations and customers impatience. We utilise the steady-state probabilities obtained by Altman and Yechiali (2006) [2] for our analysis. In the sequel, we describe the fluid queueing system as follows: In the fluid queueing system under consideration, the server goes for a vacation when there are no customers in the system and comes back to service when the vacation period ends. After the vacation period, if there are customers in the system the server starts the service otherwise goes for another vacation. We assume that the vacation period follows exponential distribution with parameter η. The arrival pattern of the customers follows Poisson process with rate λ. The service time of the customers are exponentially distributed with mean $1/\mu$. In addition, every customer who arrives during the vacation period activates the independent impatience timer which is exponentially distributed with rate ξ. The impatient customer who activated the independent timer completes his service if the server returns from vacation before the timer period ends. In case, if the timer period expires when the server is on vacation the customer abandons the queue and never returns. Also let us assume that the random variables representing the service time, the inter-arrival time and the vacation time are mutually independent. Let the customers are served on First

Come First Served (FCFS) basis. Let $X(t)$ denote the number of customers in the system of the $M/M/1$ queue at time t. $Y(t)$ denote the state of the server such that

$$Y(t) = \begin{cases} 0, & \text{at time } t, \text{ when the server is on vacation} \\ 1, & \text{at time } t, \text{ when the server is busy.} \end{cases}$$

It is clear that the process $\{(X(t),\ Y(t)),\ t \geq 0\}$ is a stochastic process with state space $S = \{(0,0) \bigcup (j,k),\ j = 1,2,3,\ldots,\ k = 0,1\}$. Further let $\{W(t),\ t \geq 0\}$ denote the content of an infinite buffer at time t whose input and output are regulated by the $M/M/1$ vacation queue with impatient customers. When there are no customers in the $M/M/1$ vacation queue the buffer content of the fluid system reduces with rate $c_0 < 0$. On the other hand, the buffer occupancy increases with rate $c_1 > 0$ when customers are present in the system. Thus, the $M/M/1$ vacation queue acts as a background queueing process and regulates the buffer content.
Thus,

$$\frac{dW(t)}{dt} = \begin{cases} 0, & X(t) = 0, W(t) = 0 \\ c_0, & X(t) = 0, W(t) > 0 \\ c_1, & X(t) > 0. \end{cases}$$

Now the fluid queue driven by an $M/M/1$ queue with multiple vacations and impatient customers is represented by the three-dimensional process $\{(X(t), Y(t), W(t)),\ t \geq 0\}$. When the buffer is infinite the fluid system is stable if and only if

$$c_0 p_{0,0} + c_1 \sum_{j=1}^{\infty} p_{j,0}(x) + c_1 \sum_{j=1}^{\infty} p_{j,1}(x) < 0 \text{ and } \frac{\lambda}{\mu} < 1 \tag{2.1}$$

As said earlier, the explicit expressions for the steady-state probabilities $p_{j,k}$ of the background queueing process can be found by replacing γ as η in equations (2.17) and (2.18) of Altman and Yechiali (2006) [2].
Let us denote the joint probability distribution function at time t as

$$F_{0,0}(t,x) = P\{X(t) = 0, Y(t) = 0, W(t) \leq x\}, x \geq 0$$

and

$$F_{j,k}(t,x) = P\{X(t) = j, Y(t) = k, W(t) \leq x\}\ , x \geq 0, j = 1,2,3,\ldots, k = 0,1.$$

The governing Chapman-Kolmogorov forward differential-difference equations for the model under consideration are

$$\begin{aligned}
\frac{\partial F_{0,0}(t,x)}{\partial t} + c_0 \frac{\partial F_{0,0}(t,x)}{\partial x} &= -\lambda F_{0,0}(t,x) + \xi F_{1,0}(t,x) \\
&\quad + \mu F_{1,1}(t,x) \\
\frac{\partial F_{j,0}(t,x)}{\partial t} + c_1 \frac{\partial F_{j,0}(t,x)}{\partial x} &= \lambda F_{j-1,0}(t,x) - (\lambda + j\xi + \eta) F_{j,0}(t,x) \\
&\quad + (j+1)\xi F_{j+1,0}(t,x),\ j \geq 1, \\
\frac{\partial F_{1,1}(t,x)}{\partial t} + c_1 \frac{\partial F_{1,1}(t,x)}{\partial x} &= \eta F_{1,0}(t,x) - (\lambda + \mu) F_{1,1}(t,x) \\
&\quad + \mu F_{2,1}(t,x) \\
\text{and } \frac{\partial F_{j,1}(t,x)}{\partial t} + c_1 \frac{\partial F_{j,1}(t,x)}{\partial x} &= \eta F_{j,0}(t,x) - (\lambda + \mu) F_{j,1}(t,x) \\
&\quad + \lambda F_{j-1,1}(t,x) + \mu F_{j+1,1}(t,x), j \geq 2.
\end{aligned}$$

When the system is in equilibrium, the process $\{(X(t), Y(t), W(t)), t \geq 0\}$ is stable and we have

$$F_{j,k}(x) = \lim_{t\to\infty} P\{X(t) = j, Y(t) = k, W(t) \leq x\}, x \geq 0,\ j, k \in S$$

and also, the distribution function of the buffer content in steady-state is

$$F(x) = P[W \leq x] = F_{0,0}(x) + \sum_{j=1}^{\infty} F_{j,0}(x) + \sum_{j=1}^{\infty} F_{j,1}(x).$$

As $t \to \infty$, $F_{j,k}(t,x) = F_{j,k}(x)$ and $\frac{\partial F_{j,k}(t,x)}{\partial t} = 0$ and therefore the governing differential-difference equations reduce to

$$\begin{aligned}
c_0 \frac{dF_{0,0}(x)}{dx} &= -\lambda F_{0,0}(x) + \xi F_{1,0}(x) + \mu F_{1,1}(x), && (2.2)\\
c_1 \frac{dF_{j,0}(x)}{dx} &= \lambda F_{j-1,0}(x) - (\lambda + j\xi + \eta) F_{j,0}(x) \\
&\quad +(j+1)\xi F_{j+1,0}(x),\ j \geq 1, && (2.3)\\
c_1 \frac{dF_{1,1}(x)}{dx} &= \eta F_{1,0}(x) - (\lambda + \mu) F_{1,1}(x) + \mu F_{2,1}(x) && (2.4)\\
\text{and } c_1 \frac{dF_{j,1}(x)}{dx} &= \eta F_{j,0}(x) + \lambda F_{j-1,1}(x) - (\lambda + \mu) F_{j,1}(x), \\
&\quad +\mu F_{j+1,1}(x),\ j \geq 2. && (2.5)
\end{aligned}$$

Obviously the boundary conditions of the sytem under consideration are

$$\begin{aligned}
F_{j,k}(0) &= 0,\ j = 1, 2, 3, \ldots, k = 0, 1, && (2.6)\\
\lim_{x\to\infty} F_{j,k}(x) &= p_{j,k},\ j, k \in S && (2.7)\\
\text{and } F_{0,0}(0) &= A,\ 0 < A < 1 && (2.8)
\end{aligned}$$

where the constant A is determined using (2.1).

3. Stationary buffer content distribution

In this section, the stationary distribution of the buffer content for the state 0 is found using continued fraction methodology and we used generating function method to find the buffer content distribution in state 1. Analytical expressions are obtained for the joint probability distribution function $F_{0,0}(x)$ and $F_{j,k}(x), j = 1, 2, 3, \ldots, k = 0, 1$ in terms of modified Bessel function of first kind. In the following sequel, we denote $H^*(s)$ as the Laplace transform of $H(.)$.

3.1. The buffer content distribution during busy period ($\mathbf{F_{j,1}(x)}$).

Taking Laplace transform of the governing equations (2.2) – (2.5), we get

$$\begin{aligned}
sc_0 F^*_{0,0}(s) - c_0 F_{0,0}(0) &= -\lambda F^*_{0,0}(s) + \xi F^*_{1,0}(s) + \mu F^*_{1,1}(s), && (3.1)\\
sc_1 F^*_{j,0}(s) - c_1 F_{j,0}(0) &= \lambda F^*_{j-1,0}(s) - (\lambda + j\xi + \eta) F^*_{j,0}(s) \\
&\quad +(j+1)\xi F^*_{j+1,0}(s),\ j \geq 1, && (3.2)
\end{aligned}$$

$$\begin{aligned}
sc_1 F^*_{1,1}(s) - c_1 F_{1,1}(0) &= \eta F^*_{1,0}(s) - (\lambda + \mu) F^*_{1,1}(s) + \mu F^*_{2,1}(s) && (3.3)\\
\text{and } sc_1 F^*_{j,1}(s) - c_1 F_{j,1}(0) &= \eta F^*_{j,0}(s) + \lambda F^*_{j-1,1}(s) - (\lambda + \mu) F^*_{j,1}(s) \\
&\quad +\mu F^*_{j+1,1}(s),\ j \geq 2. && (3.4)
\end{aligned}$$

Let us define the Generating function for $F^*_{j,1}(s)$ as

$$Q^*(z,s) = \sum_{j=1}^{\infty} F^*_{j,1}(s) z^j.$$

Adding (3.3) and (3.4) and multiplying with z^j yields

$$\begin{aligned} sc_1 \sum_{j=1}^{\infty} F^*_{j,1}(s) z^j &= \eta \sum_{j=1}^{\infty} F^*_{j,0}(s) z^j + \lambda \sum_{j=2}^{\infty} F^*_{j-1,1}(s) z^j + \mu \sum_{j=1}^{\infty} F^*_{j+1,1}(s) z^j \\ &\quad -(\lambda+\mu) \sum_{j=1}^{\infty} F^*_{j,1}(s) z^j \end{aligned}$$

and thus

$$Q^*(z,s) = \frac{\frac{\eta}{c_1} \sum_{j=1}^{\infty} F^*_{j,0}(s) z^j - \frac{\mu}{c_1} F^*_{1,1}(s)}{s - \left[-\left(\frac{\lambda+\mu}{c_1}\right) + \frac{\lambda z}{c_1} + \frac{\mu}{z c_1} \right]},$$

which on inversion gives $Q(z,x)$

$$\begin{aligned} Q(z,x) &= \frac{\eta}{c_1} \int_0^x \sum_{j=1}^{\infty} F_{j,0}(y) z^j e^{-\left(\frac{\lambda+\mu}{c_1}\right)(x-y)} e^{\left(\frac{\lambda z}{c_1} + \frac{\mu}{z c_1}\right)(x-y)} dy \\ &\quad -\frac{\mu}{c_1} \int_0^x F_{1,1}(y) e^{-\left(\frac{\lambda+\mu}{c_1}\right)(x-y)} e^{\left(\frac{\lambda z}{c_1} + \frac{\mu}{z c_1}\right)(x-y)} dy. \end{aligned} \quad (3.5)$$

It is well known that the generating function of the modified Bessel function of the first kind is

$$e^{(t+\frac{1}{t})(\frac{x}{2})} = \sum_{n=-\infty}^{\infty} I_n(x)(t)^n.$$

By means of the above equation, we obtain

$$e^{\left(\frac{\lambda z}{c_1} + \frac{\mu}{z c_1}\right)(x-y)} = \sum_{j=-\infty}^{\infty} I_j(a(x-y))(bz)^j \quad (3.6)$$

where $a = \frac{2\sqrt{\lambda\mu}}{c_1}$ and $b = \sqrt{\frac{\lambda}{\mu}}$.
Substituting (3.6) in (3.5), $Q(z,x)$ is obtained as

$$\begin{aligned} Q(z,x) &= \frac{\eta}{c_1} \int_0^x \sum_{j=1}^{\infty} F_{j,0}(y) z^j e^{-\left(\frac{\lambda+\mu}{c_1}\right)(x-y)} \sum_{j=-\infty}^{\infty} I_j(a(x-y))(bz)^j dy \\ &\quad -\frac{\mu}{c_1} \int_0^x F_{1,1}(y) e^{-\left(\frac{\lambda+\mu}{c_1}\right)(x-y)} \sum_{j=-\infty}^{\infty} I_j(a(x-y))(bz)^j dy. \end{aligned}$$

Comparing the coefficients of z^j and after some algebraic manipulation, we obtain

$$\begin{aligned} F_{j,1}(x) &= \frac{\eta}{c_1} \int_0^x \sum_{m=1}^{\infty} F_{m,0}(y) I_{j-m}(a(x-y)) b^{j-m} e^{-\left(\frac{\lambda+\mu}{c_1}\right)(x-y)} dy \\ &\quad -\frac{\mu}{c_1} \int_0^x F_{1,1}(y) e^{-\left(\frac{\lambda+\mu}{c_1}\right)(x-y)} I_j(a(x-y)) b^j dy, j = 1,2,3,\ldots. \end{aligned} \quad (3.7)$$

Now comparing the negative powers of z, the above equation holds for $j = -1, -2, -3, \ldots$ with $F_{j,1}(x) = 0$. Employing $I_{-j}(a(x-y)) = I_j(a(x-y))$ for $j = 1, 2, 3, \ldots$, we get

$$\begin{aligned}\frac{\eta}{c_1}\int_0^x \sum_{m=1}^{\infty} F_{m,0}(y) I_{j+m}(a(x-y)) b^{j-m} e^{-\left(\frac{\lambda+\mu}{c_1}\right)(x-y)} dy \\ -\frac{\mu}{c_1}\int_0^x F_{1,1}(y) e^{-\left(\frac{\lambda+\mu}{c_1}\right)(x-y)} I_j(a(x-y)) b^j dy = 0. \qquad (3.8)\end{aligned}$$

Equations (3.7) and (3.8) lead to

$$\begin{aligned} F_{j,1}(x) &= \frac{\eta}{c_1}\int_0^x \sum_{m=1}^{\infty} F_{m,0}(y)[I_{j-m}(a(x-y)) - I_{j+m}(a(x-y))] b^{j-m} \\ &\quad \times e^{-\left(\frac{\lambda+\mu}{c_1}\right)(x-y)} dy, \ j = 1, 2, 3, \ldots, \qquad (3.9)\end{aligned}$$

where $I_j(.)$ is the modified Bessel function of the first kind of order j. Thus, equation (3.9) completely provides the buffer content distribution when there are j customers in the system and the server is in busy period.

3.2. **The buffer content distribution during the vacation period ($\mathbf{F_{j,0}(x)}$).**

Substituting the boundary condition (2.6) in (3.2) and after some algebra, we get

$$\frac{F^*_{j,0}(s)}{F^*_{j-1,0}(s)} = \frac{\lambda}{sc_1 + \lambda + j\xi + \eta - (j+1)\xi \frac{F^*_{j+1,0}(s)}{F^*_{j,0}(s)}},$$

which on iteration gives a continued fraction expression as

$$\frac{F^*_{j,0}(s)}{F^*_{j-1,0}(s)} = \frac{\lambda}{sc_1 + \lambda + j\xi + \eta - \frac{(j+1)\xi\lambda}{sc_1+\lambda+(j+1)\xi+\eta - \frac{(j+2)\xi\lambda}{sc_1+\lambda+(j+1)\xi+\eta-\ldots}}}. \qquad (3.10)$$

Using (A.4) the above continued fraction can be expressed as

$$\begin{aligned}&\frac{{}_1F_1\left(j+1; \frac{sc_1+\eta}{\xi} + j + 1; \frac{-\lambda}{\xi}\right)}{\xi(\frac{sc_1+\eta}{\xi} + j)\, {}_1F_1\left(j; \frac{sc_1+\eta}{\xi} + j; \frac{-\lambda}{\xi}\right)} \\ &\quad = \frac{1}{sc_1 + \lambda + j\xi + \eta - \frac{(j+1)\xi\lambda}{sc_1+\lambda+(j+1)\xi+\eta - \frac{(j+2)\xi\lambda}{sc_1+\lambda+(j+1)\xi+\eta-\ldots}}}.\end{aligned}$$

Thus explicitly

$$\frac{F^*_{j,0}(s)}{F^*_{j-1,0}(s)} = \frac{\left(\frac{\lambda}{\xi}\right)}{\left(\frac{sc_1+\eta}{\xi} + j\right)} \frac{{}_1F_1\left(j+1; \frac{sc_1+\eta}{\xi} + j + 1; \frac{-\lambda}{\xi}\right)}{{}_1F_1\left(j; \frac{sc_1+\eta}{\xi} + j; \frac{-\lambda}{\xi}\right)} \qquad (3.11)$$

which on iteration gives

$$F^*_{j,0}(s) = \sigma^*_j(s) F^*_{0,0}(s) \qquad (3.12)$$

$$\text{where } \sigma^*_j(s) = \frac{\left(\frac{\lambda}{\xi}\right)^j}{\prod_{i=1}^{j}\left(\frac{sc_1+\eta}{\xi} + i\right)} \frac{{}_1F_1\left(j+1; \frac{sc_1+\eta}{\xi} + j + 1; \frac{-\lambda}{\xi}\right)}{{}_1F_1\left(1; \frac{sc_1+\eta}{\xi} + 1; \frac{-\lambda}{\xi}\right)}.$$

On inverting (3.12), we get

$$F_{j,0}(x) = \sigma_j(x) * F_{0,0}(x). \tag{3.13}$$

3.3. **The buffer content distribution during the vacation period when there are no customers in the background queue ($\mathbf{F_{0,0}(x)}$).**

From (3.1) $F^*_{0,0}(s)$ can be rewritten as

$$F^*_{0,0}(s) = \frac{1}{(sc_0+\lambda)}[Ac_0 + \xi F^*_{1,0}(s) + \mu F^*_{1,1}(s)]. \tag{3.14}$$

Substituting $j = 1$ in (3.9), we get

$$\begin{aligned} F_{1,1}(x) &= \frac{\eta}{c_1}\int_0^x \sum_{m=1}^{\infty} F_{m,0}(y)[I_{m-1}(a(x-y)) - I_{m+1}(a(x-y))]b^{1-m} \\ &\quad \times e^{-\left(\frac{\lambda+\mu}{c_1}\right)(x-y)}dy. \end{aligned} \tag{3.15}$$

Laplace transform of the above equation gives

$$F^*_{1,1}(s) = \frac{\eta}{\mu}\sum_{m=1}^{\infty} b^{-m} F^*_{0,0}(s)\sigma^*_m(s)\left(\frac{p-\sqrt{(p^2-a^2)}}{a}\right)^m$$

where $p = \left(s + \frac{\lambda+\mu}{c_1}\right)$. From (3.12), we get $F^*_{1,0}(s) = \sigma^*_1(s)F^*_{0,0}(s)$. Substituting $F^*_{1,1}(s)$ and $F^*_{1,0}(s)$ in (3.14), $F^*_{0,0}(s)$ modifies to

$$\begin{aligned} F^*_{0,0}(s) &= Ac_0\sum_{q=0}^{\infty}\frac{1}{(sc_0+\lambda)^{q+1}} \\ &\quad \times \sum_{r=0}^{q}\binom{q}{r}\left[\eta\sum_{m=1}^{\infty}\sigma^*_m(s)\left(\frac{p-\sqrt{(p^2-a^2)}}{ab}\right)^m\right]^{(q-r)}[\xi\sigma^*_1(s)]^r. \end{aligned}$$

Inversion of the above equation yields

$$\begin{aligned} F_{0,0}(x) &= A\left(\frac{\mu}{c_1}\right)\sum_{q=0}^{\infty}\sum_{r=0}^{q}\xi^r\eta^{q-r}\binom{q}{r}\frac{e^{-\left(\frac{\lambda}{c_0}\right)x}}{c_0^q}\frac{x^q}{q!} * \sigma_1^r(x) \\ &\quad * \left\{\sum_{m=1}^{\infty} b^{1-m}e^{-\left(\frac{\lambda+\mu}{c_1}\right)x}[I_{m-1}(a(x)) - I_{m+1}(a(x))] * \sigma_m(x)\right\}^{*(q-r)} \end{aligned} \tag{3.16}$$

where $*$ denotes the convolution and $*(q-r)$ is the $(q-r)^{th}$ fold convolution. Inversion of $\sigma^*_j(s)$ can be carried out by using the identities given in the appendix. We found $\sigma^*_j(s)$ as

$$\sigma^*_j(s) = \frac{\left(\frac{\lambda}{\xi}\right)^j}{{}_1F_1\left(1; \frac{sc_1+\eta}{\xi}+1; \frac{-\lambda}{\xi}\right)}\frac{{}_1F_1\left(j+1; \frac{sc_1+\eta}{\xi}+j+1; \frac{-\lambda}{\xi}\right)}{\prod_{i=1}^{j}\left(\frac{sc_1+\eta}{\xi}+i\right)}.$$

Now we use (A.1) and (A.2) in $\frac{{}_1F_1\left(j+1;\frac{sc_1+\eta}{\xi}+j+1;\frac{-\lambda}{\xi}\right)}{\prod_{i=1}^{j}\left(\frac{sc_1+\eta}{\xi}+i\right)}$ to invert $\sigma_j^*(s)$

$$\frac{{}_1F_1\left(j+1;\frac{sc_1+\eta}{\xi}+j+1;\frac{-\lambda}{\xi}\right)}{\prod_{i=1}^{j}\left(\frac{sc_1+\eta}{\xi}+i\right)}=\frac{1}{\prod_{i=1}^{j}\left(\frac{sc_1+\eta}{\xi}+i\right)}\sum_{n=0}^{\infty}\frac{(-\lambda)^n}{\xi^n n!}$$

$$\times\frac{(j+1)(j+2)(j+3)...(j+n)}{\left(\frac{sc_1+\eta}{\xi}+j+1\right)\left(\frac{sc_1+\eta}{\xi}+j+2\right)\left(\frac{sc_1+\eta}{\xi}+j+3\right)...\left(\frac{sc_1+\eta}{\xi}+j+n\right)}.$$

It is evident that

$$\frac{1}{\prod_{i=1}^{j+n}\left(\frac{sc_1+\eta}{\xi}+i\right)}=\sum_{i=1}^{j+n}\frac{(-1)^{i-1}}{(i-1)!(j+n-i)!(sc_1+\eta+i\xi)}. \tag{3.17}$$

Hence $\frac{{}_1F_1\left(j+1;\frac{sc_1+\eta}{\xi}+j+1;\frac{-\lambda}{\xi}\right)}{\prod_{i=1}^{j}\left(\frac{sc_1+\eta}{\xi}+i\right)}$

$$=\sum_{n=0}^{\infty}\frac{(-\lambda)^n}{\xi^{n-1}n!}\prod_{i=1}^{n}(j+i)\sum_{i=1}^{j+n}\frac{(-1)^{i-1}}{(i-1)!(j+n-i)!(sc_1+\eta+i\xi)} \tag{3.18}$$

where we have used (3.17).

Now consider $\frac{\left(\frac{\lambda}{\xi}\right)^j}{{}_1F_1\left(1;\frac{sc_1+\eta}{\xi}+1;\frac{-\lambda}{\xi}\right)}$,

$$\frac{\left(\frac{\lambda}{\xi}\right)^j}{{}_1F_1(1;\frac{sc_1+\eta}{\xi}+1;\frac{-\lambda}{\xi})}=\frac{\left(\frac{\lambda}{\xi}\right)^j}{\sum_{n=0}^{\infty}\frac{(-\lambda)^n}{\xi^{n-1}n!}\prod_{i=1}^{n}(i)\sum_{i=1}^{n}\frac{(-1)^{i-1}}{(i-1)!(n-i)!(sc_1+\eta+i\xi)}} \tag{3.19}$$

$$=\frac{\left(\frac{\lambda}{\xi}\right)^j}{\sum_{n=0}^{\infty}\lambda^n v_n^*(s)}$$

where $v_n^*(s)=\frac{(-1)^n}{\xi^{n-1}n!}\prod_{i=1}^{n}(i)\sum_{i=1}^{n}\frac{(-1)^{i-1}}{(i-1)!(n-i)!(sc_1+\eta+i\xi)}$, thus

$${}_1F_1\left(1;\frac{sc_1+\eta}{\xi}+1;\frac{-\lambda}{\xi}\right)=\sum_{n=0}^{\infty}\lambda^n v_n^*(s).$$

Let us rewrite the above equation as

$$\left[{}_1F_1\left(1;\frac{sc_1+\eta}{\xi}+1;\frac{-\lambda}{\xi}\right)\right]^{-1}=\sum_{n=0}^{\infty}\lambda^n u_n^*(s) \tag{3.20}$$

where $u_0^*(s)=1$ and for $n=1,2,3,\ldots$

$$u_n^*(s)=\begin{vmatrix} v_1^*(s) & 1 & & & & \\ v_2^*(s) & v_1^*(s) & 1 & & & \\ v_3^*(s) & v_2^*(s) & v_1^*(s) & 1 & & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ v_{n-1}^*(s) & v_{n-2}^*(s) & v_{n-3}^*(s) & & v_1^*(s) & 1 \\ v_n^*(s) & v_{n-1}^*(s) & v_{n-2}^*(s) & & v_2^*(s) & v_1^*(s) \end{vmatrix}$$

$$=\sum_{i=1}^{n}(-1)^{i-1}v_i^*(s)u_{n-i}^*(s).$$

On inverting the above equation, we get

$$u_n(x) = \sum_{i=1}^{n} (-1)^{i-1} v_i(x) * u_{n-i}(x)$$

$$\text{where } v_n(x) = \frac{(-1)^n}{\xi^{n-1} n!} \prod_{i=1}^{n} (i) \sum_{i=1}^{n} \frac{(-1)^{i-1}}{c_1 (i-1)!(n-i)!} e^{-\left(\frac{\eta + i\xi}{c_1}\right)x}.$$

Substituting (3.18) and (3.19) in $\sigma_j^*(s)$, we obtain

$$\begin{aligned}\sigma_j^*(s) &= \left(\frac{\lambda}{\xi}\right)^j \sum_{n=0}^{\infty} \frac{(-\lambda)^n}{\xi^{n-1} n!} \prod_{i=1}^{n} (j+i) \sum_{i=1}^{j+n} \frac{(-1)^{i-1}}{(i-1)!(j+n-i)!(sc_1 + \eta + i\xi)} \\ &\quad \times \sum_{n=0}^{\infty} \lambda^n u_n^*(s).\end{aligned}$$

Inverting the above equation, $\sigma_j(x)$ is obtained as

$$\begin{aligned}\sigma_j(x) &= \left(\frac{\lambda}{\xi}\right)^j \sum_{n=0}^{\infty} \frac{(-\lambda)^n}{\xi^{n-1} n!} \prod_{i=1}^{n} (j+i) \sum_{i=1}^{j+n} \frac{(-1)^{i-1}}{c_1 (i-1)!(j+n-i)!} e^{-\left(\frac{\eta + i\xi}{c_1}\right)x} \\ &\quad * \sum_{n=0}^{\infty} \lambda^n u_n(x).\end{aligned}$$

4. Mean buffer content

In this section, the mean buffer content of the fluid queue fed by multiple exponential vacations and impatient customers is obtained using Laplace-Stieltjes transform. The cumulative distribution function of the buffer content is

$$F(x) = P[W \le x] = F_{0,0}(x) + \sum_{j=1}^{\infty} F_{j,0}(x) + \sum_{j=1}^{\infty} F_{j,1}(x).$$

The Laplace transform of the above equation yields

$$F^*(s) = F_{0,0}^*(s) + \sum_{j=1}^{\infty} F_{j,0}^*(s) + \sum_{j=1}^{\infty} F_{j,1}^*(s).$$

The Laplace-Stieltjes transform $\widehat{F}(s)$ is given by

$$\widehat{F}(s) = \int_0^{\infty} e^{-sx} dF(x) = sF^*(s).$$

Substituting $F_{j,0}^*(s)$ and $F_{j,1}^*(s)$ in $F^*(s)$ gives

$$\begin{aligned}\widehat{F}(s) &= s\Bigg[\Bigg(1 + \sum_{j=1}^{\infty} \sigma_j^*(s) + \frac{\eta}{sc_1} \sum_{j=1}^{\infty} \sigma_j^*(s) - \frac{\eta}{sc_1} \sum_{m=1}^{\infty} \sigma_m^*(s) \\ &\quad \times \left(\frac{p - \sqrt{(p^2 - a^2)}}{a}\right)^m \Bigg) F_{0,0}^*(s)\Bigg].\end{aligned}$$

The mean of the buffer content $E(W)$ is

$$E(W) = -\frac{d\widehat{F}(s)}{ds}\bigg|_{s=0}$$

differentiating $\widehat{F}(s)$ with respect to s and setting $s=0$ gives

$$\begin{aligned}
E(W) = &-\frac{A\eta^2 c_0}{c_1}\left\{\sum_{j=1}^{\infty}\sigma_j^*(0)-\sum_{m=1}^{\infty}\sigma_m^*(0)\left(\sqrt{\frac{\mu}{\lambda}}\right)^m\right\}\left\{\sum_{q=0}^{\infty}\frac{1}{(c_0+\lambda)^{q+1}}\right.\\
&\times\sum_{r=0}^{q}R[\xi\sigma_1^*(0)]^r\left\{\left[r\sigma_1^{*'}(0)[\sigma_1^*(0)]^{-1}\right]-\left[\frac{(q+1)c_0}{(c_0+\lambda)}\right]+\left[R^{-2}\right.\right.\\
&\left.\left.\left.\times(q-r)\left(\binom{q}{r}\sum_{m=1}^{\infty}\left(\frac{\mu}{\lambda}\right)^m\left(\sigma_m^{*'}(0)-\sigma_m^*(0)\frac{mc_1}{(\lambda-\mu)}\right)\right)\right]\right\}\right\}\\
&-\left\{Ac_0\sum_{q=0}^{\infty}\frac{\eta}{(c_0+\lambda)^{q+1}}\sum_{r=0}^{q}R[\xi\sigma_1^*(0)]^r\left\{1+\sum_{j=1}^{\infty}\sigma_j^*(0)+\frac{\eta}{c_1}\right.\right.\\
&\left.\times\sum_{j=1}^{\infty}\sigma_j^{*'}(0)\right\}-\frac{\eta}{c_1}\sum_{m=1}^{\infty}\sigma_m^{*'}(0)\left(\sqrt{\frac{\mu}{\lambda}}\right)^m-\frac{\eta}{(\lambda-\mu)}\sum_{m=1}^{\infty}m\sigma_m^*(0)\\
&\left.\times\left(\sqrt{\frac{\mu}{\lambda}}\right)^m\right\}
\end{aligned}$$

where $R=\left[\binom{q}{r}\sum_{m=1}^{\infty}\sigma_m^*(0)\left(\frac{\mu}{\lambda}\right)^m\right]^{q-r}$,

$$\sigma_j^*(0)=\frac{[(\frac{\lambda}{\xi})^j\sum_{n=0}^{\infty}\frac{(-\lambda)^n}{\xi^{n-1}n!}\sum_{i=1}^{j+n}\frac{(i)(-1)^i}{(\eta+i\xi)}\binom{j+n}{i}]}{[\sum_{n=0}^{\infty}\frac{(-\lambda)^n}{\xi^{n-1}n!}\sum_{i=1}^{n}\frac{(i)(-1)^{i-1}}{(\eta+i\xi)}\binom{n}{i}]}$$

and

$$\begin{aligned}
\sigma_j^{*'}(0) = &\frac{[(\frac{\lambda}{\xi})^j\sum_{n=0}^{\infty}\frac{(-\lambda)^n}{\xi^{n-1}n!}\sum_{i=1}^{j+n}\frac{(i)(-1)^i c_1}{(\eta+i\xi)}\binom{j+n}{i}]}{[\sum_{n=0}^{\infty}\frac{(-\lambda)^n}{\xi^{n-1}n!}\sum_{i=1}^{n}\frac{(i)(-1)^i c_1^2}{(\eta+i\xi)^2}\binom{n}{i}]}\\
&-\frac{[\sum_{n=0}^{\infty}\frac{(-\lambda)^n}{\xi^{n-1}n!}\sum_{i=1}^{n}\frac{(i)(-1)^i}{(\eta+i\xi)}\binom{n}{i}]}{[\sum_{n=0}^{\infty}\frac{(-\lambda)^n}{\xi^{n-1}n!}\sum_{i=1}^{n}\frac{(i)(-1)^i c_1^2}{(\eta+i\xi)^2}\binom{n}{i}]^2}\\
&\times\left[\left(\frac{\lambda}{\xi}\right)^j\sum_{n=0}^{\infty}\frac{(-\lambda)^n}{\xi^{n-1}n!}\sum_{i=1}^{j+n}\frac{(i)(-1)^i c_1}{(\eta+i\xi)^2}\binom{j+n}{i}\right].
\end{aligned}$$

To determine the constant A, consider the equation

$$c_0\frac{d}{dx}F_{0,0}(x)+c_1\sum_{j=1}^{\infty}\frac{d}{dx}F_{j,0}(x)+c_1\sum_{j=1}^{\infty}\frac{d}{dx}F_{j,1}(x)=0.$$

Integrating the above equation from 0 to ∞, A can be obtained as

$$A=F_{0,0}(0)=\frac{p_{0,0}(c_0-c_1)+c_1}{c_0}.$$

Remark. *When the impatient timer $\xi = 0$, our model reduces to a single server fluid queue regulated by an $M/M/1$ queue with mutiple exponential vacations. By definition,*

$$ {}_1F_1\left(j+1; \frac{sc_1+\eta}{\xi}+j+1; \frac{-\lambda}{\xi}\right) = \sum_{k=0}^{\infty} \frac{(j+1)_k}{(\frac{sc_1+\eta}{\xi}+j+1)_k} \frac{(\frac{-\lambda}{\xi})^k}{k!}. $$

When $\xi = 0$ the above equation reduces to

$$ \begin{aligned} {}_1F_1\left(j+1; \tfrac{sc_1+\eta}{\xi}+j+1; \tfrac{-\lambda}{\xi}\right)\Big|_{\xi=0} & \\ &= \sum_{k=0}^{\infty} \frac{(j+1)(j+2)(j+3)\dots(j+k)}{(sc_1+\eta)^k} \frac{(-\lambda)^k}{k!} \\ &= \left(\frac{sc_1+\eta+\lambda}{sc_1+\eta}\right)^{-(j+1)}. \end{aligned} $$

Similarly

$$ {}_1F_1\left(1; \frac{sc_1+\eta}{\xi}+1; \frac{-\lambda}{\xi}\right)\Big|_{\xi=0} = \frac{sc_1+\eta}{sc_1+\eta+\lambda} $$

Hence $\sigma_j^*(s)$ when $\xi = 0$ becomes

$$ \begin{aligned} \sigma_j^*(s)\big|_{\xi=0} &= \frac{\lambda^j}{(sc_1+\eta)^j} \frac{\left(\frac{sc_1+\eta+\lambda}{sc_1+\eta}\right)^{-(j+1)}}{\left(\frac{sc_1+\eta}{sc_1+\eta+\lambda}\right)} \\ &= \left(\frac{\lambda}{sc_1+\eta+\lambda}\right)^j. \end{aligned} $$

Now, for $\xi = 0$ $F_{0,0}^*(s)$ becomes

$$ F_{0,0}^*(s)\big|_{\xi=0} = \frac{Ac_0[sc_1+\lambda(1-R)+\eta]}{(sc_0+\lambda)[sc_1+\lambda(1-R)+\eta]-\eta\lambda R} $$

where

$$ R = \frac{sc_1+\lambda+\mu-\sqrt{(sc_1+\lambda+\mu)^2-4\lambda\mu}}{2\lambda} $$

which is same as the equation (16) *of Mao et al.* (2010) [8]. *Further in the above obtained result, when we substitute η (vacation rate) $= 0$ the model under consideration reduces to a single server fluid queue fed by an $M/M/1$ queue.*

Application

The motivation of this fluid queueing model has vast applications in computer systems and communication networks. For example, in online gaming data packets are transmitted from the games server or host to the receiver (customers). Let the arrival of the data packets follow Poisson process with rate λ and the service times are exponenetially distributed with rate μ. The data packets are stored in a jitter buffer, which is a temporary storage buffer used in packet-based networks. They are mainly used to guarantee the packet continuity by controlling packet arrival rates during network congestion. Let us assume that the data packets accumulate in the jitter at a rate $c_1 > 0$ and deplete from the jitter at a rate $c_0 < 0$. Most of the online games use UDP (User Datagram Protocol), which is the transport protocol that facilitates the transmission of packets from the host to the destination and it is highly preferred because of its speed, but it is vulnerable to data loss. It is common that the host sets a timer for a packet and sends it to the receiver. Let ξ be the exponential timer set by the host and let the vacation periods of the server are eponentially distributed with rate η. Sometimes the arriving packets during the vacation period will be waiting for the server to get transmitted. If the server did not return within the predetermined time period, the packets are dropped and are not retransmitted. The above scenario can be modeled as a fluid queue with multiple exponential vacations and impatient customers.

Appendix

The confluent hypergeometric function is defined by

$$ {}_1F_1(a;c;z) = 1 + \frac{a}{c}\frac{z}{1!} + \frac{a}{c}\frac{(a+1)}{(c+1)}\frac{z^2}{2!} + \cdots = \sum_{k=0}^{\infty} \frac{(a)_k}{(c)_k}\frac{z^k}{k!} \tag{A.1} $$

for $z \in C$, parameters $a, c \in C$ (c, not a negative integer). Now $(\alpha)_n$, the Pochhammer symbol is described as

$$ (\alpha)_n = \frac{\Gamma(\alpha+1)}{\Gamma(\alpha-n+1)}, n = 0, 1, 2, \ldots . \tag{A.2} $$

We observe that

$$ {}_1F_1(0;c;z) = 1. $$

The recurrence relation for the confluent hypergeometric function is given by

$$ c(c-1){}_1F_1(a-1;c-1;z) - az{}_1F_1(a+1;c+1;z) = c(c-1-z){}_1F_1(a;c;z) $$

(see Abramowitz and Stegun [1]).
The quotient of two hypergeometric functions may be expressed as continued fractions. The following identity is obtained from Lorentzen and Waadeland [8] and Andrews [5]:

$$ \frac{{}_1F_1(a+1;c+1;z)}{{}_1F_1(a;c;z)} = \frac{c}{c-z+}\frac{(a+1)z}{c-z+1+}\frac{(a+2)z}{c-z+2+}\cdots, \tag{A.3} $$

which can be rewritten as

$$ c\frac{{}_1F_1(a;c;z)}{{}_1F_1(a+1;c+1;z)} - (c-z) = \frac{(a+1)z}{c-z+1+}\frac{(a+2)z}{c-z+2+}\cdots . \tag{A.4} $$

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S. Sophia
Department of Mathematics, Sri Sivasubramaniya Nadar College of Engineering, Tamil Nadu, India.

E-mail address: sophias@ssn.edu.in

B. Muthu Deepika
Department of Mathematics, Sri Sivasubramaniya Nadar College of Engineering, Tamil Nadu, India.

E-mail address: muthudeepikab@ssn.edu.in

S. R. Ananatha Lakshmi
Department of Mathematics, Easwari Engineering College, Tamil Nadu, India.

E-mail address: sr.ananthalakshmi@gmail.com