Bulletin of Mathematical Analysis and Applications
ISSN: 1821-1291, URL: http://www.bmathaa.org
Volume 15 Issue 4(2023), Pages 43-52
https://doi.org/10.54671/BMAA-2023-4-5

ON L^1-CONVERGENCE OF MODIFIED COMPLEX TRIGONOMETRIC SUMS WITH NEW GENERALIZED CLASSES

SANDEEP KAUR GILL, JATINDERDEEP KAUR, S.S. BHATIA

Abstract. In this paper, new classes of numerical sequences denoted by $\mathbb{JS}^c$ and $\mathbb{JS}_2^c$ are defined which are the generalization of Krasniqi classes $\mathbb{K}$ and $\mathbb{K}^2$. Also, complex form of modified cosine and sine sums which were introduced by chouhan, Kaur and Bhatia [2] is obtained. Moreover, L^1-convergence of complex trigonometric series has been studied under the new generalized classes of numerical sequences.

1. Introduction

Let $\{q_k^c,\ k = 0, \pm1, \pm2, ...\}$ be a numerical sequence of complex numbers and let

$$\sum_{k=-\infty}^{k=\infty} q_k^c e^{ikt} \tag{1.1}$$

be the complex trigonometric series with its partial sums

$$S_n(C,t) = \sum_{k=-n}^{k=n} q_k^c e^{ikt}\ ;\ n \in \{0,1,2,...\}. \tag{1.2}$$

L^1-convergence of complex trigonometric series has been studied by various authors such as Stanojević Č.V. and Stanojević V. B. [10], Sheng Shu Yun [9], Móricz F. [8], Chen C.P. [1], Bhatia S.S. and Ram B. [3], Bhatia S.S., Kaur K. and Ram B. [4], Tomovski Ź. ([11], [12]), Kaur J. and Bhatia S.S. [5], Krasniqi X.Z. ([6], [7]) by defining new classes of coefficient sequences or by introducing modified trigonometric complex sums. In 2010, Kaur J. and Bhatia S.S. [5] have introduced complex trigonometric sums as

$$g_n(C,t) = S_n(C,t) + \frac{i}{2\sin t}\begin{bmatrix} q_n^c e^{i(n+1)t} - q_{-n}^c e^{-i(n+1)t} + q_{n+1}^c e^{int} - q_{-(n+1)}^c e^{-int} \\ +(q_n^c - q_{n+2}^c)E_n(t) + (q_{-(n+2)}^c - q_{-n}^c)E_{-n}(t) \end{bmatrix} \tag{1.3}$$

and studied its L^1-convergence with class J^* of coefficient sequences.

2010 *Mathematics Subject Classification.* 42A16, 42A20, 42A32, 42B05.
Key words and phrases. L^1- convergence, modified complex trigonometric sums, Dirichlet kernel.

Submitted August 18, 2023. Published November 16, 2023.
Communicated by F. Marcellan.

Definition 1.1. [5] A null sequence $\{q_n^c\}$ of complex numbers belongs to the class J^* if there exists a sequence $\{Q_k\}$ such that

$$(i) \qquad Q_k \downarrow 0, \ as \ k \to \infty,$$
$$(ii) \qquad \sum_{k=1}^{\infty} kQ_k < \infty,$$
$$(iii) \qquad \left|\Delta\left(\frac{q_k^c - q_{-k}^c}{k}\right)\right| \leq \frac{Q_k}{k}, \ \forall \ k.$$

Krasniqi X.Z. ([6],[7]) defined the new classes $\mathbb{K}$ and $\mathbb{K}^2$ of numerical sequence of complex numbers as:

Definition 1.2. [6] A sequence $\{q_k^c\}$ of complex numbers belongs to class $\mathbb{K}$ if $\lim_{k\to\infty} q_k^c = 0$, and there exists a sequence $\{Q_k\}$ such that

$$(i) \qquad Q_k \downarrow 0, \ as \ k \to \infty,$$
$$(ii) \qquad \sum_{k=1}^{\infty} kQ_k < \infty,$$
$$(iii) \qquad \max\left\{\left|\Delta\left(\frac{q_k^c}{k}\right)\right|, \left|\Delta\left(\frac{q_{-k}^c}{k}\right)\right|\right\} \leq \frac{Q_k}{k}, \qquad k \in \{1, 2, ...\}.$$

Definition 1.3. [7] A sequence $\{q_k^c\}$ of complex numbers belongs to class $\mathbb{K}^2$ if $\lim_{k\to\infty} q_k^c = 0$, and there exists a sequence $\{Q_k\}$ such that

$$(i) \qquad Q_k \downarrow 0, \ as \ k \to \infty,$$
$$(ii) \qquad \sum_{k=1}^{\infty} k^2 Q_k < \infty,$$
$$(iii) \qquad \max\left\{\left|\Delta^2\left(\frac{q_k^c}{k}\right)\right|, \left|\Delta^2\left(\frac{q_{-k}^c}{k}\right)\right|\right\} \leq \frac{Q_k}{k^2}, \qquad k \in \{1, 2, ...\}.$$

and studied the L^1-convergence of complex trigonometric sums (1.3).

Theorem 1.4. *([6], [7]) Let q_k^c belongs to the class $\mathbb{K}$(or $\mathbb{K}^2$). Then*
(i) $\lim_{n\to\infty} g_n(C,t) = f(t)$ exists for $|x| \in (0,\pi]$,
(ii) $f \in L^1(0,\pi]$ and $||g_n(C,t) - f(t)||_{L^1} \to 0$ as $n \to \infty$,
(iii) $||S_n(C,t) - f(t)||_{L^1} \to 0$ as $n \to \infty$.

In 2019, Chouhan S.K., Kaur J and Bhatia S.S. [2] have introduced the modified trigonometric cosine and sine sums as

$$f_n(t) = \sum_{k=1}^{n}\left(\frac{q_{k+1}}{k+1} + \sum_{j=k}^{n}\Delta^2\left(\frac{q_j}{j}\right)\right)k\cos kt \tag{1.4}$$

and

$$g_n(t) = \sum_{k=1}^{n}\left(\frac{q_{k+1}}{k+1} + \sum_{j=k}^{n}\Delta^2\left(\frac{q_j}{j}\right)\right)k\sin kt \tag{1.5}$$

and studied their L^1-convergence. The complex form of (1.4) and (1.5) is given by

$$\begin{aligned} K_n(C,t) &= S_n(C,t) + \frac{i}{n+1}\left[q^c_{n+1}E'_n(t) - q^c_{-(n+1)}E'_{-n}(t)\right] \\ &\quad -\frac{i}{n+2}\left[q^c_{n+2}E'_n(t) - q^c_{-(n+2)}E'_{-n}(t)\right]. \end{aligned} \tag{1.6}$$

The new classes $\mathbb{JS}^c$ and $\mathbb{JS}^c_2$ of numerical sequences which are the generalization of the classes $\mathbb{K}$ and $\mathbb{K}^2$ respectively are defined as follows:

Definition 1.5. A sequence $\{q^c_k\}$ of complex numbers belongs to the class $\mathbb{JS}^c$ if $q^c_k \to 0$ as $k \to \infty$, and there exists a nonincreasing sequence $\{Q_k\}$ such that $\sum\limits_{k=1}^{\infty} Q_k \log k < \infty$ and

$$\max\left\{\left|\Delta\left(\frac{q^c_k}{k}\right)\right|, \left|\Delta\left(\frac{q^c_{-k}}{k}\right)\right|\right\} \le \frac{Q_k}{k}, \qquad k \in \{1,2,...\}.$$

Remark. If $\{q^c_k\}$ belongs to the class $\mathbb{K}$ then $\{q^c_k\}$ belongs to the class $\mathbb{JS}^c$. But the converse need not be true.

Example 1.6. Let $\{q^c_k\}$ be a sequence whose general term is $q^c_k = \frac{1}{k}$, $k \in \{1,2,3,..\}$. Then $\left|\Delta\left(\frac{q^c_{\pm k}}{k}\right)\right| \le \frac{2}{k^3} = \frac{Q_k}{k}$, $Q_k = \frac{2}{k^2} \downarrow 0$, and $\sum\limits_{k=1}^{n} \frac{\log k}{k^2} < \infty$ which implies $q^c_n \in \mathbb{JS}^c$. But $\sum\limits_{k=1}^{n} \frac{1}{k} \not< \infty$ which means $\{q^c_n\}$ does not belong to the class $\mathbb{K}$.
This example shows $\{q^c_n\} \in \mathbb{JS}^c$ but it does not belong to the class $\mathbb{K}$.

Definition 1.7. A sequence $\{q^c_k\}$ of complex numbers to class $\mathbb{JS}^c_2$ if $q^c_k \to 0$ as $k \to \infty$, and there exists a nonincreasing sequence $\{Q_k\}$ such that $\sum\limits_{k=1}^{\infty} Q_k \log k < \infty$ and

$$\max\left\{\left|\Delta^2\left(\frac{q^c_k}{k}\right)\right|, \left|\Delta^2\left(\frac{q^c_{-k}}{k}\right)\right|\right\} \le \frac{Q_k}{k^2}, \qquad k \in \{1,2,...\}.$$

Remark. $\mathbb{K}^2 \subset \mathbb{JS}^c_2$. But the converse need not hold.

Example 1.8. Let $\{q^c_k\}$ be a sequence whose general term is $q^c_k = \frac{1}{k}$, $k \in \{1,2,3,..\}$. Then $\left|\Delta^2\left(\frac{q^c_{\pm k}}{k}\right)\right| \le \frac{6}{k^4} = \frac{Q_k}{k^2}$, $Q_k = \frac{6}{k^2} \downarrow 0$, and $\sum\limits_{k=1}^{n} \frac{\log k}{k^2} < \infty$.
Hence $\{q^c_n\} \in \mathbb{JS}^c_2$. But $\sum\limits_{k=1}^{n} 1 \not< \infty$ which means $\{q^c_n\}$ does not belongs to the class $\mathbb{K}^2$.
This example shows $\{q^c_n\} \in \mathbb{JS}^c_2$ but it does not belongs to the class $\mathbb{K}^2$.

The aim of this paper is to study the L^1-convergence of trigonometric series using complex modified trigonometric sums (1.6) under the new generalized classes of coefficient sequences.

2. LEMMAS

The proof of the main result is based on the following lemmas out of which first two are given by Sheng [9]

Lemma 2.1. [9] $\|D'_n(t)\| = \frac{4}{\pi}(n \log n) + o(n)$.

Lemma 2.2. [9] *$||\tilde{D}'_n(t)|| = o(n \log n)$.*

Lemma 2.3. [3] *Let r be a non-negative integer and $0 < \epsilon < \pi$. Then there exists $M_{r\epsilon} > 0$ such that for all $\epsilon \leq |x| \leq \pi$ and all $n \geq 1$,*

(i) $|E_n^{(r)}(t)| \leq \frac{M_{r\epsilon} n^r}{|t|}$,

(ii) $|E_{-n}^{(r)}(t)| \leq \frac{M_{r\epsilon} n^r}{|t|}$,

(iii) $|D_n^{(r)}(t)| \leq \frac{2M_{r\epsilon} n^r}{|t|}$,

(iv) $|\tilde{D}_n^{(r)}(t)| \leq \frac{M_{r\epsilon} n^r}{|t|}$,
where $D_n(t)$ and $\tilde{D}_n(t)$ denotes the Dirichlet kernel and conjugate Dirichlet kernel and $E_n(t) = \sum_{k=0}^{n} e^{ikt}$.

Lemma 2.4. [7]*Let r be a non-negative integer and $0 < \epsilon < \pi$. Then there exists $M_{r\epsilon} > 0$ such that for all $\epsilon \leq |x| \leq \pi$ and all $n \geq 1$,*

(i) $|\bar{E}'_n(t)| \leq \frac{M_{r\epsilon} n^2}{|t|}$,

(ii) $|\bar{E}'_{-n}(t)| \leq \frac{M_{r\epsilon} n^2}{|t|}$,

where $\bar{E}_n(t) = \sum_{k=1}^{n} E_k(t)$.

3. MAIN RESULTS

Theorem 3.1. *Let $\{q_k^c\}$ belongs to $\mathbb{JS}^c$. Then*

(i) $\lim_{n\to\infty} K_n(C,t) = f(t)$ exists for all $x \in (0,\pi]$,

(ii) $f \in L^1(0,\pi]$ and $||K_n(C,t) - f(t)||_{L^1} = o(1)$ as $n \to \infty$,

(iii) $||S_n(C,t) - f(t)||_{L^1} = o(1)$ as $n \to \infty$.

Proof. Firstly, we will show that $f(t)$ exists in $(0,\pi]$. The complex form of the modified sums is

$$\begin{aligned}
K_n(C,t) &= S_n(C,t) + \frac{i}{n+1}\left[q_{n+1}^c E'_n(t) - q_{-(n+1)}^c E'_{-n}(t)\right] \\
&\qquad - \frac{i}{n+2}\left[q_{n+2}^c E'_n(t) - q_{-(n+2)}^c E'_{-n}(t)\right] \\
&= q_0^c + \sum_{k=-n}^{k=n} q_k^c e^{ikt} + \frac{i}{n+1}\left[q_{n+1}^c E'_n(t) - q_{-(n+1)}^c E'_{-n}(t)\right] - \frac{i}{n+2}\left[q_{n+2}^c E'_n(t) - q_{-(n+2)}^c E'_{-n}(t)\right] \\
&= q_0^c + \sum_{k=1}^{k=n}\left(\frac{q_k^c}{k} k e^{ikt} + \frac{q_{-k}^c}{k} k e^{-ikt}\right) + \frac{i}{n+1}\left[q_{n+1}^c E'_n(t) - q_{-(n+1)}^c E'_{-n}(t)\right] \\
&\qquad - \frac{i}{n+2}\left[q_{n+2}^c E'_n(t) - q_{-(n+2)}^c E'_{-n}(t)\right].
\end{aligned}$$

Apply Abel's transformation, we get

$$\begin{aligned} K_n(C,t) &= q_0^c - i\left[\sum_{k=1}^{n-1}\Delta\left(\frac{q_k^c}{k}\right)E_k'(t) + \frac{q_n^c}{n}E_n'(t)\right] + i\left[\sum_{k=1}^{n-1}\Delta\left(\frac{q_{-k}^c}{k}\right)E_{-k}'(t) + \frac{q_{-n}^c}{n}E_{-n}'(t)\right] \\ &\quad + \frac{i}{n+1}\left[q_{n+1}^c E_n'(t) - q_{-(n+1)}^c E_{-n}'(t)\right] - \frac{i}{n+2}\left[q_{n+2}^c E_n'(t) - q_{-(n+2)}^c E_{-n}'(t)\right] \\ &= q_0^c - i\sum_{k=1}^{n}\Delta\left(\frac{q_k^c}{k}\right)E_k'(t) + i\sum_{k=1}^{n}\Delta\left(\frac{q_{-k}^c}{k}\right)E_{-k}'(t) - \frac{i}{n+2}\left[q_{n+2}^c E_n'(t) - q_{-(n+2)}^c E_{-n}'(t)\right] \end{aligned}$$

$$\begin{aligned} |K_n(C,t)| &\le |q_0^c| + \sum_{k=1}^{n}\left[\left|\Delta\left(\frac{q_k^c}{k}\right)\right||E_k'(t)| + \left|\Delta\left(\frac{q_{-k}^c}{k}\right)\right|\left|E_{-k}'(t)\right|\right] + \left|\frac{q_{n+2}^c}{n+2}\right||E_n'(t)| \\ &\quad + \left|\frac{q_{-(n+2)}^c}{n+2}\right|\left|E_{-n}'(t)\right|. \end{aligned}$$

Using Lemma 2.3, we have

$$\begin{aligned} |K_n(C,t)| &\le |q_0^c| + \frac{M_{r\epsilon}}{|t|}\left\{\sum_{k=1}^{n} k\left[\left|\Delta\left(\frac{q_k^c}{k}\right)\right| + \left|\Delta\left(\frac{q_{-k}^c}{k}\right)\right|\right] + |q_{n+2}^c| + |q_{-(n+2)}^c|\right\} \\ &\le |q_0^c| + \frac{2M_{r\epsilon}}{|t|}\left\{\sum_{k=1}^{\infty} Q_k + 2\bar{M}\right\} < \infty, \end{aligned}$$

where $\bar{M}$ is a positive absolute constant.

Since $\{q_k^c\} \in \mathbb{JS}^c$. Therefore, $\lim_{n\to\infty} K_n(C,t) = f(t)$ exists.

For proving (ii), consider

$$\begin{aligned} f(t) - K_n(C,t) &= \sum_{k=n+1}^{\infty}\left(\frac{q_k^c}{k}ke^{ikt} + \frac{q_{-k}^c}{k}ke^{-ikt}\right) - \frac{i}{n+1}\left[q_{n+1}^c E_n'(t) - q_{-(n+1)}^c E_{-n}'(t)\right] \\ &\quad + \frac{i}{n+2}\left[q_{n+2}^c E_n'(t) - q_{-(n+2)}^c E_{-n}'(t)\right]. \end{aligned}$$

Apply Abel's transformation, we have

$$\begin{aligned} f(t) - K_n(C,t) &= \sum_{k=n+1}^{\infty}\left[\Delta\left(\frac{q_k^c}{k}\right)(-iE_k'(t)) + \Delta\left(\frac{q_{-k}^c}{k}\right)(iE_{-k}'(t))\right] \\ &\quad + \frac{i}{n+2}\left[q_{n+2}^c E_n'(t) - q_{-(n+2)}^c E_{-n}'(t)\right] \end{aligned}$$

$$\begin{aligned} |f(t) - K_n(C,t)| &\le \sum_{k=n+1}^{\infty}\left[\left|\Delta\left(\frac{q_k^c}{k}\right)\right||E_k'(t)| + \left|\Delta\left(\frac{q_{-k}^c}{k}\right)\right|\left|E_{-k}'(t)\right|\right] + \left|\frac{q_{n+2}^c}{n+2}\right||E_n'(t)| \\ &\quad + \left|\frac{q_{-(n+2)}^c}{n+2}\right|\left|E_{-n}'(t)\right| \\ &\le \frac{M_{r\epsilon}}{|t|}\left\{\sum_{k=n+1}^{\infty} k\left[\left|\Delta\left(\frac{q_k^c}{k}\right)\right| + \left|\Delta\left(\frac{q_{-k}^c}{k}\right)\right|\right] + \frac{n}{n+2}|q_{n+2}^c| + \frac{n}{n+2}|q_{-(n+2)}^c|\right\} \\ &\le \frac{M_{r\epsilon}}{|t|}\left[\sum_{k=n+1}^{\infty} k\frac{Q_k}{k} + \sum_{k=n+1}^{\infty} k\frac{Q_k}{k} + |q_{n+2}^c| + |q_{-(n+2)}^c|\right] \end{aligned}$$

$$
\begin{aligned}
|f(t)-K_n(C,t)| &\le \frac{M_{r\epsilon}}{|t|}\left[2\sum_{k=n+1}^{\infty}Q_k+|q^c_{n+2}|+|q^c_{-(n+2)}|\right]\\
||f(t)-K_n(C,t)||_{L^1} &\le M_{r\epsilon}\left[2\sum_{k=n+1}^{\infty}Q_k\int_0^{\pi}\frac{dt}{|t|}+\left(|q^c_{n+2}|+|q^c_{-(n+2)}|\right)\int_0^{\pi}\frac{dt}{|t|}\right]\\
&\le M_{r\epsilon}\left[2\sum_{k=n+1}^{\infty}Q_k\log k+\left(|q^c_{n+2}|+|q^c_{-(n+2)}|\right)\log n\right].
\end{aligned}
$$

Now, we note that

$$\sum_{k=n+1}^{\infty}Q_k\log k=o(1)$$

and

$$
\begin{aligned}
|q^c_{\pm(n+2)}|\log n &= (n+2)\log n\left|\frac{q^c_{\pm(n+2)}}{n+2}\right|=(n+2)\log n\sum_{k=n+2}^{\infty}\left|\Delta\left(\frac{q^c_{\pm k}}{k}\right)\right|\\
&\le \sum_{k=n+2}^{\infty}k\log k\left|\Delta\left(\frac{q^c_{\pm k}}{k}\right)\right|\\
&\le \sum_{k=n+2}^{\infty}Q_k\log k=o(1).
\end{aligned}
$$

Subsequently, we get

$$||f(t)-K_n(C,t)||_{L^1}=o(1) \text{ as } n\to\infty.$$

To prove (iii),

$$
\begin{aligned}
||f_n(t)-S_n(C,t)|| &= ||f(t)-K_n(C,t)+K_n(C,t)-S_n(C,t)||\\
&\le \int_0^{\pi}|f(t)-K_n(C,t)|dt+\int_0^{\pi}|K_n(C,t)-S_n(t)|dt\\
&\le \int_0^{\pi}|f(t)-K_n(C,t)|dt+\int_0^{\pi}\left|\frac{i}{n+1}\left[q^c_{n+1}E'_n(t)-q^c_{-(n+1)}E'_{-n}(t)\right]-\frac{i}{n+2}\left[q^c_{n+2}E'_n(t)-q^c_{-(n+2)}E'_{-n}(t)\right]\right|dt\\
&\le \int_0^{\pi}|f(t)-K_n(C,t)|dt+M_{r\epsilon}\left[|q^c_{n+1}|+|q^c_{-(n+1)}|+|q^c_{n+2}|+|q^c_{-(n+2)}|\right]\int_0^{\pi}\frac{dt}{|t|}\\
&\le \int_0^{\pi}|f(t)-K_n(C,t)|dt+M_{r\epsilon}\left[|q^c_{n+1}|+|q^c_{-(n+1)}|+|q^c_{n+2}|+|q^c_{-(n+2)}|\right]o(\log n)\\
&= o(1)\ ,\ n\to\infty.
\end{aligned}
$$

This completes the proof of Theorem 3.1. □

The second main result of this paper is to study the L^1-convergence of modified complex form (1.6) under new class $\mathbb{JS}^c_2$ of numerical sequences is as follows:

Theorem 3.2. *Let $\{q^c_k\}$ belongs to $\mathbb{JS}^c_2$. Then*

(i) $\lim_{n\to\infty}K_n(C,t)=f(t)$ *exists for all* $x\in(0,\pi]$,

(ii) $f \in L^1(0,\pi]$ *and* $||K_n(C,t) - f(t)||_{L^1} = o(1)$ *as* $n \to \infty$,

(iii) $||S_n(C,t) - f(t)||_{L^1} = o(1)$ *as* $n \to \infty$.

Proof. Firstly, we will show that $f(t)$ exists in $(0,\pi]$. The complex form of the modified sums is

$$\begin{aligned} K_n(C,t) &= S_n(C,t) + \frac{i}{n+1}\left[q^c_{n+1}E'_n(t) - q^c_{-(n+1)}E'_{-n}(t)\right] \\ &\qquad -\frac{i}{n+2}\left[q^c_{n+2}E'_n(t) - q^c_{-(n+2)}E'_{-n}(t)\right] \\ &= q^c_0 + \sum_{k=-n}^{k=n} q^c_k e^{ikt} + \frac{i}{n+1}\left[q^c_{n+1}E'_n(t) - q^c_{-(n+1)}E'_{-n}(t)\right] - \frac{i}{n+2}\left[q^c_{n+2}E'_n(t)\right. \\ &\qquad \left. -q^c_{-(n+2)}E'_{-n}(t)\right] \\ &= q^c_0 + \sum_{k=1}^{k=n}\left(\frac{q^c_k}{k}ke^{ikt} + \frac{q^c_{-k}}{k}ke^{-ikt}\right) + \frac{i}{n+1}\left[q^c_{n+1}E'_n(t) - q^c_{-(n+1)}E'_{-n}(t)\right] \\ &\qquad -\frac{i}{n+2}\left[q^c_{n+2}E'_n(t) - q^c_{-(n+2)}E'_{-n}(t)\right]. \end{aligned}$$

Apply Abel's transformation, we get

$$\begin{aligned} K_n(C,t) &= q^c_0 - i\left[\sum_{k=1}^{n-1}\Delta\left(\frac{q^c_k}{k}\right)E'_k(t) + \frac{q^c_n}{n}E'_n(t)\right] + i\left[\sum_{k=1}^{n-1}\Delta\left(\frac{q^c_k}{k}\right)E'_{-k}(t) + \frac{q^c_{-n}}{n}E'_{-n}(t)\right] \\ &\qquad +\frac{i}{n+1}\left[q^c_{n+1}E'_n(t) - q^c_{-(n+1)}E'_{-n}(t)\right] - \frac{i}{n+2}\left[q^c_{n+2}E'_n(t) - q^c_{-(n+2)}E'_{-n}(t)\right] \\ &= q^c_0 - i\sum_{k=1}^{n}\Delta\left(\frac{q^c_k}{k}\right)E'_k(t) + i\sum_{k=1}^{n}\Delta\left(\frac{q^c_{-k}}{k}\right)E'_{-k}(t) - \frac{i}{n+2}\left[q^c_{n+2}E'_n(t) - q^c_{-(n+2)}E'_{-n}(t)\right]. \end{aligned}$$

Again apply Abel's transformation, we obtain

$$\begin{aligned} K_n(C,t) &= q^c_0 - i\left[\sum_{k=1}^{n-1}\Delta^2\left(\frac{q^c_k}{k}\right)\bar{E}'_k(t) + \Delta\left(\frac{q^c_n}{n}\right)\bar{E}'{}_n(t)\right] + i\left[\sum_{k=1}^{n-1}\Delta^2\left(\frac{q^c_{-k}}{k}\right)\bar{E}'_{-k}(t)\right. \\ &\qquad \left.+\Delta\left(\frac{q^c_{-n}}{n}\right)\bar{E}'{}_{-n}(t)\right] - \frac{i}{n+2}\left[q^c_{n+2}E'_n(t) - q^c_{-(n+2)}E'_{-n}(t)\right] \\ |K_n(C,t)| &\leq |q^c_0| + \sum_{k=1}^{n-1}\left[\left|\Delta^2\left(\frac{q^c_k}{k}\right)\right|\left|\bar{E}'{}_k(t)\right| + \left|\Delta^2\left(\frac{q^c_{-k}}{k}\right)\right|\left|\bar{E}'{}_{-k}(t)\right|\right] + \left[\left|\Delta\left(\frac{q^c_n}{n}\right)\right|\left|\bar{E}'{}_n(t)\right|\right. \\ &\qquad \left.+\left|\Delta\left(\frac{q^c_{-n}}{n}\right)\right|\left|\bar{E}'{}_{-n}(t)\right|\right] + \left|\frac{q^c_{n+2}}{n+2}\right|\left|E'_n(t)\right| + \left|\frac{q^c_{-(n+2)}}{n+2}\right|\left|E'_{-n}(t)\right|. \end{aligned}$$

Using Lemma 2.3 and 2.4, we have

$$\begin{aligned}
&\leq |q_0^c| + \frac{M_{r\epsilon}}{|t|}\left\{\sum_{k=1}^{n-1} k^2\left[\left|\Delta^2\left(\frac{q_k^c}{k}\right)\right| + \left|\Delta^2\left(\frac{q_{-k}^c}{k}\right)\right|\right] + n^2\left[\left|\Delta\left(\frac{q_n^c}{n}\right)\right| + \left|\Delta\left(\frac{q_{-n}^c}{n}\right)\right|\right] + |q_{n+2}^c| + |q_{-(n+2)}^c|\right\}\\
&\leq |q_0^c| + \frac{M_{r\epsilon}}{|t|}\left\{\sum_{k=1}^{n-1} k^2\left[\left|\Delta^2\left(\frac{q_k^c}{k}\right)\right| + \left|\Delta^2\left(\frac{q_{-k}^c}{k}\right)\right|\right] + \sum_{k=n}^{\infty} k^2\left[\left|\Delta^2\left(\frac{q_k^c}{k}\right)\right| + \left|\Delta^2\left(\frac{q_{-k}^c}{k}\right)\right|\right] + |q_{n+2}^c| + |q_{-(n+2)}^c|\right\}\\
&\leq |q_0^c| + \frac{2M_{r\epsilon}}{|t|}\left\{\sum_{k=1}^{n-1} k^2\frac{Q_k}{k^2} + \sum_{k=n}^{\infty} k^2\frac{Q_k}{k^2} + 2\bar{M}\right\}\\
&\leq |q_0^c| + \frac{2M_{r\epsilon}}{|t|}\left\{2\sum_{k=1}^{\infty} Q_k + 2\bar{M}\right\} < \infty.
\end{aligned}$$

Since $\{q_k^c\} \in \mathbb{JS}_2^c$, where $\bar{M}$ is a positive constant.
Therefore, $\lim_{n\to\infty} K_n(C,t) = f(t)$ exists.

For proving (ii), consider

$$\begin{aligned}
f(t) - K_n(C,t) \quad = \quad & \sum_{k=n+1}^{\infty}\left(\frac{q_k^c}{k}ke^{ikt} + \frac{q_{-k}^c}{k}ke^{-ikt}\right) - \frac{i}{n+1}\left[q_{n+1}^c E_n'(t) - q_{-(n+1)}^c E_{-n}'(t)\right]\\
&+\frac{i}{n+2}\left[q_{n+2}^c E_n'(t) - q_{-(n+2)}^c E_{-n}'(t)\right].
\end{aligned}$$

Apply Abel's transformation, we have

$$= \sum_{k=n+1}^{\infty}\left[\Delta\left(\frac{q_k^c}{k}\right)(-iE_k'(t)) + \Delta\left(\frac{q_{-k}^c}{k}\right)(iE_{-k}'(t))\right] + \frac{i}{n+2}\left[q_{n+2}^c E_n'(t) - q_{-(n+2)}^c E_{-n}'(t)\right].$$

Again apply Abel's transformation, we get

$$\begin{aligned}
= \quad & -i\sum_{k=n+1}^{\infty}\left[\Delta^2\left(\frac{q_k^c}{k}\right)\bar{E}'_k(t) - \Delta\left(\frac{q_{n+1}^c}{n+1}\right)\bar{E}'_n(t)\right] + \frac{i}{n+2}\left[q_{n+2}^c E_n'(t) - q_{-(n+2)}^c E_{-n}'(t)\right]\\
&+i\left[\sum_{k=n+1}^{\infty}\Delta^2\left(\frac{q_{-k}^c}{k}\right)\bar{E}'_{-k}(t) - \Delta\left(\frac{q_{-(n+1)}^c}{n+1}\right)\bar{E}'_{-n}(t)\right]
\end{aligned}$$

$$\begin{aligned}
|f(t) - K_n(C,t)| \quad \leq \quad & \sum_{k=n+1}^{\infty}\left[\left|\Delta^2\left(\frac{q_k^c}{k}\right)\right||\bar{E}'_k(t)| + \left|\Delta^2\left(\frac{q_{-k}^c}{k}\right)\right||\bar{E}'_{-k}(t)|\right] + \left|\Delta\left(\frac{q_{n+1}^c}{n+1}\right)\right||\bar{E}'_n(t)|\\
&+\left|\Delta\left(\frac{q_{-(n+1)}^c}{n+1}\right)\right||\bar{E}'_{-n}(t)| + \left|\frac{q_{n+2}^c}{n+2}\right||E_n'(t)| + \left|\frac{q_{-(n+2)}^c}{n+2}\right||E_{-n}'(t)|
\end{aligned}$$

$$\begin{aligned}
&\leq \frac{M_{r\epsilon}}{|t|}\left\{\sum_{k=n+1}^{\infty} k^2\left[\left|\Delta^2\left(\frac{q_k^c}{k}\right)\right| + \left|\Delta^2\left(\frac{q_{-k}^c}{k}\right)\right|\right] + n^2\left[\left|\Delta\left(\frac{q_{n+1}^c}{n+1}\right)\right| + \left|\Delta\left(\frac{q_{-(n+1)}^c}{n+1}\right)\right|\right]\right.\\
&\qquad\left. +|q_{n+2}^c| + |q_{-(n+2)}^c|\right\}\\
&\leq \frac{M_{r\epsilon}}{|t|}\left[2\sum_{k=n+1}^{\infty} k^2\frac{Q_k}{k^2} + 2\sum_{k=n+1}^{\infty} k^2\frac{Q_k}{k^2} + |q_{n+2}^c| + |q_{-(n+2)}^c|\right]\\
&\leq \frac{M_{r\epsilon}}{|t|}\left[4\sum_{k=n+1}^{\infty} Q_k + |q_{n+2}^c| + |q_{-(n+2)}^c|\right]
\end{aligned}$$

$$\begin{aligned}||f(t)-g_n(C,t)||_{L^1} &\leq M_{r\epsilon}\left[4\sum_{k=n+1}^{\infty}Q_k\int_0^{\pi}\frac{dt}{|t|}+\left(|q^c_{n+2}|+|q^c_{-(n+2)}|\right)\int_0^{\pi}\frac{dt}{|t|}\right]\\ &\leq M_{r\epsilon}\left[4\sum_{k=n+1}^{\infty}Q_k\log k+\left(|q^c_{n+2}|+|q^c_{-(n+2)}|\right)\log n\right].\end{aligned}$$

Now, we note that

$$\sum_{k=n+1}^{\infty}Q_k\log k=o(1)$$

and

$$\begin{aligned}|q^c_{\pm(n+2)}|\log n &= (n+2)\log n\left|\frac{q^c_{\pm(n+2)}}{n+2}\right|=(n+2)\log n\sum_{k=n+2}^{\infty}\left|\Delta\left(\frac{q^c_{\pm(k)}}{k}\right)\right|\\ &\leq \log n\sum_{k=n+2}^{\infty}k\left|\Delta\left(\frac{q^c_{\pm k}}{k}\right)\right|\\ &\leq \log n\left[\sum_{k=n+2}^{\infty}k^2\left|\Delta^2\left(\frac{q^c_{\pm k}}{k}\right)\right|+\Delta\left(\frac{q^c_{\pm(n+2)}}{n+2}\right)n^2\right]\\ &\leq \log n\left[\sum_{k=n+2}^{\infty}k^2\left|\Delta^2\left(\frac{q^c_{\pm k}}{k}\right)\right|+\sum_{k=n+2}^{\infty}k^2\left|\Delta^2\left(\frac{q^c_{\pm k}}{k}\right)\right|\right]\\ &\leq \log n\left[2\sum_{k=n+2}^{\infty}Q_k\right]\\ &\leq 2\sum_{k=n+2}^{\infty}Q_k\log k=o(1).\end{aligned}$$

Subsequently, we get

$$||f(t)-K_n(C,t)||_{L^1}=o(1) \text{ as } n\to\infty.$$

To prove (iii),

$$||f_n(t)-S_n(C,t)|| \quad = \quad ||f(t)-K_n(C,t)+K_n(C,t)-S_n(C,t)||$$

$$\begin{aligned}&\leq \int_0^{\pi}|f(t)-K_n(C,t)|dt+\int_0^{\pi}|K_n(C,t)-S_n(t)|dt\\ &\leq \int_0^{\pi}|f(t)-K_n(C,t)|dt+\int_0^{\pi}\left|\frac{i}{n+1}\left[q^c_{n+1}E'_n(t)-q^c_{-(n+1)}E'_{-n}(t)\right]-\frac{i}{n+2}\left[q^c_{n+2}E'_n(t)-q^c_{-(n+2)}E'_{-n}(t)\right]\right|dt\\ &\leq \int_0^{\pi}|f(t)-K_n(C,t)|dt+M_{r\epsilon}\left[|q^c_{n+1}|+|q^c_{-(n+1)}|+|q^c_{n+2}|+|q^c_{-(n+2)}|\right]\int_0^{\pi}\frac{dt}{|t|}\\ &\leq \int_0^{\pi}|f(t)-K_n(C,t)|dt+M_{r\epsilon}\left[|q^c_{n+1}|+|q^c_{-(n+1)}|+|q^c_{n+2}|+|q^c_{-(n+2)}|\right]o(\log n)\\ &= o(1)\ ,\ n\to\infty.\end{aligned}$$

Thus the proof of the theorem is completed. □

References

[1] C.P. Chen *Integrability and L^1-convergence of multiple trigonometric series*, Bull. Austral. Math. Soc. **49** (1994) 333-339.

[2] S.K. Chouhan,J. Kaur, S.S. Bhatia, *Convergence and Summability of Fourier sine and cosine series with its Applications*, Proceedings of National Academy of Sciences, India, Sect A Phys. Sci. **89** (2019) 141-148.

[3] S.S. Bhatia, B. Ram, *On L^1-convergence of modified complex trigonometric sums*, Proc. Indian Acad. Sci. (Math. Sci.),**105 2** (1995) 193-199.

[4] S.S. Bhatia, K. Kaur, B. Ram, *L^1-convergence of modified complex trigonometric sums*, Lobachevskii J. of Math. **12** (2003) 3-10.

[5] J. Kaur, S.S. Bhatia, *Convergence of modified complex trigonometric sums in the metric space L*, Lobachevskii J. of Math., **31 3**(2010) 290-294.

[6] X.Z. Krasniqi, *A new of sequences and L^1-convergence of some complex modified trigonometric sums*, Lobachevskii J. of Math., **39 2** (2018) 236-239.

[7] X.Z. Krasniqi, *Further results on L^1-convergence of some modified complex trigonometric sums*, J. Numer. Anal. Approx. Theory, **44 2** (2015) 180-189.

[8] F. Moricz,*On the integrability and L^1-convergence of complex trigonometric series*, Proc. Amer. Math. Soc., **113** (1991) 53-64.

[9] Sheng, Shuyum, *The extensions of theorems of C.V. Stanojevic and V.B. Stanojevic*, Proc. Amer. Math. Soc., **110** (1990) 895-904.

[10] C.V. Stanojevic, V.B. Stanojevic, *Generalizations of the Sidon-Teljakovskii theorem*, Proc. Amer. Math. Soc., **110 4** (1987), 679-684.

[11] Ž. Tomovski, *Inequalities for Walsh polynomials with semi-monotone and semi-convex coefficients*, J. of Ineq. in Pure and Applied Math., **6 4** (2005), Article 98.

[12] Ž., Tomovski, *Generalization on some theorems of L^1-convergence of certain trigonometric series*, Tamkang J. of Mathematics, **39 1** (2008), 63-74.

Sandeep Kaur Gill
Department of Applied Sciences
Guru Nanak Dev Engineering College, Ludhiana, Punjab, India
E-mail address: sandeepchouhan247@gmail.com

Jatinderdeep Kaur
School of Mathematics, TIET, Patiala,India
E-mail address: jkaur@thapar.edu

S.S. Bhatia
School of Mathematics, TIET, Patiala,India
E-mail address: ssbhatia@thapar.edu