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A NEW SUBCLASS OF MEROMORPHIC FUNCTION WITH POSITIVE COEFFICIENTS

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Abstract. In the present investigation, the authors define a new class of meromorphic functions defined in the punctured unit disk $\Delta^* := \{z \in \mathbb{C} : 0 < |z| < 1\}$. Coefficient inequalities, growth and distortion inequalities, as well as closure results are obtained. We also prove a Property using an integral operator and its inverse defined on the new class.

1. Introduction

Let Σ denote the class of normalized meromorphic functions f of the form

$$f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_n z^n, \tag{1.1}$$

defined on the punctured unit disk

$$\Delta^* := \{z \in \mathbb{C} : 0 < |z| < 1\}.$$

The Hadamard product or convolution of two functions $f(z)$ given by (1.1) and

$$g(z) = \frac{1}{z} + \sum_{n=1}^{\infty} g_n z^n \tag{1.2}$$

is defined by

$$(f * g)(z) := \frac{1}{z} + \sum_{n=1}^{\infty} a_n g_n z^n.$$

A function $f \in \Sigma$ is *meromorphic starlike of order* α $(0 \leq \alpha < 1)$ if

$$-\Re\left(\frac{zf'(z)}{f(z)}\right) > \alpha \quad (z \in \Delta := \Delta^* \cup \{0\}).$$

The class of all such functions is denoted by $\Sigma^*(\alpha)$. Similarly the class of convex functions of order α is defined. Let Σ_P be the class of functions $f \in \Sigma$ with $a_n \geq 0$. The subclass of Σ_P consisting of starlike functions of order α is denoted by $\Sigma_P^*(\alpha)$.

Now, we define a new class of functions in Definition 1.

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Definition 1. *Let $0 \leq \alpha < 1$. Further, let $f(z) \in \Sigma_p$ be given by (1.1), $0 \leq \lambda < 1$ The class $M_P(\alpha, \lambda)$ is defined by*

$$M_P(\alpha, \lambda) = \left\{ f \in \Sigma_P : \Re\left(\frac{zf'(z)}{(\lambda - 1)f(z) + \lambda zf'(z)} \right) > \alpha \right\}.$$

Clearly, $M_P(\alpha, 0)$ reduces to the class $\Sigma_P^*(\alpha)$.

The class $\Sigma_P^*(\alpha)$ and various other subclasses of Σ have been studied rather extensively by Clunie [4], Nehari and Netanyahu [8], Pommerenke ([9], [10]), Royster [11], and others (cf., e.g., Bajpai [2] , Mogra et al. [7], Uralegaddi and Ganigi [16], Cho et al. [3], Aouf [3], and Uralegaddi and Somanatha [15]; see also Duren [[5], pages 29 and 137], and Srivastava and Owa [[13], pages 86 and 429]) (see also [1]).

In this paper, we obtain the coefficient inequalities, growth and distortion inequalities, as well as closure results for the class $M_P(\alpha, \lambda)$. Properties of a certain integral operator and its inverse defined on the new class $M_P(\alpha, \lambda)$ are also discussed.

2. Coefficients Inequalities

Our first theorem gives a necessary and sufficient condition for a function f to be in the class $M_P(\alpha, \lambda)$.

Theorem 2.1. *Let $f(z) \in \Sigma_P$ be given by (1.1). Then $f \in M_P(\alpha, \lambda)$ if and only if*

$$\sum_{n=1}^{\infty} \{n + \alpha - \alpha\lambda(1 + n)\} \; a_n \leq 1 - \alpha. \tag{2.1}$$

Proof. If $f \in M_p(\alpha, \lambda)$, then

$$\Re\left(\frac{zf'(z)}{(\lambda - 1)f(z) + \lambda zf'(z)} \right) = \Re\left\{ \frac{-1 + \sum_{n=1}^{\infty} na_n z^{n+1}}{-1 + \sum_{n=1}^{\infty} (\lambda - 1 + \lambda n) a_n z^{n+1}} \right\} > \alpha.$$

By letting $z \to 1^-$, we have

$$\left\{ \frac{-1 + \sum_{n=1}^{\infty} na_n}{-1 + \sum_{n=1}^{\infty} (\lambda - 1 + \lambda n) a_n} \right\} > \alpha.$$

This shows that (2.1) holds.

Conversely assume that (2.1) holds. It is sufficient to show that

$$\left| \frac{zf'(z) - \{(\lambda - 1)f(z) + \lambda zf'(z)\}}{zf'(z) + (1 - 2\alpha)\{(\lambda - 1)f(z) + \lambda zf'(z)\}} \right| < 1 \quad (z \in \Delta).$$

Using (2.1), we see that

$$\begin{aligned}
&\left| \frac{zf'(z) - \{(\lambda - 1)f(z) + \lambda zf'(z)\}}{zf'(z) + (1 - 2\alpha)\{(\lambda - 1)f(z) + \lambda zf'(z)\}} \right| \\
&= \left| \frac{\sum_{n=1}^{\infty} (1 - \lambda)(n + 1) a_n z^{n+1}}{-2(1 - \alpha) + \sum_{n=1}^{\infty} [\{1 + (1 - 2\alpha)\lambda\} n + (1 - 2\alpha)(\lambda - 1)] \, a_n z^{n+1}} \right| \\
&\leq \frac{\sum_{n=1}^{\infty} (1 - \lambda)(n + 1) a_n}{2(1 - \alpha) - \sum_{n=1}^{\infty} [\{1 + (1 - 2\alpha)\lambda\} n + (1 - 2\alpha)(\lambda - 1)] \, a_n} \leq 1.
\end{aligned}$$

Thus we have $f \in M_p(\alpha, \lambda)$. □

For the choice of $\lambda = 0$, we get the following.

Remark 2.2. *Let $f(z) \in \Sigma_P$ be given by (1.1). Then $f \in \Sigma_P^*(\alpha)$ if and only if*

$$\sum_{n=1}^{\infty}(n+\alpha)a_n \leq 1-\alpha.$$

Our next result gives the coefficient estimates for functions in $M_P(\alpha, \lambda)$.

Theorem 2.3. *If $f \in M_P(\alpha, \lambda)$, then*

$$a_n \leq \frac{1-\alpha}{\{n+\alpha-\alpha\lambda(1+n)\}}, \qquad n = 1, 2, 3, \ldots.$$

The result is sharp for the functions $F_n(z)$ given by

$$F_n(z) = \frac{1}{z} + \frac{1-\alpha}{\{n+\alpha-\alpha\lambda(1+n)\}}z^n, \qquad n = 1, 2, 3, \ldots.$$

Proof. If $f \in M_P(\alpha, \lambda)$, then we have, for each n,

$$\{n+\alpha-\alpha\lambda(1+n)\}\, a_n \leq \sum_{n=1}^{\infty}\{n+\alpha-\alpha\lambda(1+n)\}\, a_n \leq 1-\alpha.$$

Therefore we have

$$a_n \leq \frac{1-\alpha}{\{n+\alpha-\alpha\lambda(1+n)\}}.$$

Since

$$F_n(z) = \frac{1}{z} + \frac{1-\alpha}{\{n+\alpha-\alpha\lambda(1+n)\}}z^n$$

satisfies the conditions of Theorem 2.1, $F_n(z) \in M_P(\alpha, \lambda)$ and the equality is attained for this function. □

For $\lambda = 0$, we have the following corollary.

Remark 2.4. *If $f \in \Sigma_P^*(\alpha)$, then*

$$a_n \leq \frac{1-\alpha}{n+\alpha}, \qquad n = 1, 2, 3, \ldots.$$

Theorem 2.5. *If $f \in M_P(\alpha, \lambda)$, then*

$$\frac{1}{r} - \frac{1-\alpha}{1+\alpha-2\alpha\lambda}r \leq |f(z)| \leq \frac{1}{r} + \frac{1-\alpha}{1+\alpha-2\alpha\lambda}r \quad (|z| = r).$$

The result is sharp for

$$f(z) = \frac{1}{z} + \frac{1-\alpha}{\{1+\alpha-2\alpha\lambda\}}z. \tag{2.2}$$

Proof. Since $f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_n z^n$, we have

$$|f(z)| \leq \frac{1}{r} + \sum_{n=1}^{\infty} a_n r^n \leq \frac{1}{r} + r\sum_{n=1}^{\infty} a_n.$$

Since,

$$\sum_{n=1}^{\infty} a_n \leq \frac{1-\alpha}{1+\alpha-2\alpha\lambda}.$$

Using this, we have

$$|f(z)| \leq \frac{1}{r} + \frac{1-\alpha}{1+\alpha-2\alpha\lambda}r.$$

Similarly

$$|f(z)| \geq \frac{1}{r} - \frac{1-\alpha}{1+\alpha-2\alpha\lambda} r.$$

The result is sharp for $f(z) = \frac{1}{z} + \frac{1-\alpha}{1+\alpha-2\alpha\lambda} z$. □

Similarly we have the following:

Theorem 2.6. *If $f \in M_P(\alpha, \lambda)$, then*

$$\frac{1}{r^2} - \frac{1-\alpha}{1+\alpha-2\alpha\lambda} \leq |f'(z)| \leq \frac{1}{r^2} + \frac{1-\alpha}{1+\alpha-2\alpha\lambda} \quad (|z| = r).$$

The result is sharp for the function given by (2.2).

3. Neighborhoods for the class $M_p^{(\gamma)}(\alpha, \lambda)$

In this section, we determine the neighborhood for the class $M_p^{(\gamma)}(\alpha, \lambda)$, which we define as follows:

Definition 2. *A function $f \in \Sigma_p$ is said to be in the class $M_p^{(\gamma)}(\alpha, \lambda)$ if there exists a function $g \in M_p(\alpha, \lambda)$ such that*

$$\left| \frac{f(z)}{g(z)} - 1 \right| < 1 - \gamma, \qquad (z \in \Delta, 0 \leq \gamma < 1). \tag{3.1}$$

Following the earlier works on neighborhoods of analytic functions by Goodman [6] and Ruscheweyh [14], we define the δ-neighborhood of a function $f \in \Sigma_p$ by

$$N_\delta(f) := \left\{ g \in \Sigma_p \ : \ g(z) = \frac{1}{z} + \sum_{n=1}^{\infty} b_n z^n \text{ and } \sum_{n=1}^{\infty} n|a_n - b_n| \leq \delta \right\}. \tag{3.2}$$

Theorem 3.1. *If $g \in M_p(\alpha, \lambda)$ and*

$$\gamma = 1 - \frac{\delta(1+\alpha-2\alpha\lambda)}{2\alpha - 2\alpha\lambda}, \tag{3.3}$$

then

$$N_\delta(g) \subset M_p^{(\gamma)}(\alpha, \lambda).$$

Proof. Let $f \in N_\delta(g)$. Then we find from (3.2) that

$$\sum_{n=1}^{\infty} n|a_n - b_n| \leq \delta, \tag{3.4}$$

which implies the coefficient inequality

$$\sum_{n=1}^{\infty} |a_n - b_n| \leq \delta, \qquad (n \in \mathbb{N}). \tag{3.5}$$

Since $g \in M_p(\alpha, \lambda)$, we have [cf. equation (2.1)]

$$\sum_{n=1}^{\infty} b_n \leq \frac{1-\alpha}{1+\alpha-2\alpha\lambda}, \tag{3.6}$$

so that

$$\begin{aligned}\left|\frac{f(z)}{g(z)}-1\right| &< \frac{\sum_{n=1}^{\infty}|a_n-b_n|}{1-\sum_{n=1}^{\infty}b_n}\\ &= \frac{\delta(1+\alpha-2\alpha\lambda)}{2\alpha-2\alpha\lambda}\\ &= 1-\gamma,\end{aligned}$$

provided γ is given by (3.3). Hence, by definition, $f \in M_p^{(\gamma)}(\alpha,\lambda)$ for γ given by (3.3), which completes the proof. □

4. Closure Theorems

Let the functions $F_k(z)$ be given by

$$F_k(z)=\frac{1}{z}+\sum_{n=1}^{\infty}f_{n,k}z^n, \quad k=1,2,...,m. \tag{4.1}$$

We shall prove the following closure theorems for the class $M_P(\alpha,\lambda)$.

Theorem 4.1. *Let the function $F_k(z)$ defined by (4.1) be in the class $M_P(\alpha,\lambda)$ for every $k=1,2,...,m$. Then the function $f(z)$ defined by*

$$f(z)=\frac{1}{z}+\sum_{n=1}^{\infty}a_nz^n \qquad (a_n\geq 0)$$

belongs to the class $M_P(\alpha,\lambda)$, where $a_n=\frac{1}{m}\sum_{k=1}^{m}f_{n,k}$ $(n=1,2,..)$

Proof. Since $F_n(z)\in M_P(\alpha,\lambda)$, it follows from Theorem 2.1 that

$$\sum_{n=1}^{\infty}\{n+\alpha-\alpha\lambda(1+n)\}f_{n,k}\leq 1-\alpha \tag{4.2}$$

for every $k=1,2,..,m$. Hence

$$\begin{aligned}\sum_{n=1}^{\infty}\{n+\alpha-\alpha\lambda(1+n)\}a_n &= \sum_{n=1}^{\infty}\{n+\alpha-\alpha\lambda(1+n)\}\left(\frac{1}{m}\sum_{k=1}^{m}f_{n,k}\right)\\ &= \frac{1}{m}\sum_{k=1}^{m}\left(\sum_{n=1}^{\infty}\{n+\alpha-\alpha\lambda(1+n)\}f_{n,k}\right)\\ &\leq 1-\alpha.\end{aligned}$$

By Theorem 2.1, it follows that $f(z)\in M_P(\alpha,\lambda)$. □

Theorem 4.2. *The class $M_P(\alpha,\lambda)$ is closed under convex linear combination.*

Proof. Let the function $F_k(z)$ given by (4.1) be in the class $M_P(\alpha,\lambda)$. Then it is enough to show that the function

$$H(z)=\lambda F_1(z)+(1-\lambda)F_2(z) \quad (0\leq\lambda\leq 1)$$

is also in the class $M_P(\alpha,\lambda)$. Since for $0\leq\lambda\leq 1$,

$$H(z)=\frac{1}{z}+\sum_{n=1}^{\infty}[\lambda f_{n,1}+(1-\lambda)f_{n,2}]z^n,$$

we observe that

$$\sum_{n=1}^{\infty} \{n+\alpha-\alpha\lambda(1+n)\} [\lambda f_{n,1} + (1-\lambda) f_{n,2}]$$

$$\begin{aligned} &= \lambda \sum_{n=1}^{\infty} \{n+\alpha-\alpha\lambda(1+n)\} f_{n,1} + (1-\lambda) \sum_{n=1}^{\infty} \{n+\alpha-\alpha\lambda(1+n)\} f_{n,2} \\ &\leq 1-\alpha. \end{aligned}$$

By Theorem 2.1, we have $H(z) \in M_P(\alpha, \lambda)$. □

Theorem 4.3. *Let $F_0(z) = \frac{1}{z}$ and $F_n(z) = \frac{1}{z} + \frac{1-\alpha}{\{n+\alpha-\alpha\lambda(1+n)\}} z^n$ for $n = 1, 2, \ldots$. Then $f(z) \in M_P(\alpha, \lambda)$ if and only if $f(z)$ can be expressed in the form $f(z) = \sum_{n=0}^{\infty} \lambda_n F_n(z)$ where $\lambda_n \geq 0$ and $\sum_{n=0}^{\infty} \lambda_n = 1$.*

Proof. Let

$$f(z) = \sum_{n=0}^{\infty} \lambda_n F_n(z)$$

$$= \frac{1}{z} + \sum_{n=1}^{\infty} \frac{\lambda_n (1-\alpha)}{\{n+\alpha-\alpha\lambda(1+n)\}} z^n.$$

Then

$$\sum_{n=1}^{\infty} \lambda_n \frac{1-\alpha}{\{n+\alpha-\alpha\lambda(1+n)\}} \frac{\{n+\alpha-\alpha\lambda(1+n)\}}{(1-\alpha)} = \sum_{n=1}^{\infty} \lambda_n = 1 - \lambda_0 \leq 1.$$

By Theorem 2.1, we have $f(z) \in M_P(\alpha, \lambda)$.

Conversely, let $f(z) \in M_P(\alpha, \lambda)$. From Theorem 2.3, we have

$$a_n \leq \frac{1-\alpha}{\{n+\alpha-\alpha\lambda(1+n)\}} \quad \text{for} \quad n = 1, 2, ..$$

we may take

$$\lambda_n = \frac{\{n+\alpha-\alpha\lambda(1+n)\}}{1-\alpha} a_n \quad \text{for} \quad n = 1, 2, ...$$

and

$$\lambda_0 = 1 - \sum_{n=1}^{\infty} \lambda_n.$$

Then

$$f(z) = \sum_{n=0}^{\infty} \lambda_n F_n(z).$$

□

5. Partial Sums

Silverman [12] determined sharp lower bounds on the real part of the quotients between the normalized starlike or convex functions and their sequences of partial sums. As a natural extension, one is interested to search results analogous to those of Silverman for meromorphic univalent functions. In this section, motivated essentially by the work of silverman [12] and Cho and Owa [3] we will investigate the ratio of a function of the form

$$f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_n z^n, \tag{5.1}$$

to its sequence of partial sums

$$f_1(z) = \frac{1}{z} \text{ and } f_k(z) = \frac{1}{z} + \sum_{n=1}^{k} a_n z^n \tag{5.2}$$

when the coefficients are sufficiently small to satisfy the condition analogous to

$$\sum_{n=1}^{\infty} \{n + \alpha - \alpha\lambda(1+n)\} \ a_n \leq 1 - \alpha.$$

For the sake of brevity we rewrite it as

$$\sum_{n=1}^{\infty} d_n |a_n| \leq 1 - \alpha, \tag{5.3}$$

where

$$d_n := n + \alpha - \alpha\lambda(1+n) \tag{5.4}$$

More precisely we will determine sharp lower bounds for $\Re\{f(z)/f_k(z)\}$ and $\Re\{f_k(z)/f(z)\}$. In this connection we make use of the well known results that$\Re\left\{\frac{1+w(z)}{1-w(z)}\right\} > 0 \ \ (z \in \Delta)$ if and only if $\omega(z) = \sum\limits_{n=1}^{\infty} c_n z^n$ satisfies the inequality $|\omega(z)| \leq |z|$. Unless otherwise stated, we will assume that f is of the form (1.1) and its sequence of partial sums is denoted by $f_k(z) = \frac{1}{z} + \sum_{n=1}^{k} a_n z^n$.

Theorem 5.1. *Let $f(z) \in M_P(\alpha, \lambda)$ be given by (5.1)satisfies condition, (2.1)*

$$Re\left\{\frac{f(z)}{f_k(z)}\right\} \geq \frac{d_{k+1}(\lambda,\alpha) - 1 + \alpha}{d_{k+1}(\lambda,\alpha)} \qquad (z \in U) \tag{5.5}$$

where

$$d_n(\lambda,\alpha) \geq \begin{cases} 1-\alpha, & if \quad n = 1,2,3,\ldots,k \\ d_{k+1}(\lambda,\alpha), & if \quad n = k+1, k+2, \ldots. \end{cases} \tag{5.6}$$

The result (5.5) is sharp with the function given by

$$f(z) = \frac{1}{z} + \frac{1-\alpha}{d_{k+1}(\lambda,\alpha)} z^{k+1}. \tag{5.7}$$

Proof. Define the function $w(z)$ by

$$\frac{1+w(z)}{1-w(z)} = \frac{d_{k+1}(\lambda,\alpha)}{1-\alpha}\left[\frac{f(z)}{f_k(z)} - \frac{d_{k+1}(\lambda,\alpha) - 1 + \alpha}{d_{k+1}(\lambda,\alpha)}\right]$$

$$= \frac{1 + \sum\limits_{n=1}^{k} a_n z^{n+1} + \left(\frac{d_{k+1}(\lambda,\alpha)}{1-\alpha}\right) \sum\limits_{n=k+1}^{\infty} a_n z^{n+1}}{1 + \sum\limits_{n=1}^{k} a_n z^{n+1}} \tag{5.8}$$

It suffices to show that $|w(z)| \leq 1$. Now, from (5.8) we can write

$$w(z) = \frac{\left(\frac{d_{k+1}(\lambda,\alpha)}{1-\alpha}\right) \sum\limits_{n=k+1}^{\infty} a_n z^{n+1}}{2 + 2\sum\limits_{n=1}^{k} a_n z^{n+1} + \left(\frac{d_{k+1}(\lambda,\alpha)}{1-\alpha}\right) \sum\limits_{k=n+1}^{\infty} a_n z^{n+1}}.$$

Hence we obtain

$$|w(z)| \leq \frac{\left(\frac{d_{k+1}(\lambda,\alpha)}{1-\alpha}\right) \sum\limits_{k=n+1}^{\infty} |a_n|}{2 - 2\sum\limits_{n=1}^{k} |a_n| - \left(\frac{d_{k+1}(\lambda,\alpha)}{1-\alpha}\right) \sum\limits_{n=k+1}^{\infty} |a_n|}$$

Now $|w(z)| \leq 1$ if

$$2\left(\frac{d_{k+1}(\lambda,\alpha)}{1-\alpha}\right) \sum_{n=k+1}^{\infty} |a_n| \leq 2 - 2\sum_{n=1}^{k} |a_n|$$

or, equivalently,

$$\sum_{n=1}^{k} |a_n| + \frac{d_{k+1}(\lambda,\alpha)}{1-\alpha} \sum_{n=k+1}^{\infty} |a_n| \leq 1.$$

From the condition (2.1), it is sufficient to show that

$$\sum_{n=1}^{k} |a_n| + \frac{d_{k+1}(\lambda,\alpha)}{1-\alpha} \sum_{n=k+1}^{\infty} |a_n| \leq \sum_{n=1}^{\infty} \frac{d_n(\lambda,\alpha)}{1-\alpha} |a_n|$$

which is equivalent to

$$\begin{aligned} &\sum_{n=1}^{k} \left(\frac{d_n(\lambda,\alpha) - 1 + \alpha}{1-\alpha}\right) |a_n| \\ &+ \sum_{n=k+1}^{\infty} \left(\frac{d_n(\lambda,\alpha) - d_{k+1}(\lambda,\alpha)}{1-\alpha}\right) |a_n| \\ \geq\ & 0 \end{aligned} \tag{5.9}$$

To see that the function given by (5.7) gives the sharp result, we observe that for $z = re^{i\pi/k}$

$$\begin{aligned} \frac{f(z)}{f_k(z)} &= 1 + \frac{1-\alpha}{d_{k+1}(\lambda,\alpha)} z^n \to 1 - \frac{1-\alpha}{d_{k+1}(\lambda,\alpha)} \\ &= \frac{d_{k+1}(\lambda,\alpha) - 1 + \alpha}{d_{k+1}(\lambda,\alpha)} \quad \text{when } r \to 1^-. \end{aligned}$$

which shows the bound (5.5) is the best possible for each $k \in \mathbb{N}$. □

The proof of the next theorem is much akin to that of the earlier theorem and hence we state the theorem without proof.

Theorem 5.2. *Let $f(z) \in M_P(\alpha, \lambda)$ be given by (5.1)satisfies condition, (2.1)*

$$Re\left\{\frac{f_k(z)}{f(z)}\right\} \geq \frac{d_{k+1}(\lambda,\alpha)}{d_{k+1}(\lambda,\alpha)+1-\alpha} \qquad (z \in U) \tag{5.10}$$

where

$$d_{k+1}(\lambda,\alpha) \geq 1-\alpha$$

$$d_n(\lambda,\alpha) \geq \begin{cases} 1-\alpha, & if \quad n = 1,2,3,\dots,k \\ d_{k+1}(\lambda,\alpha), & if \quad n = k+1, k+2, \dots. \end{cases} \tag{5.11}$$

The result (5.10) is sharp with the function given by

$$f(z) = \frac{1}{z} + \frac{1-\alpha}{d_{k+1}(\lambda,\alpha)} z^{k+1}. \tag{5.12}$$

6. Radius of meromorphic starlikeness and meromorphic convexity

The radii of starlikeness and convexity for the class are given by the following theorems for the class $M_P(\alpha, \lambda)$.

Theorem 6.1. *Let the function f be in the class $M_P(\alpha, \lambda)$. Then f is meromorphically starlike of order $\rho (0 \leq \rho < 1)$, in $|z| < r_1(\alpha, \lambda, \rho)$, where*

$$r_1(\alpha,\lambda,\rho) = \inf_{n\geq 1}\left[\frac{(1-\rho)(1-\alpha)}{(n+2-\rho)\left\{n+\alpha-\alpha\lambda(1+n)\right\}}\right]^{\frac{1}{n+1}}, \tag{6.1}$$

Proof. Since,

$$f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_n z^n,$$

we get

$$f'(z) = -\frac{1}{z^2} + \sum_{n=1}^{\infty} n a_n z^{n-1}.$$

It is sufficient to show that

$$\left| -\frac{zf'(z)}{f(z)} - 1 \right| \leq 1-\rho \tag{6.2}$$

or equivalently

$$\left|\frac{zf'(z)}{f(z)} + 1\right| = \left|\frac{\sum_{n=1}^{\infty}(n+1)a_n z^n}{\frac{1}{z} - \sum_{n=1}^{\infty} a_n z^n}\right| \leq 1-\rho$$

or

$$\sum_{n=1}^{\infty}\left(\frac{n+2-\rho}{1-\rho}\right) a_n |z|^{n+1} \leq 1,$$

for $0 \leq \rho < 1$, and $|z| < r_1(\alpha, \lambda, \rho)$. By Theorem 2.1, (6.2) will be true if

$$\left(\frac{n+2-\rho}{1-\rho}\right)|z|^{n+1} \leq \frac{1-\alpha}{\{n+\alpha-\alpha\lambda(1+n)\}}$$

or, if

$$|z| \leq \left[\frac{(1-\rho)(1-\alpha)}{(n+2-\rho)\left\{n+\alpha-\alpha\lambda(1+n)\right\}}\right]^{\frac{1}{n+1}}, \quad n \geq 1. \tag{6.3}$$

This completes the proof of Theorem 6.1. □

Theorem 6.2. *Let the function f in the class $M_P(\alpha,\lambda)$. Then f is meromorphically convex of order ρ, $(0 \leq \rho < 1)$, in $|z| < r_2(\alpha,\lambda,\rho)$, where*

$$r_2(\alpha,\lambda,\rho) = \inf_{n\geq 1}\left[\frac{(1-\rho)(1-\alpha)}{n\,(n+2-\rho)\left\{n+\alpha-\alpha\lambda(1+n)\right\}}\right]^{\frac{1}{n+1}}, \qquad n \geqq 1, \tag{6.4}$$

Proof. Since,

$$f(z) = \frac{1}{z} - \sum_{n=1}^{\infty} a_n z^n,$$

we get

$$f'(z) = -\frac{1}{z^2} - \sum_{n=1}^{\infty} n a_n z^{n-1}.$$

It is sufficient to show that

$$\left|-1 - \frac{zf''(z)}{f'(z)} - 1\right| \leq 1-\rho \text{ or equivalently} \tag{6.5}$$

$$\left|\frac{zf''(z)}{f'(z)} + 2\right| = \left|\frac{\sum_{n=1}^{\infty} n(n+1)a_n z^{n-1}}{-\frac{1}{z^2} - \sum_{n=1}^{\infty} n a_n z^{n-1}}\right| \leq 1-\rho \text{ or}$$

$$\sum_{n=1}^{\infty}\left(\frac{n(n+2-\rho)}{1-\rho}\right) a_n |z|^{n+1} \leq 1,$$

for $0 \leq \rho < 1$, and $|z| < r_2(\alpha,\lambda,\rho)$. By Theorem 2.1, (6.5) will be true if

$$\left(\frac{n(n+2-\rho)}{1-\rho}\right)|z|^{n+1} \leq \frac{(1-\alpha)}{\left\{n+\alpha-\alpha\lambda(1+n)\right\}}$$

or, if

$$|z| \leq \left[\frac{(1-\rho)(1-\alpha)}{n(n+2-\rho)\left\{n+\alpha-\alpha\lambda(1+n)\right\}}\right]^{\frac{1}{n+1}}, \quad n \geq 1. \tag{6.6}$$

This completes the proof of Theorem 6.2. □

7. Integral Operators

In this section, we consider integral transforms of functions in the class $M_p(\alpha,\lambda)$.

Theorem 7.1. *Let the function $f(z)$ given by (1) be in $M_p(\alpha,\lambda)$. Then the integral operator*

$$F(z) = c\int_0^1 u^c f(uz)du \qquad (0 < u \leq 1, 0 < c < \infty)$$

is in $M_p(\delta,\lambda)$, where

$$\delta = \frac{(c+2)\left\{1+\alpha-2\alpha\lambda\right\} - c(1-\alpha)}{c(1-\alpha)\left\{1-2\lambda\right\} + (1+\alpha)\left\{1-2\lambda\right\}(c+2)}.$$

The result is sharp for the function $f(z) = \frac{1}{z} + \frac{1-\alpha}{\{1+\alpha-2\alpha\lambda\}}z$.

Proof. Let $f(z) \in M_p(\alpha, \lambda)$. Then

$$\begin{aligned} F(z) &= c\int_0^1 u^c f(uz)du \\ &= c\int_0^1 \left(\frac{u^{c-1}}{z} + \sum_{n=1}^{\infty} f_n u^{n+c} z^n\right) du \\ &= \frac{1}{z} + \sum_{n=1}^{\infty} \frac{c}{c+n+1} f_n z^n. \end{aligned}$$

It is sufficient to show that

$$\sum_{n=1}^{\infty} \frac{c\{n+\delta-\delta\lambda(1+n)\}}{(c+n+1)(1-\delta)} a_n \leq 1. \tag{7.1}$$

Since $f \in M_p(\alpha, \lambda)$, we have

$$\sum_{n=1}^{\infty} \frac{\{n+\alpha-\alpha\lambda(1+n)\}}{(1-\alpha)} a_n \leq 1.$$

Note that (7.1) is satisfied if

$$\frac{c\{n+\delta-\delta\lambda(1+n)\}}{(c+n+1)(1-\delta)} \leq \frac{\{n+\alpha-\alpha\lambda(1+n)\}}{(1-\alpha)}.$$

Rewriting the inequality, we have

$$c\{n+\delta-\delta\lambda(1+n)\}(1-\alpha) \leq (c+n+1)(1-\delta)\{n+\alpha-\alpha\lambda(1+n)\}.$$

Solving for δ, we have

$$\delta \leq \frac{(c+n+1)\{n+\alpha-\alpha\lambda(1+n)\} - cn(1-\alpha)}{c(1-\alpha)\{1-\lambda(1+n)\} + \{(n+\alpha-\alpha\lambda(1+n)\}(c+n+1)} = F(n).$$

A simple computation will show that $F(n)$ is increasing and $F(n) \geq F(1)$. Using this, the results follows. □

For the choice of $\lambda = 0$, we have the following result of Uralegaddi and Ganigi [15].

Remark 7.2. *Let the function* $f(z)$ *defined by (1) be in* $\Sigma_p^*(\alpha)$. *Then the integral operator*

$$F(z) = c\int_0^1 u^c f(uz)du \qquad (0 < u \leq 1, 0 < c < \infty)$$

is in $\Sigma_p^*(\delta)$, *where* $\delta = \frac{1+\alpha+c\alpha}{1+\alpha+c}$. *The result is sharp for the function*

$$f(z) = \frac{1}{z} + \frac{1-\alpha}{1+\alpha}z.$$

Also we have the following:

Theorem 7.3. *Let $f(z)$, given by (1), be in $M_p(\alpha,\lambda)$,*

$$F(z)=\frac{1}{c}[(c+1)f(z)+zf'(z)]=\frac{1}{z}+\sum_{n=1}^{\infty}\frac{c+n+1}{c}f_nz^n, \qquad c>0. \tag{7.2}$$

Then $F(z)$ is in $M_p(\alpha,\lambda)$ for $|z|\le r(\alpha,\lambda,\beta)$ where

$$r(\alpha,\lambda,\beta)=\inf_n\left(\frac{c(1-\beta)\{n+\alpha-\alpha\lambda(1+n)\}}{(1-\alpha)(c+n+1)\{n+\beta-\beta\lambda(1+n)\}}\right)^{1/(n+1)}, \quad n=1,2,3,\dots.$$

The result is sharp for the function $f_n(z)=\frac{1}{z}+\frac{1-\alpha}{\{n+\alpha-\alpha\lambda(1+n)\}}z^n, \quad n=1,2,3,\dots.$

Proof. Let $w=\frac{zf'(z)}{(\lambda-1)f(z)+\lambda zf'(z)}$. Then it is sufficient to show that

$$\left|\frac{w-1}{w+1-2\beta}\right|<1.$$

A computation shows that this is satisfied if

$$\sum_{n=1}^{\infty}\frac{\{n+\beta-\beta\lambda(1+n)\}(c+n+1)}{(1-\beta)c}a_n|z|^{n+1}\le 1. \tag{7.3}$$

Since $f\in M_p(\alpha,\lambda)$, by Theorem 2.1, we have

$$\sum_{n=1}^{\infty}\{n+\alpha-\alpha\lambda(1+n)\}a_n\le 1-\alpha.$$

The equation (7.3) is satisfied if

$$\frac{\{n+\beta-\beta\lambda(1+n)\}(c+n+1)}{(1-\beta)c}a_n|z|^{n+1}\le\frac{\{n+\alpha-\alpha\lambda(1+n)\}a_n}{1-\alpha}.$$

Solving for $|z|$, we get the result. □

For the choice of $\lambda=0$, we have the following result of Uralegaddi and Ganigi [15].

Remark 7.4. *Let the function $f(z)$ defined by (1) be in $\Sigma_p^*(\alpha)$ and $F(z)$ given by (7.2). Then $F(z)$ is in $\Sigma_p^*(\alpha)$ for $|z|\le r(\alpha,\beta)$ where*

$$r(\alpha,\beta)=\inf_n\left(\frac{c(1-\beta)(n+\alpha)}{(1-\alpha)(c+n+1)(n+\beta)}\right)^{1/(n+1)}, \quad n=1,2,3,\dots.$$

The result is sharp for the function $f_n(z)=\frac{1}{z}+\frac{1-\alpha}{n+\alpha}z^n, \quad n=1,2,3,\dots.$

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