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ON CERTAIN SUBCLASS OF MEROMORPHIC HARMONIC FUNCTIONS WITH FIXED RESIDUE α

(DEDICATED IN OCCASION OF THE 70-YEARS OF PROFESSOR HARI M. SRIVASTAVA)

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Abstract. In this paper, we consider some properties such as growth and distortion theorem, coefficient problems, linear combinations for certain subclass of meromorphic harmonic functions with positive coefficients.

1. Introduction

Let $A(p)$ denote the set of function analytic in $D\backslash\{p\}$, Where $D = \{z : |z| < 1\}$. In the annulus $\{z : p < |z| < 1\}$ every function h in S_p has an expansion of the form

$$h(z) = \frac{\alpha}{z-p} + \sum_{n=1}^{\infty} a_n z^n, \tag{1.1}$$

where $\alpha = Res(f, p)$, with $0 < \alpha \leq 1$, $z \in D\backslash\{p\}$.

The function h given in (1.1) was studied by Jinxi Ma [8] and Ghanim and Darus [1].

A continuous function $f = u + iv$ is a complex valued harmonic function in a complex domain D if both u and v are real harmonic in D. In any simply connected domain $D \subset \mathbb{C}$ we can write $f = h + \overline{g}$, where h and g are analytic in D. A necessary and sufficient condition for f to be locally univalent and sense preserving in D is that $|\ h'(z)| > |\ g'(z)|$ in D (see [5]). In [7], there is a more comprehensive study on harmonic univalent functions.

Denote by SH_p the class of the functions $f = h + \overline{g}$ that are harmonic univalent and sense preserving in the punctured unit disk $D\backslash\{p\}$.

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Then for $f = h + \overline{g}$ we may express the analytic function h as the form (1.1) and g as

$$g\ (z) = \sum_{n=1}^{\infty} b_n z^n$$

then, we have

$$f\ (z) = h\ (z) + \overline{g}\ (z) = \frac{\alpha}{z-p} + \sum_{n=1}^{\infty} a_n z^n + \overline{\sum_{n=1}^{\infty} b_n z^n}, \tag{1.2}$$

where $\alpha = Res(f,p)$, with $0 < \alpha \leq 1$, $z \in D\backslash\{p\}$.

Let $\mathbb{SH}_p$ be subclass of SH_p consisting of function of the form

$$f\ (z) = h\ (z) + \overline{g}\ (z) = \frac{\alpha}{z-p} + \sum_{n=1}^{\infty} a_n z^n + \overline{\sum_{n=1}^{\infty} b_n z^n}, \quad (a_n, b_n \geq 0) \tag{1.3}$$

where $\alpha = Res(f,p)$, with $0 < \alpha \leq 1$, $z \in D\backslash\{p\}$, which are univalent harmonic in the punctured unit disc $D\backslash\{p\}$. $h(z)$ and $g(z)$ are analytic in $D\backslash\{p\}$ and D, respectively and h(z) has a simple pole at the point p with residue α.

For $\alpha = 1$ and $p = 0$ the function f studied by Bostanci, Yalçin and Öztürk [4].

A function $f \in SH_p$ is said to be in the subclass SH_p^* of meromorphically harmonic starlike in $D\backslash\{p\}$ if it satisfies the condition

$$\Re\left\{ -\frac{zh'\ (z) + \overline{zg'\ (z)}}{h\ (z) + \overline{g\ (z)}} \right\} > 0, \qquad (z : p < |z| < 1). \tag{1.4}$$

Also, a function $f \in SH_p$ is said to be in the subclass CH_p of meromorphically harmonic convex in $D\backslash\{p\}$ if it satisfies the condition

$$\Re\left\{ -\frac{z^2h''\ (z) + zh'\ (z) + \overline{z^2g''\ (z) + zg'\ (z)}}{zh'\ (z) + \overline{zg'\ (z)}} \right\} > 0, \qquad (z : p < |z| < 1). \tag{1.5}$$

This classification (1.4) for univalent functions was studied by Ghanim and Daus [[1], [2]], and the classification (1.5) with $\alpha = 1$ and $p = 0$ was first used by Jahangiri [6].

Next, we define the operator I^k on the class SH_p as follows:

$$I^0 f\ (z) = f\ (z)\,,$$

$$I^k f\ (z) = I^k h\ (z) + \overline{I^k g\ (z)}\,, \qquad k = 1, 2, 3, \ldots\ , \tag{1.6}$$

where

$$I^k h\ (z) = z\left(I^{k-1} h\ (z)\right)' + \frac{\alpha\ (2z-p)}{(z-p)^2} = \frac{\alpha}{z-p} + \sum_{n=1}^{\infty} n^k a_n z^n.$$

and

$$I^k g\ (z) = z\left(I^{k-1} g\ (z)\right)' = \sum_{n=1}^{\infty} n^k b_n z^n.$$

With the help of the differential operator I^k, we define the class $SH_p^*\ (k,\alpha,\beta)$

Definition 1.1. *The function* $f \in SH_p$ *is said to be a member of the class* $SH_p^*\ (k,\alpha,\beta)$ *if it satisfies*

$$\left|\frac{z\left(I^k h(z)\right)' + \overline{z\left(I^k g\ (z)\right)'}}{I^k f(z)} + 1\right| \leq \left|\frac{z\left(I^k h(z)\right)' + \overline{z\left(I^k g\ (z)\right)'}}{I^k f(z)} + 2\beta - 1\right|, \quad (1.7)$$

$(k \in N_0 = N \cup 0)$ *for some* $\beta (0 \leq \beta < 1)$ *and for all* z *in* $D \backslash \{p\}$.

It is easy to check that $SH_p^*(0,1,\beta)$ is the class of meromorphically starlike functions of order β and $SH_p^*(0,1,0)$gives the meromorphically starlike functions for all $z \in D\backslash\{p\}$.

Let us write

$$SH_p^*[k,\alpha,\beta] = SH_p^*(k,\alpha,\beta) \cap \mathbb{SH}_p \quad (1.8)$$

where $\mathbb{SH}_p$ is the class of functions of the form (1.3) that are analytic and harmonic in $D\backslash\{p\}$.

Next, our first results will concern on the coefficient estimates for the classes $SH_p^*\ (k,\alpha,\beta)$ and $SH_p^*[k,\alpha,\beta]$.

2. Main Results

Here we provide a sufficient condition for a function, analytic in $D\backslash\{p\}$ to be in $SH_p^*\ (k,\alpha,\beta)$.

Theorem 2.1. *If* $f(z) = h(z) + \overline{g\ (z)}$ *is of the form (1.2) and satisfies the condition*

$$\sum_{n=1}^{\infty} n^k (n+\beta)(1-p)(|a_n| + |b_n|) \leq \alpha (1-\beta) \qquad (k \in N_0), \quad (2.1)$$

where $(0 \leq \beta < 1)$*, then* f *is harmonic univalent sense preserving in* $D\backslash\{p\}$ *and* $f \in SH_p^*(k,\alpha,\beta)$.

Proof: Suppose that (2.1) holds true for $0 \leq \beta < 1$. Consider the expression

$$M(z) = \left|z\left(I^k h(z)\right)' + \overline{z\left(I^k g\ (z)\right)'} + I^k f(z)\right|$$
$$-\left|z\left(I^k h(z)\right)' + \overline{z\left(I^k g\ (z)\right)'} + (2\beta - 1) I^k f(z)\right|$$

then for $|z| = r$, and since $|z-p| \geq |z| - p = r - p$, we have

$$M(z) = \left|-\frac{\alpha z}{(z-p)^2} + \frac{\alpha}{z-p} + \sum_{n=1}^{\infty} n^k (n+1)\left(a_n z^n + \overline{b_n z^n}\right)\right|$$
$$-\left|\frac{-\alpha z + \alpha(z-p)(2\beta-1)}{(z-p)^2} + \sum_{n=1}^{\infty} n^k (n+2\beta-1)\left(a_n z^n + \overline{b_n z^n}\right)\right|$$
$$= \left|-\frac{\alpha p}{(z-p)^2} + \sum_{n=1}^{\infty} n^k (n+1)\left(a_n z^n + \overline{b_n z^n}\right)\right|$$

$$-\left|\frac{-2\alpha z+2\alpha\beta z-2\alpha\beta p+\alpha p}{(z-p)^2}+\sum_{n=1}^{\infty}n^k\left(n+2\beta-1\right)\left(a_nz^n+\overline{b_nz^n}\right)\right|$$

and

$$M\left(r\right)\leq\frac{\alpha p}{(r-p)^2}+\sum_{n=1}^{\infty}n^k\left(n+1\right)\left(|a_n|+|b_n|\right)r^n$$
$$-\frac{2\alpha\left[(1-\beta)r+\beta p\right]-\alpha p}{\left(r-p\right)^2}+\sum_{n=1}^{\infty}n^k\left(n+2\beta-1\right)\left(|a_n|+|b_n|\right)r^n$$
$$=\sum_{n=1}^{\infty}2n^k\left(n+\beta\right)\left(|a_n|+|b_n|\right)r^n-\frac{2\alpha\left(1-\beta\right)}{\left(r-p\right)}.$$

That is

$$\left(r-p\right)M\left(r\right)\leq\sum_{n=1}^{\infty}2n^k\left(n+\beta\right)\left(|a_n|+|b_n|\right)\left(r-p\right)r^n-2\alpha\left(1-\beta\right)\tag{2.2}$$

The inequality in (2.2) holds true for all r $(0\leq r<1)$. Therefore, letting $r\to 1$ in (2.2), we obtain

$$\left(1-p\right)M\left(r\right)\leq\sum_{n=1}^{\infty}2n^k\left(n+\beta\right)\left(|a_n|+|b_n|\right)\left(1-p\right)-2\alpha\left(1-\beta\right).$$

By the hypothesis (2.1) it follows that (1.7) holds, so that $f\in SH_p^*(k,\alpha,\beta)$. Note that f is sense-preserving in $U\backslash\{p\}$. This is because

$$\begin{aligned}
&\left|f'\left(z\right)\right|\geq\tfrac{1}{|z-p|^2}-\sum_{n=1}^{\infty}n\left|a_n\right|\left|z\right|^{n-1}\\
&\geq\tfrac{1}{|z|^2}-\sum_{n=1}^{\infty}n\left|a_n\right|\left|z\right|^{n-1} \qquad\qquad (|z-p|\leq|z|)\\
&\geq\tfrac{1}{r^2}-\sum_{n=1}^{\infty}n\left|a_n\right|r^{n-1}\geq 1-\sum_{n=1}^{\infty}n\left|a_n\right|\\
&\geq 1-\sum_{n=1}^{\infty}n\left(n+\beta\right)\left(1-p\right)\left|a_n\right|\\
&\geq\sum_{n=1}^{\infty}n\left(n+\beta\right)\left(1-p\right)\left|b_n\right|\\
&\geq\sum_{n=1}^{\infty}n\left|b_n\right|\geq\sum_{n=1}^{\infty}n\left|b_n\right|\left|z\right|^{n-1}\geq\left|g'\left(z\right)\right|
\end{aligned}$$

Hence the theorem.

Corollary 2.2. *Let $k=\beta=0$ and $p\to 0$ in the Theorem 2.1, then we have*

$$\sum_{n=1}^{\infty}n\left(|a_n|+|b_n|\right)\leq\alpha.$$

Corollary 2.3. *Let $k=\beta=0$, $\alpha=1$ and $p\to 0$ in the Theorem 2.1, then we have*

$$\sum_{n=1}^{\infty}n\left(|a_n|+|b_n|\right)\leq 1,$$

the result was achieved by Bostanci, Yalçin and Öztürk [4].

Corollary 2.4. *Let $k=1$, $\beta=0$ and $p\to 0$ in the Theorem 2.1, then we have*

$$\sum_{n=1}^{\infty} n^2\left(|a_n|+|b_n|\right)\le \alpha$$

.

Corollary 2.5. *Let $k=1$, $\beta=0$, $\alpha=1$ and $p\to 0$ in Theorem 2.1 , then we have*

$$\sum_{n=1}^{\infty} n^2\left(|a_n|+|b_n|\right)\le 1$$

the result was achieved by Bostanci, Yalçin and Öztürk [4].

Next we give a necessary and sufficient condition for a function $f\in \mathbb{SH}_p$ to be in the class $SH_p^*[k,\alpha,\beta]$.

Theorem 2.6. *Let $f\in\mathbb{SH}_p$ be a function defined by (1.3). Then $f\in SH_p^*[k,\alpha,\beta]$ if and only if the inequality*

$$\sum_{n=1}^{\infty} n^k\left(n+\beta\right)\left(1-p\right)\left(a_n+b_n\right)\le \alpha\left(1-\beta\right)\qquad (k\in N_0) \tag{2.3}$$

is satisfied. The result is sharp.

Proof: In view of Theorem 2.1, it sufficies to show that the 'only if ' part is true. Assume that $f\in SH_p^*[k,\alpha,\beta]$. Then

$$\left|\frac{\dfrac{z\left(I^k h\left(z\right)\right)'+\overline{z\left(I^k g\ \left(z\right)\right)'}}{I^k f\left(z\right)}+1}{\dfrac{z\left(I^k h\left(z\right)\right)'+\overline{z\left(I^k g\ \left(z\right)\right)'}}{I^k f\left(z\right)}+2\beta-1}\right|$$

$$=\left|\frac{\dfrac{-\alpha p}{(z-p)^2}+\displaystyle\sum_{n=1}^{\infty} n^k\left(n+1\right)\left(a_n z^n+\overline{b_n z^n}\right)z^n}{\dfrac{-2\alpha z+2\alpha\beta z-2\alpha\beta p+\alpha p}{\left(z-p\right)^2}+\displaystyle\sum_{n=1}^{\infty} n^k\left(n+2\beta-1\right)\left(a_n z^n+\overline{b_n z^n}\right)}\right|\le 1, \tag{2.4}$$

$z\in D\backslash\left\{p\right\}$.
Since $\Re(z)\le |z|$ for all z, it follows from (2.4) that

$$\Re\left\{\frac{\dfrac{-\alpha p}{(z-p)^2}+\displaystyle\sum_{n=1}^{\infty} n^k\left(n+1\right)\left(a_n z^n+\overline{b_n z^n}\right)z^n}{\dfrac{-2\alpha[(1-\beta)z+\beta p]+\alpha p}{\left(z-p\right)^2}+\displaystyle\sum_{n=1}^{\infty} n^k\left(n+2\beta-1\right)\left(a_n z^n+\overline{b_n z^n}\right)}\right\}\le 1, \tag{2.5}$$

$z\in D\backslash\left\{p\right\}$. We now choose the values z on the real axis. Upon clearing the denominator in (2.5) and letting $z\to 1$ through real values, we obtain

$$\sum_{n=1}^{\infty} n^k\left(n+1\right)\left(1-p\right)\left(a_n+b_n\right)\le$$

$$2\alpha\left(1-\beta\right)-\sum_{n=1}^{\infty}n^{k}\left(n+2\beta-1\right)\left(1-p\right)\left(a_{n}+b_{n}\right),$$

which immediately yields the required condition (2.3).

A distortion property for functions in the class $SH_p^*[k,\alpha,\beta]$ is contained in the following theorem:

Theorem 2.7. *If the function f defined by (1.3) is in the class $SH_p^*[k,\alpha,\beta]$, then, for $|z|=r$, we have*

$$|f(z)|\leq\frac{\alpha}{r-p}+\frac{\alpha(1-\beta)}{(1+\beta)\left(1-p\right)}r.$$

Proof: Let $f\in SH_p^*[k,\alpha,\beta]$. Taking the absolute value of f we obtain

$$|f\left(z\right)|\leq\frac{\alpha}{r-p}+\sum_{n=1}^{\infty}\left(a_{n}+b_{n}\right)r^{n}$$

$$\leq\frac{\alpha}{r-p}+\frac{\alpha(1-\beta)}{(1+\beta)\left(1-p\right)}\sum_{n=1}^{\infty}\frac{n^{k}\left(n+\beta\right)\left(1-p\right)}{\alpha(1-\beta)}\left(a_{n}+b_{n}\right)r$$

$$\leq\frac{\alpha}{r-p}+\frac{\alpha(1-\beta)}{(1+\beta)\left(1-p\right)}r.$$

The functions

$$f\left(z\right)=\frac{\alpha}{z-p}+\frac{\alpha\left(1-\beta\right)}{(1+\beta)\left(1-p\right)}z\quad\text{and}\quad f\left(z\right)=\frac{\alpha}{z-p}+\frac{\alpha\left(1-\beta\right)}{(1+\beta)\left(1-p\right)}\overline{z}$$

for $0<\alpha\leq1$ and $0\leq\beta<1$ show that the bound given in Theorem 2.7 are sharp in $D\backslash\left\{p\right\}$.

Theorem 2.8. *Set*

$$h_{0}\left(z\right)=\frac{\alpha}{z-p},\quad g_{0}(z)=0,$$

$$h_{n}\left(z\right)=\frac{\alpha}{z-p}+\frac{\alpha\left(1-\beta\right)}{n^{k}\left(n+\beta\right)\left(1-p\right)}z^{n}\tag{2.6}$$

for $n=1,2,3,...,$ and

$$g_{n}\left(z\right)=\frac{\alpha\left(1-\beta\right)}{n^{k}\left(n+\beta\right)\left(1-p\right)}\overline{z}^{n}\tag{2.7}$$

for $n=1,2,3,...,$.
Then $f\in SH_p^*[k,\alpha,\beta]$ if and only if it can be expressed in the form

$$f\left(z\right)=\sum_{n=0}^{\infty}\left(\lambda_{n}h_{n}+\gamma_{n}g_{n}\right),\tag{2.8}$$

where $\lambda_n\geq0,\ \gamma_n\geq0$ and $\sum\limits_{n=0}^{\infty}\left(\lambda_n+\gamma_n\right)=1.$
In particular, the extreme points of $SH_p^*[k,\alpha,\beta]$ are $\{h_n\}$ and $\{g_n\}$.

Proof: From (2.6), (2.7) and (2.8), we have

$$f\left(z\right)=\sum_{n=0}^{\infty}\left(\lambda_{n}h_{n}+\gamma_{n}g_{n}\right)$$

$$= \sum_{n=0}^{\infty} (\lambda_n + \gamma_n) \frac{\alpha}{z-p} + \sum_{n=1}^{\infty} \frac{\alpha (1-\beta)}{n^k (n+\beta)(1-p)} \lambda_n z^n + \sum_{n=0}^{\infty} \frac{\alpha (1-\beta)}{n^k (n+\beta)(1-p)} \gamma_n \overline{z}^n.$$

Then

$$\sum_{n=1}^{\infty} \left(n^k (n+\beta)(1-p)\right) \frac{\lambda_n}{n^k (n+\beta)(1-p)} + \sum_{n=0}^{\infty} \left(n^k (n+\beta)(1-p)\right) \frac{\gamma_n}{n^k (n+\beta)(1-p)} = \sum_{n=1}^{\infty} (\lambda_n + \gamma_n) - \lambda_0 = 1 - \lambda_0 \leq 1$$

So $f \in SH_p^*[k, \alpha, \beta]$.

Conversely, suppose that $f \in SH_p^*[k, \alpha, \beta]$. Set

$$\lambda_n = \frac{n^k (n+\beta)(1-p)}{\alpha (1-\beta)} a_n, \qquad n \geq 1,$$

and

$$\gamma_n = \frac{n^k (n+\beta)(1-p)}{\alpha (1-\beta)} b_n, \qquad n \geq 0.$$

Then by Theorem 2.6, $0 \leq \lambda_n \leq 1$ $(n = 1, 2, 3, ...)$ and $0 \leq \gamma_n \leq 1$, $(n = 0, 1, 2, ...)$. We define

$$\lambda_0 = 1 - \sum_{n=1}^{\infty} \lambda_n - \sum_{n=0}^{\infty} \gamma_n$$

and note that, by Theorem 2.6, $\lambda_0 \geq 0$.
Consequently, we obtain

$$f(z) = \sum_{n=0}^{\infty} (\lambda_n h_n + \gamma_n g_n),$$

as required.

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