Bulletin of Mathematical Analysis and Applications
ISSN: 1821-1291, URL: http://www.bmathaa.org
Volume 2 Issue 4(2010), Pages 130-136.

A QUADRATIC TYPE FUNCTIONAL EQUATION

(DEDICATED IN OCCASION OF THE 70-YEARS OF PROFESSOR HARI M. SRIVASTAVA)

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Abstract. In this paper, the solution and the Hyers–Ulam stability of the following quadratic type functional equation

$$\sum_{i=2}^{k}\sum_{\varepsilon_j\in\{-1,1\}} f\left(x_1+\varepsilon_j x_i\right)=2(k-1)f(x_1)+2\sum_{i=2}^{k}f(x_i)$$

is investigated.

1. Introduction and preliminaries

A classical question in the theory of functional equations is the following: "When is it true that a function which approximately satisfies a functional equation $\mathcal{E}$ must be close to an exact solution of $\mathcal{E}$?" If there exists an affirmative answer, we say that the equation $\mathcal{E}$ is stable [9]. During the last decades several stability problems for various functional equations have been investigated by numerous mathematicians. We refer the reader to the survey articles [10, 9, 21] and monographs [11, 12, 8] and references therein.

Let $\mathcal{X}$ and $\mathcal{Y}$ be normed spaces. A function $f:\mathcal{X}\to\mathcal{Y}$ satisfying the functional equation

$$f(x+y)+f(x-y)=2f(x)+2f(y)\quad (x,y\in\mathcal{X}) \tag{1.1}$$

is called the quadratic functional equation. It is well known that a function f between real vector spaces is quadratic if and only if there exists a unique symmetric bi-additive function B such that $f(x)=B(x,x)$ for all $x\in\mathcal{X}$; see [9]. The bi-additive function B is given by

$$B(x,x)=\frac{1}{4}\left(f(x+y)-f(x-y)\right).$$

The Hyers–Ulam stability of the quadratic equation (1.1) was proved by Skof [22]. Cholewa [6] noticed that the theorem of Skof is still true if the relevant domain $\mathcal{X}$ is replaced by an abelian group. Furthermore, Czerwik [7] deal with stability problem of the quadratic functional equation (1.1) in the spirit of Hyers–Ulam–

2000 *Mathematics Subject Classification.* 39B22, 39B82.
Key words and phrases. Hyers–Ulam stability; quadratic equation; normed space.

Submitted August 1, 2010. Published November 15, 2010.

Rassias. Also, Jung [13] proved the stability of (1.1) on a restricted domain. For more information on the stability of the quadratic equation, we refer the reader to [2, 3, 16, 4, 14].

Theorem 1.1. *(Czerwik) Let $\varepsilon \geq 0$ be fixed. If a mapping $f : \mathcal{X} \to \mathcal{Y}$ satisfies the inequality*

$$\|f(x+y) + f(x-y) - 2f(x) - 2f(y)\| \leq \varepsilon \qquad (x \in \mathcal{X}) \tag{1.2}$$

then there exists a unique quadratic mapping $Q : \mathcal{X} \to \mathcal{Y}$ such that

$$\|f(x) - Q(x)\| \leq \frac{1}{2}\varepsilon \qquad (x \in \mathcal{X}).$$

Moreover, if f is measurable or if $f(tx)$ is continuous in t for each fixed $x \in \mathcal{X}$, then $Q(tx) = t^2 Q(tx)$ for all $x \in \mathcal{X}$ and $t \in \mathbb{R}$.

The Hyers–Ulam stability of equation (1.1) on a certain restricted domain was investigated by Jung [13] in the following theorem,

Theorem 1.2. *(Jung) Let $d > 0$ and $\varepsilon \geq 0$ be given. Assume that a mapping $f : \mathcal{X} \to \mathcal{Y}$ satisfies the inequality (1.2) for all $x, y \in \mathcal{X}$ with $\|x\| + \|y\| \geq d$. Then there exists a unique quadratic mapping $Q : \mathcal{X} \to \mathcal{Y}$ such that*

$$\|f(x) - Q(x)\| \leq \frac{7}{2}\varepsilon \qquad (x \in \mathcal{X}). \tag{1.3}$$

If, moreover, f is measurable or $f(tx)$ is continuous in t for each fixed $x \in \mathcal{X}$ then $Q(tx) = t^2 Q(tx)$ for all $x \in \mathcal{X}$ and $t \in \mathbb{R}$.

The quadratic functional equation was used to characterize the inner product spaces [1]. A square norm on an inner product space satisfies the important parallelogram equality

$$\|x+y\|^2 + \|x-y\|^2 = 2(\|x\|^2 + \|y\|^2).$$

It was shown by Moslehian and Rassias [19] that a normed space $(\mathcal{X}, \|.\|)$ is an inner product space if and only if for any finite set of vectors $x_1, x_2, \cdots, x_k \in \mathcal{X}$,

$$\sum_{\varepsilon_j \in \{-1,1\}} \left\| x_1 + \sum_{i=2}^{k} \varepsilon_j x_i \right\|^2 = \sum_{\varepsilon_j \in \{-1,1\}} \left(\|x_1\| + \sum_{i=2}^{k} \varepsilon_j \|x_i\| \right)^2. \tag{1.4}$$

Motivated by (1.4), we introduce the following functional equation deriving from the quadratic function

$$\sum_{i=2}^{k} \sum_{\varepsilon_j \in \{-1,1\}} f\left(x_1 + \varepsilon_j x_i\right) = 2(k-1) f(x_1) + 2 \sum_{i=2}^{k} f(x_i), \tag{1.5}$$

where $k \geq 2$ is a fixed integer. It is easy to see that the function $f(x) = x^2$ is a solution of functional equation (1.5).

2. Solution of the equation (1.5)

Theorem 2.1. *A mapping $f : \mathcal{X} \to \mathcal{Y}$ satisfies the equation (1.5) for all $x_1, x_2, \cdots, x_k \in \mathcal{X}$ if and only if f is quadratic.*

Proof. If we replace $x_1, x_2, \cdots, x_k$ in (1.5) by 0, then we get $f(0) = 0$. Putting $x_3 = x_4 = \cdots = x_k = 0$ in the equation (1.5) we see that

$$f(x_1 - x_2) + f(x_1 + x_2) + 2(k-2)f(x_1) = 2(k-1)f(x_1) + 2f(x_2)\,.$$

Hence $f(x_1 - x_2) + f(x_1 + x_2) = 2f(x_1) + 2f(x_2)$. The converse is trivial. □

Remark. *We can prove the theorem above on the punching space $\mathcal{X} - \{0\}$. If we consider $x_2 = x_3 = \cdots = x_k$, then we observe that*

$$\sum_{i=2}^{k} \sum_{\varepsilon_j \in \{-1,1\}} f\left(x_1 + \varepsilon_j x_2\right) = 2(k-1)f(x_1) + 2\sum_{i=2}^{k} f(x_2)\,,$$

whence

$$(k-1)\left(f(x_1 - x_2) + f(x_1 + x_2)\right) = 2(k-1)f(x_1) + 2(k-1)f(x_2).$$

Hence f is quadratic.

3. Stability Results

Throughout this section, let $\mathcal{X}$ and $\mathcal{Y}$ be normed and Banach spaces also, we prove the Hyers–Ulam stability of equation (1.5). From now on, we use the following abbreviation

$$\mathfrak{D}f(x_1, x_2, \cdots, x_k) = \sum_{i=2}^{k} \sum_{\varepsilon_j \in \{-1,1\}} f\left(x_1 + \varepsilon_j x_i\right) - 2(k-1)f(x_1) - 2\sum_{i=2}^{k} f(x_i)\,. \quad (3.1)$$

Theorem 3.1. *Let $\varepsilon \geq 0$ be fixed. If a mapping $f : \mathcal{X} \to \mathcal{Y}$ with $f(0) = 0$ satisfies*

$$\|\mathfrak{D}f(x_1, x_2, \cdots x_k)\| \leq \varepsilon \quad (3.2)$$

for all $x_1, x_2, \cdots x_k \in \mathcal{X}$, then there exists a unique quadratic mapping $Q : \mathcal{X} \to \mathcal{Y}$ such that

$$\|f(x) - Q(x)\| \leq \frac{1}{2}\varepsilon\,.$$

Moreover, if f is measurable or if $f(tx)$ is continuous in t for each fixed $x \in \mathcal{X}$, then $Q(tx) = t^2 Q(tx)$ for all $x \in \mathcal{X}$ and $t \in \mathbb{R}$.

Proof. It is enough to put $x_3 = x_4 = \cdots x_k = 0$ in (3.2) and use Theorem 1.1. □

By using an idea from the paper [13], we will prove the Hyers–Ulam stability of equation (1.5) on a restricted domain.

Theorem 3.2. *Let $d > 0$ and $\varepsilon \geq 0$ be given. Suppose that a mapping $f : \mathcal{X} \to \mathcal{Y}$ satisfies the inequality (3.2) for all $x_1, x_2, \cdots x_k \in \mathcal{X}$ with $\|x_1\| + \|x_2\| + \cdots + \|x_k\| \geq d$. Then there exists a unique quadratic mapping $Q : \mathcal{X} \to \mathcal{Y}$ such that*

$$\|f(x) - Q(x)\| \leq \frac{3 + 2k}{2}\varepsilon \quad (3.3)$$

for all $x \in \mathcal{X}$. Moreover, if f is measurable or if $f(tx)$ is continuous in t for each fixed $x \in \mathcal{X}$, then $Q(tx) = t^2 Q(tx)$ for all $x \in \mathcal{X}$ and $t \in \mathbb{R}$.

Proof. Assume $\|x_1\| + \|x_2\| + \cdots + \|x_k\| < d$. If $x_1 = x_2 = \cdots = x_k = 0$, then we chose a $t \in \mathcal{X}$ with $\|t\| = d$. Otherwise, let $t = (1 + \frac{d}{\|x_{i_0}\|})x_{i_0}$, where $\|x_{i_0}\| = \max\{\|x_j\| : 1 \leq j \leq k\}$. Clearly, we see that

$$\begin{aligned} \|x_1 - t\| + \|x_2 + t\| + \cdots + \|x_k + t\| &\geq d \\ \|x_1 + t\| + \|x_2 + t\| + \cdots + \|x_k + t\| &\geq d \\ \|x_1\| + \|x_2 + 2t\| + \cdots + \|x_k + 2t\| &\geq d \\ \|x_2 + t\| + \|x_3 + t\| + \cdots + \|x_k + t\| + \|t\| &\geq d \\ \|x_1\| + \|t\| &\geq d, \end{aligned} \tag{3.4}$$

since $\|x_j + t\| \geq d$ and $\|x_j + 2t\| \geq d$, for $1 \leq j \leq k$.

From (3.2) and (3.4) and the relations

$$\begin{aligned} & f(x_1 + x_2) + f(x_1 - x_2) - 2f(x_1) - 2f(x_2) \\ = \ & f(x_1 + x_2) + f(x_1 - x_2 - 2t) - 2f(x_1 - t) - 2f(x_2 + t) \\ + \ & f(x_1 + x_2 + 2t) + f(x_1 - x_2) - 2f(x_1 + t) - 2f(x_2 + t) \\ - \ & 2f(x_2 + 2t) - 2f(x_2) + 4f(x_2 + t) + 4f(t) \\ - \ & f(x_1 + x_2 + 2t) - f(x_1 - x_2 - 2t) + 2f(x_1) + 2f(x_2 + 2t) \\ + \ & 2f(x_1 + t) + 2f(x_1 - t) - 4f(x_1) - 4f(t) \end{aligned}$$

we get

$$\begin{aligned} \|\mathfrak{D}f(x_1, x_2, \cdots, x_k)\| \ \leq \ & \left\| \sum_{i=2}^{k} \sum_{\varepsilon_j \in \{-1,1\}} f(\alpha_1 + \varepsilon_j \alpha_i) - 2(k-1)f(\alpha_1) - 2\sum_{i=2}^{k} f(\alpha_i) \right\| \\ + \ & \left\| \sum_{i=2}^{k} \sum_{\varepsilon_j \in \{-1,1\}} f(\beta_1 + \varepsilon_j \beta_i) - 2(k-1)f(\beta_1) - f(\beta_i) \right\| \\ + \ & 2\left\| \sum_{i=2}^{k} \sum_{\varepsilon_j \in \{-1,1\}} f(\gamma_1 + \varepsilon_j \gamma_i) - 2(k-1)f(\gamma_1) - 2\sum_{i=2}^{k} f(\gamma_i) \right\| \\ + \ & \left\| \sum_{i=2}^{k} \sum_{\varepsilon_j \in \{-1,1\}} f(\theta_1 + \varepsilon_j \theta_i) - 2(k-1)f(\theta_1) - 2\sum_{i=2}^{k} f(\theta_i) \right\| \\ + \ & 2(k-1)\left\| \sum_{i=2}^{k} \sum_{\varepsilon_j \in \{-1,1\}} f(\eta_1 + \varepsilon_j \eta_i) - 2(k-1)f(\eta_1) - 2\sum_{i=2}^{k} f(\eta_i) \right\|, \end{aligned}$$

where

$$\begin{aligned} \alpha_1 = x_1 - t \quad &, \quad \alpha_i = x_i + t \ , \quad 2 \leq i \leq k \\ \beta_1 = x_1 + t \quad &, \quad \beta_i = x_i + t \ , \quad 2 \leq i \leq k \\ \gamma_1 = t \quad &, \quad \gamma_i = x_i + t \ , \quad 2 \leq i \leq k \\ \theta_1 = x_1 \quad &, \quad \theta_i = x_i + 2t \ , \quad 2 \leq i \leq k \\ \eta_i = x_1 \quad &, \quad \eta_{i+1} = t \ , \quad 2 \leq i \leq k. \end{aligned}$$

Hence we have

$$\begin{aligned}\|\mathfrak{D}f(x_1,x_2,\cdots,x_k)\| &\le \|\mathfrak{D}f(\alpha_1,\alpha_2,\cdots,\alpha_k)\| + \|\mathfrak{D}f(\beta_1,\beta_2,\cdots,\beta_k)\|\\ &+ 2\|\mathfrak{D}f(\gamma_1,\gamma_2,\cdots,\gamma_k)\| + \|\mathfrak{D}f(\theta_1,\theta_2,\cdots,\theta_k)\|\\ &+ 2(k-1)\|\mathfrak{D}f(\eta_1,\eta_2,\cdots,\eta_k)\|\\ &\le (3+2k)\varepsilon. \qquad (3.5)\end{aligned}$$

Obviously, inequality (3.2) holds for all $x, y \in \mathcal{X}$. According to (3.5) and Theorem 3.1, there exists a unique quadratic mapping $Q : \mathcal{X} \to \mathcal{Y}$ which satisfies the inequality (3.3) for all $x_1, x_2, \cdots, x_k \in \mathcal{X}$. □

Now we study asymptotic behavior of function equation (1.5).

Theorem 3.3. *Suppose that $f : \mathcal{X} \to \mathcal{Y}$ is a mapping. Then f is quadratic if and only if for $k \in \mathbb{N}$ $(k \ge 2)$*

$$\|\mathfrak{D}f(x_1,x_2,\cdots,x_k)\| \to 0 \qquad (3.6)$$

as $\|x_1\| + \|x_2\| + \cdots \|x_k\| \to \infty$.

Proof. If f is quadratic then (3.6) evidently holds. Conversely, by using the limits (3.6) we can find for every $n \in \mathbb{N}$ a sequence ε_n such that $\|\mathfrak{D}f(x_1,x_2,\cdots,x_k)\| \le \frac{1}{n}$ for all $x_1, x_2, \cdots, x_k \in \mathcal{X}$ with $\|x_1\| + \|x_2\| + \cdots \|x_k\| \ge \varepsilon_n$.

By Theorem 3.2 for every $n \in \mathbb{N}$ there exists a unique quadratic mapping Q_n such that

$$\|f(x) - Q_n(x)\| \le \frac{3+2k}{2n} \qquad (3.7)$$

for all $x \in \mathcal{X}$. Since $\|f(x) - Q_1(x)\| \le \frac{3+2k}{2}$ and $\|f(x) - Q_n(x)\| \le \frac{3+2k}{2n} \le \frac{3+2k}{2}$, by the uniqueness of Q_1 we conclude that $Q_n = Q_1$ for all $n \in \mathcal{N}$. Now, by tending n to the infinity in (3.7) we deduce that $f = Q_1$. Therefore f is quadratic. □

4. STABILITY ON BOUNDED DOMAINS

Throughout this section, we denote by $B_r(0)$ the closed ball of radius r around the origin and $B_r = B_r(0) - \{0\}$. In this section we used some ideas from the paper's Moslehian et al [18].

Theorem 4.1. *Let $\mathcal{X}$ and $\mathcal{Y}$ be normed and Banach spaces $p > 2, r > 0, \varphi : X^k \to [0,\infty)(k \ge 2)$ be a function such that $\varphi(\frac{x_1}{2}, \frac{x_2}{2}, \cdots, \frac{x_k}{2}) \le \frac{1}{2^p}\varphi(x_1,x_2,\cdots,x_k)$ for all*
$x_1, x_2, \cdots, x_k \in B_r$. Suppose that $f : \mathcal{X} \to \mathcal{Y}$ is a mapping satisfying $f(0) = 0$ and

$$\|\mathfrak{D}f(x_1,x_2,\cdots,x_k)\| \le \varphi(x_1,x_2,\cdots,x_k) \qquad (4.1)$$

for all $x_1, x_2, \cdots, x_K \in B_r$ with $x_i \pm x_j \in B_r$ for $1 \le i, j \le k$. Then there exists a unique quadratic mapping $Q : \mathcal{X} \to \mathcal{Y}$ such that

$$\|f(x) - Q(x)\| \le \frac{1}{(2^p-4)(k-1)}\varphi(x,x,\cdots,x), \qquad (4.2)$$

where $x \in B_r$.

Proof. Let $x_1, x_2, \cdots, x_K \in B_r$. If we consider $x_2 = x_3 = \cdots = x_k$ in (4.1), then we see that

$$\|f(x_1+x_2) + f(x_1-x_2) - 2f(x_1) - 2f(x_2)\| \le \frac{1}{k-1}\varphi(x_1,x_2,\cdots,x_2). \qquad (4.3)$$

Replacing x_1, x_2 in (4.3) by $\frac{x}{2}$, we get

$$\|f(x) - 4f(\frac{x}{2})\| \le \frac{1}{k-1}\varphi\left(\frac{x}{2}, \frac{x}{2}, \cdots, \frac{x}{2}\right). \tag{4.4}$$

Replacing x by $\frac{x}{2^n} \in B_r$ and multiplying with 4^n in (4.4), we obtain

$$\|4^n f(\frac{x}{2^n}) - 4^{n+1} f(\frac{x}{2^{n+1}})\| \le \frac{4^n}{k-1}\varphi\left(\frac{x}{2^{n+1}}, \frac{x}{2^{n+1}}, \cdots, \frac{x}{2^{n+1}}\right). \tag{4.5}$$

It follows from (4.5) that

$$\begin{aligned} \|4^n f(\frac{x}{2^n}) - 4^{n+m} f(\frac{x}{2^{n+m}})\| &\le \frac{1}{k-1}\sum_{i=1}^{m} 4^{n+i-1}\varphi\left(\frac{x}{2^{n+i}}, \frac{x}{2^{n+i}}, \cdots, \frac{x}{2^{n+i}}\right) \\ &\le \frac{2^{2(n-1)}}{2^{pn}(k-1)}\varphi(x, x, \cdots, x)\sum_{i=1}^{m}\frac{1}{2^{(p-2)i}}. \end{aligned} \tag{4.6}$$

It follows that $\{4^n f(\frac{x}{2^n})\}$ is Cauchy and so is convergent. Therefore we see that a mapping

$$\widehat{Q}(x) := \lim_{n\to\infty} 4^n f(\frac{x}{2^n}) \quad (x \in B_r),$$

satisfies

$$\|f(x) - \widehat{Q}(x)\| \le \frac{1}{(2^p - 4)(k-1)}\varphi(x, x, \cdots, x),$$

and $\widehat{Q}(0) = 0$, when taking the limit $m \to \infty$ in (4.6) with $n = 0$.

Next fix $x \in B_r$. Because of $\frac{x}{2} \in B_r$, we have

$$4\widehat{Q}(\frac{x}{2}) = \lim_{n\to\infty} 4^{n+1} f(\frac{x}{2^{n+1}}) = \lim_{n\to\infty} 4^n f(\frac{x}{2^n}) = \widehat{Q}(x).$$

Therefore $4^{n+m}\widehat{Q}(\frac{x}{2^{n+m}}) = \widehat{Q}(x)$ and so the mapping $Q : \mathcal{X} \to \mathcal{Y}$ given by $Q(x) := 4^n\widehat{Q}(\frac{x}{2^n})$, where n is least non-negative integer such that $\frac{x}{2^n} \in B_r$ is well-defined.

It is easy to see that $Q(x) = \lim_{n\to\infty} 4^n f(\frac{x}{2^n}) \quad (x \in \mathcal{X})$ and $Q|_{B_r(0)} = \widehat{Q}$.

Now let $x, y \in \mathcal{X}$. There is a large enough n such that $\frac{x}{2^n}, \frac{y}{2^n}, \frac{x+y}{2^n}, \frac{x-y}{2^n} \in B_r(0)$. Put $x_1 = \frac{x}{2^n}$ and $x_2 = \frac{y}{2^n}$ in (4.3) and multiplying both sides with 4^n to obtain

$$\begin{aligned} \|4^n f(\frac{x+y}{n}) + 4^n f(\frac{x-y}{2^n}) - 4^n 2f(\frac{x}{2^n}) - 4^n 2f(\frac{y}{2^n})\| &\le \frac{4^n}{k-1}\varphi\left(\frac{x}{2^n}, \frac{y}{2^n}, \cdots, \frac{y}{2^n}\right) \\ &\le \frac{4^n}{2^{np}(k-1)}\varphi(x, y, y, \cdots, y). \end{aligned}$$

whence, by taking the limit as $n \to \infty$, we get $Q(x+y) + Q(x-y) = 2q(x) + 2Q(y)$. Hence Q is quadratic. Uniqueness of Q can be proved by using the strategy used in the proof of Theorem 3.2. □

Corollary 4.2. *Let $\mathcal{X}$ and $\mathcal{Y}$ be normed and Banach spaces $p > 2, r > 0, \theta > 0$. Suppose that $f : \mathcal{X} \to \mathcal{Y}$ is a mapping satisfying $f(0) = 0$ and*

$$\|\mathfrak{D}f(x_1, x_2, \cdots, x_k)\| \le \theta\|x_1\|^{\frac{p}{k}}\|x_2\|^{\frac{p}{k}}\cdots\|x_k\|^{\frac{p}{k}} \tag{4.7}$$

for all $x_1, x_2, \cdots, x_k \in B_r$ with $x_i \pm x_j \in B_r$ for $2 \le i, j \le k$. Then there exists a unique quadratic mapping $Q : \mathcal{X} \to \mathcal{Y}$ such that

$$\|f(x) - Q(x)\| \le \frac{\theta r^p}{(2^p - 4)(k-1)}, \tag{4.8}$$

where $x \in B_r$.

Proof. Apply Theorem 4.1 with $\varphi(x_1, x_2, \cdots, x_k) = \theta\|x_1\|^{\frac{p}{k}}\|x_2\|^{\frac{p}{k}} \cdots \|x_k\|^{\frac{p}{k}}$. □

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