Bulletin of Mathematical Analysis and Applications
ISSN: 1821-1291, URL: http://www.bmathaa.org
Volume 3 Issue 1(2011), Pages 77-88.

GROWTH OF A CLASS OF ITERATED ENTIRE FUNCTIONS

(COMMUNICATED BY VICENTIU RADULESCU)

DIBYENDU BANERJEE, RATAN KUMAR DUTTA

Abstract. In this paper we generalise a result of J. Sun to n-th iterations of $f(z)$ with respect to $g(z)$.

1. Introduction and Notation

We first consider two entire functions $f(z)$ and $g(z)$ and following Lahiri and Banerjee [5] form the iterations of $f(z)$ with respect to $g(z)$ as follows:

$$\begin{aligned}
f_1(z) &= f(z)\\
f_2(z) &= f(g(z)) = f(g_1(z))\\
f_3(z) &= f(g(f(z))) = f(g_2(z)) = f(g(f_1(z)))\\
&\quad .... \qquad .... \qquad ....\\
&\quad .... \qquad .... \qquad ....\\
f_n(z) &= f(g(f........(f(z) \text{ or } g(z))........))\\
&\quad \text{according as } n \text{ is odd or even}\\
&= f(g_{n-1}(z)) = f(g(f_{n-2}(z))),
\end{aligned}$$

and so

$$\begin{aligned}
g_1(z) &= g(z)\\
g_2(z) &= g(f(z)) = g(f_1(z))\\
&\quad .... \qquad ....\\
&\quad .... \qquad ....\\
g_n(z) &= g(f_{n-1}(z)) = g(f(g_{n-2}(z))).
\end{aligned}$$

Clearly all $f_n(z)$ and $g_n(z)$ are entire functions.

Notation 1.1. *Let $f(z)$ and $g(z)$ be two entire functions. Throughout the paper we use the notations $M_{f_1}(r), M_{f_2}(r), M_{f_3}(r)$ etc., to mean $M(r,f), M(M(r,f),g)$,*

2000 *Mathematics Subject Classification.* 30D35.
Key words and phrases. Entire function; growth; iteration; order.

Submitted January 10, 2011. Published January 20, 2011.

$M(M(M(r,f),g),f)$ respectively and $F(r)=O^*(G(r))$ to mean that there exist two positive constants K_1 and K_2 such that $K_1 \le \frac{F(r)}{G(r)} \le K_2$ for any r big enough.

In [2], C. Chuang and C. C. Yang posed the question:
For four entire functions f_1, f_2 and g_1, g_2, when is $T(r, f_1og_1) \sim T(r, f_2og_2)$ as $r \to \infty$, provided $T(r, f_1) \sim T(r, f_2)$ and $T(r, g_1) \sim T(r, g_2)$?

In 2003, Sun [7] showed that in general there is no positive answer and he gave a condition under which there is a positive answer by proving the following theorem.

Theorem A. *Let f_1, f_2 and g_1, g_2 be four transcendental entire functions with*
$$T(r,f_1)=O^*((\log r)^{\nu}e^{(\log r)^{\alpha}}) \text{ and } T(r,g_1)=O^*((\log r)^{\beta}).$$
If $T(r,f_1)\sim T(r,f_2)$ and $T(r,g_1)\sim T(r,g_2)$ $(r\to\infty)$, then
$$T(r,f_1(g_1))\sim T(r,f_2(g_2)) \qquad (r\to\infty,\ r\notin E),$$
where $\nu>0, 0<\alpha<1, \beta>1$ and $\alpha\beta<1$ and E is a set of finite logarithmic measure.

We extend Theorem A to iterated entire functions.

Theorem 1.2. *Let f, g and u, v be four transcendental entire functions with $T(r,f)\sim T(r,u)$, $T(r,g)\sim T(r,v)$, $T(r,f)=O^*((\log r)^{\nu}e^{(\log r)^{\alpha}})$ $(0<\alpha<1,\ \nu>0)$ and $T(r,g)=O^*((\log r)^{\beta})$ where $\beta>1$ is a constant, then $T(r,f_n)\sim T(r,u_n)$ for $n\ge 2$, where $u_n(z)=u(v(u(v........(u(z)$ or $v(z))........)))$ according as n is odd or even.*

We do not explain the standard notations and definitions of the theory of meromorphic functions because they are available in [4].

2. Lemmas

The following lemmas will be needed in the sequel.

Lemma 2.1. [4] *Let $f(z)$ be an entire function. For $0\le r<R<\infty$, we have*
$$T(r,f)\le \log^+ M(r,f)\le \frac{R+r}{R-r}T(R,f).$$

Lemma 2.2. [3] *Let $f(z)$ be an entire function of order $\rho(\rho<\infty)$. If $k>\rho-1$, then*
$$\log M(r,f)\sim \log M(r-r^{-k},f) \qquad (r\to\infty).$$

Lemma 2.3. [6] *Let $g(z)$ and $f(z)$ be two entire functions. Suppose that $|g(z)|>R>|g(0)|$ on the circumference $\{|z|=r\}$ for some $r>0$. Then we have*
$$T(r,f(g))\ge \frac{R-|g(0)|}{R+|g(0)|}T(R,f).$$

Lemma 2.4. [1] *Let f be an entire function of order zero and $z=re^{i\theta}$. Then for any $\zeta>0$ and $\eta>0$, there exist $R_0=R_0(\zeta,\eta)$ and $k=k(\zeta,\eta)$ such that for all $R>R_0$ it holds*
$$\log|f(re^{i\theta})|-N(2R)-\log|c| \ >-kQ(2R), \qquad \zeta R\le r\le R,$$

except in a set of circles enclosing the zeros of f, the sum of whose radii is at most ηR. Here

$$Q(r) = r\int_r^\infty \frac{n(t,1/f)}{t^2}dt \quad \text{and } N(r) = \int_0^r \frac{n(t,1/f)}{t}dt.$$

Lemma 2.5. [7] *Let f be a transcendental entire function with*

$$T(r,f) = O^*((\log r)^\beta e^{(\log r)^\alpha})\ (0<\alpha<1, \beta>0).$$

Then

$$\begin{aligned} 1.\ T(r,f) &\sim \log M(r,f) \quad (r\to\infty, r\notin E),\\ 2.\ T(\sigma r,f) &\sim T(r,f) \quad (r\to\infty, \sigma\geq 2, r\notin E), \end{aligned}$$

where E is a set of finite logarithmic measure.

Lemma 2.6. *Let f be a transcendental entire function with $T(r,f) = O^*((\log r)^\beta)$ where $\beta > 1$. Then*

$$\begin{aligned} 1.\ T(r,f) &\sim \log M(r,f) \quad (r\to\infty, r\notin E),\\ 2.\ T(\sigma r,f) &\sim T(r,f) \quad (r\to\infty, \sigma\geq 2, r\notin E), \end{aligned}$$

where E is a set of finite logarithmic measure.

Proof. Without loss of generality we may assume that $f(0) = 1$, otherwise we set $F(z) = f(z) - f(0) + 1$.
By Jensen's theorem,

$$N(r,1/f) = \int_0^r \frac{n(t,1/f)}{t}dt = \frac{1}{2\pi}\int_0^{2\pi} \log|f(re^{i\theta})|d\theta \leq \log M(r,f)$$

and so,

$$n(r,1/f)\log A \leq \int_r^{Ar} \frac{n(t,1/f)}{t}dt \leq \int_0^{Ar} \frac{n(t,1/f)}{t}dt \leq \log M(Ar,f),$$

for $r>1$ and $A>1$.

Therefore

$$n(r,1/f) \leq \frac{\log M(Ar,f)}{\log A}. \tag{2.1}$$

Since $T(r,f) = O^*((\log r)^\beta), \beta>1$, by Lemma 2.1 we have

$$\log M(r,f) = O^*((\log r)^\beta). \tag{2.2}$$

Take $A = r^{\sigma(r)}$ and $\sigma(r) = \frac{1}{(\log r)^{1/2}}$. Then by (2.1) we have

$$n(r,1/f) \leq \frac{\log M(r^{1+\sigma(r)},f)}{\sigma(r)\log r}. \tag{2.3}$$

Therefore, putting $r=e^u$ we have

$$\begin{aligned}
\frac{\left(\log r^{1+\sigma(r)}\right)^{\beta}}{r^{1/2}\sigma(r)\log r} &= \frac{(1+\sigma(r))^{\beta}(\log r)^{\beta}}{r^{1/2}\sigma(r)\log r}\\
&= \frac{\left(1+\frac{1}{u^{1/2}}\right)^{\beta}u^{\beta}}{e^{u/2}u^{-1/2}u}\\
&= \frac{\left(1+\frac{1}{u^{1/2}}\right)^{\beta}}{e^{u/2}u^{1-1/2-\beta}}\\
&= \frac{\left(1+\frac{1}{u^{1/2}}\right)^{\beta}}{e^{u/2}e^{(1/2-\beta)\log u}}\\
&= \frac{\left(1+\frac{1}{u^{1/2}}\right)^{\beta}}{e^{\frac{u}{2}-(\beta-1/2)\log u}}. \qquad (2.4)
\end{aligned}$$

Since $\beta>1$, for sufficiently large values of u we have $\frac{u}{2}-(\beta-1/2)\log u>0$ and $\frac{u}{2}-(\beta-1/2)\log u$ increases. By (2.4) for sufficiently large value of r, $\frac{\left(\log r^{1+\sigma(r)}\right)^{\beta}}{r^{1/2}\sigma(r)\log r}$ decreases.

From Lemma 2.4, using (2.2) and (2.3), we have

$$\begin{aligned}
Q(r) &= r\int_r^{\infty}\frac{n(t,1/f)dt}{t^2}\\
&\le r\int_r^{\infty}\frac{\log M(t^{1+\sigma(t)},f)}{t^2\sigma(t)\log t}dt\\
&= r\int_r^{\infty}\frac{O^*\left((\log t^{1+\sigma(t)})^{\beta}\right)}{t^2\sigma(t)\log t}dt\\
&\le O^*\left(r\int_r^{\infty}\frac{(\log t^{1+\sigma(t)})^{\beta}}{t^2\sigma(t)\log t}dt\right)\\
&\le \frac{r^{1/2}O^*\left((\log r^{1+\sigma(r)})^{\beta}\right)}{\sigma(r)\log r}\int_r^{\infty}t^{-3/2}dt\\
&= \frac{2O^*\left((\log r^{1+\sigma(r)})^{\beta}\right)}{\sigma(r)\log r}\\
&= \frac{2\log M(r^{1+\sigma(r)},f)}{\sigma(r)\log r}.
\end{aligned}$$

Therefore

$$\begin{aligned}
\frac{Q(r)}{\log M(r,f)} &\le \frac{2\log M(r^{1+\sigma(r)},f)}{\sigma(r)\log r\ \log M(r,f)}\\
&\le \frac{2K_2(\log r^{1+\sigma(r)})^{\beta}}{\sigma(r)\log r\ K_1(\log r)^{\beta}}, \quad \text{for some suitable constants } K_1 \text{ and } K_2\\
&= \frac{2K_2}{K_1}\frac{(1+\sigma(r))^{\beta}(\log r)^{\beta}}{\sigma(r)\log r\ (\log r)^{\beta}}\\
&= \frac{2K_2}{K_1}\frac{(1+\sigma(r))^{\beta}}{\sigma(r)\log r}\\
&\to 0 \text{ as } r\to\infty.
\end{aligned}$$

So

$$Q(r) = o(\log M(r, f)) \tag{2.5}$$

Since $T(r, f) = O^*((\log r)^\beta)$, $n(r, 1/f) = o(r)$.
The concluding part of the proof of the lemma is similar to that of Lemma 5 of J. Sun [7]. But still for the sake of completeness and for convenience of readers, we outline the proof.

$$\begin{aligned}
\log M(r, f) &\leq \log \Pi_{n=1}^{\infty}(1 + r/r_n) \\
&= \int_0^\infty \log(1 + r/t) dn(t, 1/f) \\
&\leq \int_0^\infty \frac{r}{t} dn(t, 1/f) \\
&= r \int_0^\infty \frac{n(t, 1/f)}{t(t + r)} dt \\
&= r \left(\int_0^r + \int_r^\infty \right) \frac{n(t, 1/f)}{t(t + r)} dt \\
&\leq r.\frac{1}{r} \int_0^r \frac{n(t, 1/f)}{t} dt + r \int_r^\infty \frac{n(t, 1/f)}{t^2} dt \\
&= N(r) + Q(r) \qquad (2.6)
\end{aligned}$$

So, from Lemma 2.4 and (2.5), (2.6) we have

$$\begin{aligned}
\log |f(re^{i\theta})| &> N(2R) - kQ(2R) \quad (\zeta R \leq r \leq R,\ r \notin E) \\
&= N(2R) + Q(2R) - (k + 1)Q(2R) \\
&\geq \log M(2R, f) + (k + 1)o(\log M(2R, f)) \\
&= \log M(2R, f)(1 - o(1)) \qquad (2.7) \\
&\geq \log M(r, f)(1 - o(1)) \qquad (2.8)
\end{aligned}$$

where E is a set of finite logarithmic measure.
On the other hand

$$\log |f(z)| \leq \log M(r, f) \leq \log M(\sigma r, f) \quad (|z| = r, \sigma \geq 2,) \tag{2.9}$$

Let $2R = \sigma r, \sigma \geq 2$ then from (2.7), (2.8) and (2.9) we have,

$$\log |f(z)| \sim \log M(\sigma r, f) \quad (r \to \infty, \sigma \geq 2,\ r \notin E) \tag{2.10}$$

and

$$\log |f(z)| \sim \log M(r, f) \quad (r \to \infty,\ r \notin E). \tag{2.11}$$

From (2.11) for sufficiently large value of r, we have,

$$\begin{aligned}
m(r, f) &= \frac{1}{2\pi} \int_0^{2\pi} \log^+ |f(re^{i\theta})| d\theta = \frac{1}{2\pi} \int_0^{2\pi} \log M(r, f)(1 + o(1)) d\theta \\
&= \log M(r, f)(1 + o(1)) \quad (r \to \infty,\ r \notin E).
\end{aligned}$$

So,

$$\lim_{r\to\infty} \frac{T(r, f)}{\log M(r, f)} = 1, \quad (r \notin E)$$

i.e.

$$T(r, f) \sim \log M(r, f), \ (r \notin E). \tag{2.12}$$

From (2.10) and (2.11) we have

$$\log M(r,f) \sim \log M(\sigma r, f) \quad (r\to\infty, \sigma \geq 2,\ r \notin E). \tag{2.13}$$

From (2.12) and (2.13) we have

$$T(\sigma r, f) \sim T(r,f) \quad (r\to\infty, \sigma \geq 2,\ r \notin E). \tag{2.14}$$

From (2.12) and (2.14) we have the required result.
This proves the lemma. □

Lemma 2.7. *Let f_1 and f_2 be two entire functions with $T(r,f_1) = O^*((\log r)^\beta)$ where $\beta > 1$ and $T(r,f_1) \sim T(r,f_2)$ then $M(r,f_1) \sim M(r,f_2)$.*

Proof. From Lemma 2.6 we have,

$$\log M(r,f_1) \sim T(r,f_1) \sim T(r,f_2) \sim \log M(r,f_2) \qquad (r\to\infty, r\notin E)$$

where E is a set of finite logarithmic measure.

Since $\log M(r,f_1) \sim \log M(r,f_2)$, so for given $\epsilon > 0$, there exist $r_1, r_2 > 0$ such that

$$\frac{\log M(r,f_1)}{\log M(r,f_2)} < 1 + \frac{\log(1+\epsilon)}{\log M(r,f_2)} \quad \text{for } r > r_1 \tag{2.15}$$

and

$$\frac{\log M(r,f_2)}{\log M(r,f_1)} < 1 + \frac{\log(1+\epsilon)}{\log M(r,f_1)} \quad \text{for } r > r_2 \tag{2.16}$$

Now from (2.15) we have

$$\begin{aligned} \log M(r,f_1) &< \log M(r,f_2) + \log(1+\epsilon). \\ \text{So,} \qquad \frac{M(r,f_1)}{M(r,f_2)} &< 1+\epsilon \quad \text{for } r > r_1. \end{aligned} \tag{2.17}$$

Similarly from (2.16)

$$\begin{aligned} \frac{M(r,f_2)}{M(r,f_1)} &< 1+\epsilon \quad \text{for } r > r_2. \\ \text{i.e.} \quad \frac{M(r,f_1)}{M(r,f_2)} &> 1-\epsilon \quad \text{for } r > r_2. \end{aligned} \tag{2.18}$$

From (2.17) and (2.18) we have

$$\begin{aligned} 1-\epsilon &< \frac{M(r,f_1)}{M(r,f_2)} < 1+\epsilon \quad \text{for } r > r_0 = \max\{r_1, r_2\}. \\ \text{So,} \quad M(r,f_1) &\sim M(r,f_2). \end{aligned}$$

This proves the lemma. □

Lemma 2.8. *Let f_1 and f_2 be two entire functions with $T(r,f_1) = O^*((\log r)^\nu e^{(\log r)^\alpha})$ where $\nu > 0$ and $0 < \alpha < 1$ and $T(r,f_1) \sim T(r,f_2)$ then $M(r,f_1) \sim M(r,f_2)$.*

Proof. From Lemma 2.5 we have,

$$\log M(r,f_1) \sim T(r,f_1) \sim T(r,f_2) \sim \log M(r,f_2) \qquad (r\to\infty, r\notin E)$$

where E is a set of finite logarithmic measure and concluding part follows from Lemma 2.7. □

3. Theorems

In [6] K. Niino and N. Suita proved the following theorem.

Theorem 3.1. *Let $f(z)$ and $g(z)$ be entire functions. If $M(r,g) > \frac{2+\epsilon}{\epsilon}|g(0)|$ for any $\epsilon > 0$, then we have*

$$T(r, f(g)) \leq (1+\epsilon)T(M(r,g), f).$$

In particular, if $g(0) = 0$, then

$$T(r, f(g)) \leq T(M(r,g), f)$$

for all $r > 0$.

The following theorem is the generalization of the above.

Theorem 3.2. *Let $f(z)$ and $g(z)$ be two entire functions. Then we have*

$$T(R_2, f) \leq T(r, f_n) \leq T(R_3, f) \tag{3.1}$$

where $|f(z)| > R_1 > \frac{2+\epsilon}{\epsilon}|f(0)|$ and $|g(z)| > R_2 > \frac{2+\epsilon}{\epsilon}|g(0)|$, $R_3 = max\{M_{f_{n-1}}(r),\ M_{g_{n-1}}(r)\}$ for sufficiently large values of r and any integer $n \geq 2$.

Proof. By Theorem 3.1 we have for odd n and any $\epsilon > 0$ arbitrary small

$$\begin{aligned}
T(r, f_n) &= T(r, f_{n-1}(f)) \\
&\leq (1+\epsilon)T(M(r,f), f_{n-1}) \\
&= (1+\epsilon)T(M_{f_1}(r), f_{n-2}(g)) \\
&\leq (1+\epsilon)^2 T(M_{f_2}(r), f_{n-2}) \\
&= (1+\epsilon)^2 T(M_{f_2}(r), f_{n-3}(f)) \\
&\leq (1+\epsilon)^3 T(M_{f_3}(r), f_{n-3}) \\
&\dots\dots \\
&\dots\dots \\
&\leq (1+\epsilon)^{n-1} T(M_{f_{n-1}}(r), f) \\
&\leq (1+\epsilon)^{n-1} T(R_3, f).
\end{aligned}$$

Similarly when n is even, we have

$$\begin{aligned}
T(r, f_n) &= T(r, f_{n-1}(g)) \\
&\leq (1+\epsilon)T(M(r,g), f_{n-1}) \\
&= (1+\epsilon)T(M_{g_1}(r), f_{n-2}(f)) \\
&\leq (1+\epsilon)^2 T(M_{g_2}(r), f_{n-2}) \\
&\dots\dots \\
&\dots\dots \\
&\leq (1+\epsilon)^{n-1} T(M_{g_{n-1}}(r), f) \\
&\leq (1+\epsilon)^{n-1} T(R_3, f).
\end{aligned}$$

Therefore

$$T(r, f_n) \leq (1+\epsilon)^{n-1} T(R_3, f) \text{ for any integer } n \geq 2.$$

Since $\epsilon > 0$ was arbitrary, we have for sufficiently large values of r

$$T(r, f_n) \leq T(R_3, f). \tag{3.2}$$

Also using Lemma 2.3 we have for odd n

$$\begin{aligned}
T(r,f_n) &= T(r,f_{n-1}(f)) \\
&\geq \left(\frac{R_1-|f(0)|}{R_1+|f(0)|}\right)T(R_1,f_{n-1}) \\
&> (1-\epsilon)T(R_1,f_{n-2}(g)) \\
&\geq (1-\epsilon)\left(\frac{R_2-|g(0)|}{R_2+|g(0)|}\right)T(R_2,f_{n-2}) \\
&> (1-\epsilon)^2T(R_2,f_{n-2}) \\
&\geq (1-\epsilon)^3T(R_1,f_{n-3}) \\
& \quad .... \qquad\qquad .... \\
& \quad .... \qquad\qquad .... \\
&\geq (1-\epsilon)^{n-2}T(R_1,f(g)) \\
&\geq (1-\epsilon)^{n-1}T(R_2,f).
\end{aligned}$$

Similarly when n is even we obtain

$$\begin{aligned}
T(r,f_n) &= T(r,f_{n-1}(g)) \\
&\geq \left(\frac{R_2-|g(0)|}{R_2+|g(0)|}\right)T(R_2,f_{n-1}) \\
&> (1-\epsilon)T(R_2,f_{n-2}(f)) \\
& .... \qquad\qquad .... \\
& .... \qquad\qquad .... \\
&\geq (1-\epsilon)^{n-2}T(R_1,f(g)) \\
&\geq (1-\epsilon)^{n-1}T(R_2,f).
\end{aligned}$$

So,

$$T(r,f_n)\geq(1-\epsilon)^{n-1}T(R_2,f).$$

Since $\epsilon>0$ was arbitrary, we have for sufficiently large values of r

$$T(r,f_n)\geq T(R_2,f). \tag{3.3}$$

Hence from (3.2) and (3.3) we obtain (3.1).
This proves the theorem. □

4. Proof of the Theorem 1.2

Proof. From Theorem 3.2 we have

$$T(R_1,f)\leq T(r,f_n)\leq T(R_2,f) \tag{4.1}$$

$$T(R_1^{'},u)\leq T(r,u_n)\leq T(R_2^{'},u) \tag{4.2}$$

and choose R_1 and $R_1^{'}$ in such way that $|g(z)|>R_1>\frac{2+\epsilon}{\epsilon}|g(0)|$, $|v(z)|>R_1^{'}>\frac{2+\epsilon}{\epsilon}|v(0)|$ and $T(R_1,f)\sim T(R_1^{'},f)$, where R_2 =max$\{M_{f_{n-1}}(r),\ M_{g_{n-1}}(r)\}$ and $R_2^{'}$ =max$\{M_{u_{n-1}}(r),\ M_{v_{n-1}}(r)\}$ for sufficiently large value of r and arbitrary small $\epsilon>0$.
Since $T(r,f)\sim T(r,u)$, so

$$\begin{aligned}
T(R_1,f) &\sim T(R_1^{'},f)\sim T(R_1^{'},u) \\
\text{i.e.}\quad T(R_1,f) &\sim T(R_1^{'},u)\quad (r\to\infty, r\notin E). \qquad (4.3)
\end{aligned}$$

Also from Lemma 2.8 we have $M(r,f) \sim M(r,u)$.
So,

$$\begin{aligned} M(M(r,f),g) &\sim M(M(r,u),v) \quad (r\to\infty), \text{ using Lemma 2.2} \\ \text{i.e. } M(M(M(r,f),g),f) &\sim M(M(M(r,u),v),u) \quad (r\to\infty). \end{aligned}$$

Finally, for odd n,

$$M_{f_{n-1}}(r) \sim M_{u_{n-1}}(r) \quad (r\to\infty). \tag{4.4}$$

Similarly, for even n,

$$M_{g_{n-1}}(r) \sim M_{v_{n-1}}(r) \quad (r\to\infty). \tag{4.5}$$

From (4.4) and (4.5) for any integer $n \geq 2$, we have $R_2 \sim R_2^{'}$ for large r. So from $T(r,f) \sim T(r,u)$ and $R_2 \sim R_2^{'}$ we have

$$T(R_2,u) \sim T(R_2^{'},f) \quad (r\to\infty) \tag{4.6}$$

So from (4.1), (4.2), (4.3) and (4.6) we have $T(r,f_n) \sim T(r,u_n)$.
This proves the theorem. □

Theorem 4.1. *Let f,g and u,v be four transcendental entire functions with $T(r,f) \sim T(r,u)$, $T(r,g) \sim T(r,v)$, $T(r,f) = O^*((\log r)^{\beta})$ and $T(r,g) = O^*((\log r)^{\beta})$ where $\beta > 1$ is a constant, then $T(r,f_n) \sim T(r,u_n)$.*

Note 4.2. *The conditions of Theorem 1.2 and Theorem 4.1 are not strictly sharp. Which are illustrated by the following examples.*

Example 4.3. *Let $f(z) = e^z, g(z) = z$ and $u(z) = 2e^z, v(z) = 2z$. Then we have $f_2 = f(g) = e^z, u_2 = u(v) = 2e^{2z}$ and $f_4 = f(g(f(g))) = e^{e^z}, u_4 = u(v(u(v))) = 2e^{4e^{2z}}$.*
Also

$$\begin{aligned} T(r,f) &= \frac{r}{\pi}, \quad T(r,\ u) = \frac{r}{\pi} + \log 2, \\ T(r,g) &= \log r, \quad T(r,v) = \log r + \log 2, \\ T(r,f_2) &= \frac{r}{\pi}, \qquad T(r,u_2) = \frac{2r}{\pi} + \log 2, \end{aligned}$$

Thus

$$T(r,f) \sim T(r,u), T(r,g) \sim T(r,v) \qquad (r\to\infty).$$

But

$$\frac{T(r,f_2)}{T(r,u_2)} = 2 \ as \ \ r\to\infty,$$

so

$$T(r,f_2) \nsim T(r,u_2).$$

Also

$$T(r,f_4) \leq \log M(r,f_4) = e^r$$

and

$$\begin{aligned} 3T(2r,u_4) &\geq \log M(r,u_4) = \log 2 + 4e^{2r} \\ \textit{i.e.}\ \ T(r,u_4) &\geq \frac{1}{3}\log 2 + \frac{4}{3}e^r \\ \textit{i.e.}\ \ \frac{1}{T(r,u_4)} &\leq \frac{1}{\frac{1}{3}\log 2 + \frac{4}{3}e^r}. \end{aligned}$$

Therefore

$$\frac{T(r,f_4)}{T(r,u_4)} \leq \frac{e^r}{\frac{1}{3}\log 2 + \frac{4}{3}e^r} = 3/4 \ as \quad r \to \infty,$$

so

$$T(r,f_4) \nsim T(r,u_4).$$

Thus, $T(r,f_n) \sim T(r,u_n)$ *does not hold for all* $n \geq 2$*. Here* $T(r,f) \neq O^*((\log r)^\beta)$ *where* $\beta > 1$ *is a constant.*

Example 4.4. *Let* $f(z) = e^z, g(z) = \log z$ *and* $u(z) = 2e^z, v(z) = \log 2z$*. Then we have*

$$\begin{aligned} f_2 &= f(g) = z, u_2 = u(v) = 4z, \\ f_3 &= f(g(f)) = e^z, u_3 = u(v(u)) = 8e^z, \\ f_4 &= f(g(f(g))) = z, u_4 = u(v(u(v))) = 16z. \end{aligned}$$

Here

$$\begin{aligned} T(r,f) &= \frac{r}{\pi}, \quad T(r,\ u) = \frac{r}{\pi} + \log 2, \\ \therefore \quad T(r,f) &\sim T(r,u) \quad (r \to \infty). \end{aligned}$$

Also

$$T(r,g) \leq \log\log r,$$

and

$$\begin{aligned} 3T(2r,v) &\geq \log\log 2r \\ i.e. \ \ T(r,v) &\geq \frac{\log\log r}{3} \\ i.e. \ \frac{1}{T(r,v)} &\leq \frac{3}{\log\log r}. \end{aligned}$$

So

$$\frac{T(r,g)}{T(r,v)} \leq 3.$$

Again

$$T(r,v) \leq \log\log 2r,$$

and

$$\begin{aligned} 3T(2r,g) &\geq \log\log r \\ i.e \quad T(r,g) &\geq \frac{\log\log r/2}{3} \\ i.e. \ \frac{1}{T(r,g)} &\leq \frac{3}{\log\log r/2}. \end{aligned}$$

So

$$\begin{aligned} \frac{T(r,v)}{T(r,g)} &\leq 3\frac{\log\log 2r}{\log\log r/2} \\ &\leq 3 \quad as \quad r \to \infty. \end{aligned}$$

$$\therefore \quad \frac{1}{3} \leq \frac{T(r,g)}{T(r,v)} \leq 3 \quad as \quad r \to \infty.$$

Also

$$\begin{aligned} T(r,f_2) &= \log r,\ T(r,u_2) = \log r\ + \log 4, \\ T(r,f_3) &= \frac{r}{\pi},\ T(r,u_3) = \frac{r}{\pi} + \log 8, \\ T(r,f_4) &= \log r,\ T(r,u_4) = \log r\ + \log 16. \end{aligned}$$

Here $T(r,g) \nsim T(r,v)$. *But still* $T(r,f_n) \sim T(r,u_n)$ *for* $n = 2,3,4$.

Example 4.5. *Let* $f(z) = e^z, g(z) = (\log z)^2$ *and* $u(z) = 2e^z, v(z) = (\log 2z)^2$. *Then we have*

$$\begin{aligned} f_2 &= f(g) = e^{(\log z)^2}, u_2 = u(v) = 2e^{(\log 2z)^2}, \\ f_3 &= f(g(f)) = e^{z^2}, u_3 = u(v(u)) = 2e^{(\log 4)^2} 4^{2z} e^{z^2}, \\ f_4 &= f(g(f(g))) = e^{(\log z)^4}, u_4 = u(v(u(v))) = 2e^{(\log 4)^2} 4^{2(\log 2z)^2} e^{(\log 2z)^4}, \\ f_5 &= f(g(f(g(f)))) = e^{z^4}, u_5 = u(v(u(v(u)))) = 32e^{(\log 4)^2} 4^{2(\log 4e^z)^2} e^{(\log 4e^z)^4}. \end{aligned}$$

Also

$$\begin{aligned} T(r,f) &= \frac{r}{\pi},\quad T(r,\ u) = \frac{r}{\pi} + \log 2, \\ \therefore &\quad T(r,f) \sim T(r,u). \end{aligned}$$

and

$$\frac{1}{3} \le \frac{T(r,g)}{T(r,v)} \le 3 \quad as \quad r \to \infty.$$

Here $T(r,f) \neq O^*((\log r)^\beta)$ *where* $\beta > 1$ *is a constant and* $T(r,g) \nsim T(r,v)$. *But*

$$T(r,f_2) = (\log r)^2 \ \ and\ T(r,u_2) = (\log r)^2 + 2\log 2 \log r + \ (\log 2)^2 + \log 2$$

so

$$T(r,f_2) \sim T(r,u_2) \quad as \quad r \to \infty,$$

and

$$T(r,f_3) = \frac{r^2}{\pi} \quad and \quad T(r,u_3) = \log 2 + (\log 4)^2 + 2r\log 4 + \frac{r^2}{\pi},$$

so

$$T(r,f_3) \sim T(r,u_3) \quad as \quad r \to \infty,$$

and

$$T(r,f_4) = (\log r)^4 and \ \ T(r,u_4) = \log 2 + (\log 4)^2 + O(\log r)^2 + (\log 2r)^4,$$

so

$$T(r,f_4) \sim T(r,u_4) \quad as \quad r \to \infty,$$

and

$$T(r,f_5) = \frac{r^4}{\pi} \quad and \quad T(r,u_5) = \log 2 + (\log 4)^2 + O(r^2) + \frac{r^4}{\pi},$$

so

$$T(r,f_5) \sim T(r,u_5) \quad as \quad r \to \infty,$$

and so on.

Acknowledgments. The authors are thankful to the referee for several suggestions which considerably improve the presentation of the paper.

References

[1] M. L. Cartwight, *Integral Function*, Cambridge University Press, 1956.

[2] C. Chuang, C. C. Yang, *The Fixpoint of Meromorphic and Factorization Theory*, Beijing University Press, 1988.

[3] J. Clunie, *The composition of entire and meromorphic functions*, Mathematical Essays dedicated to A. J. Macintyre, Ohio. Univ. Press, (1970), 75-92.

[4] W. K. Hayman, *Meromorphic Functions*, Oxford University Press, 1964.

[5] B. K. Lahiri, D. Banerjee, *Relative fix points of entire functions*, J. Indian Acad. Math., **19(1)** (1997), 87-97.

[6] K. Niino, N. Suita, *Growth of a composite function of entire functions*, Kodai Math. J., **3** (1980), 374-379.

[7] J. Sun, *Growth of a class of composite entire functions*, Acta Math. Vietnamica, **28(2)** (2003), 175-183.

Dibyendu Banerjee
Department of Mathematics, Visva-Bharati, Santiniketan-731235, West Bengal, India
E-mail address: dibyendu192@yahoo.com

Ratan Kumar Dutta
Department of Mathematics, Siliguri Institute of Technology, Post.-Sukna, Siliguri, Darjeeling-734009, West Bengal, India
E-mail address: ratan_3128@yahoo.com