Bulletin of Mathematical analysis and Applications
ISSN: 1072-6691. URL: http://www.bmathaa.org
Volume 3, Issue 1, Pages 102-107.

A new factor theorem for absolute Cesàro summability

(communicated by Naim Braha) *

HÜSEYİN BOR

Abstract

In this paper, a known theorem dealing with $\mid C, \alpha; \delta \mid_k$ summability factors has been generalized for $\mid C, \alpha, \beta\ ; \delta \mid_k$ summability factors.Our theorem also includes some known results.

1 Introduction

Let $\sum a_n$ be a given infinite series with partial sums (s_n). We denote by $u_n^{\alpha,\beta}$ and $t_n^{\alpha,\beta}$ the nth Cesàro means of order (α, β), with $\alpha+\beta > -1$, of the sequence (s_n) and (na_n), respectively, i.e., (see [3])

$$u_n^{\alpha,\beta} = \frac{1}{A_n^{\alpha+\beta}} \sum_{v=0}^{n} A_{n-v}^{\alpha-1} A_v^{\beta} s_v \tag{1}$$

$$t_n^{\alpha,\beta} = \frac{1}{A_n^{\alpha+\beta}} \sum_{v=1}^{n} A_{n-v}^{\alpha-1} A_v^{\beta} v a_v, \tag{2}$$

where

$$A_n^{\alpha+\beta} = O(n^{\alpha+\beta}), \quad A_0^{\alpha+\beta} = 1 \quad and \quad A_{-n}^{\alpha+\beta} = 0 \quad for \quad n > 0. \tag{3}$$

The series $\sum a_n$ is said to be summable $\mid C, \alpha, \beta \mid_k$, $k \geq 1$, if (see [5])

$$\sum_{n=1}^{\infty} n^{k-1} \mid u_n^{\alpha,\beta} - u_{n-1}^{\alpha,\beta} \mid^k < \infty. \tag{4}$$

* 2010 Mathematics Subject Classifications: 40D15, 40F05, 40G05,40G99.
Key words: Absolute Cesàro summability, infinite series, summability factors.

Submitted December 22, 2010. Published February 22, 2011.

Since $t_n^{\alpha,\beta} = n(u_n^{\alpha,\beta} - u_{n-1}^{\alpha,\beta})$ (see [5]), condition (4) can also be written as

$$\sum_{n=1}^{\infty} \frac{1}{n} \mid t_n^{\alpha,\beta} \mid^k < \infty. \tag{5}$$

The series $\sum a_n$ is summable $\mid C, \alpha, \beta \,; \delta \mid_k$, $k \geq 1$ and $\delta \geq 0$, if (see [2])

$$\sum_{n=1}^{\infty} n^{\delta k+k-1} \mid u_n^{\alpha,\beta} - u_{n-1}^{\alpha,\beta} \mid^k = \sum_{n=1}^{\infty} n^{\delta k-1} \mid t_n^{\alpha,\beta} \mid^k < \infty. \tag{6}$$

If we take $\delta = 0$, then $\mid C, \alpha, \beta \,; \delta \mid_k$ summability reduces to $\mid C, \alpha, \beta \mid_k$ summability.Also if we take $\beta = 0$, then we get $\mid C, \alpha; \delta \mid_k$ summability (see [7]). Furthermore, if we take $\beta = 0$ and $\delta = 0$, then $\mid C, \alpha, \beta \,; \delta \mid_k$ summability reduces to $\mid C, \alpha \mid_k$ summability (see [6]). It should be noted that obviously $(C, \alpha, 0)$ mean is the same as (C, α) mean. A sequence (λ_n) is said to be convex if $\Delta^2 \lambda_n \geq 0$, where $\Delta^2 \lambda_n = \Delta \lambda_n - \Delta \lambda_{n+1}$ and $\Delta \lambda_n = \lambda_n - \lambda_{n+1}$. In [8] Lal and Singh have proved the following theorem dealing with $\mid C, \alpha; \delta \mid_k$ summability factors of infinite series.

Theorem A. If (λ_n) is a convex sequence such that $\sum n^{-1} \lambda_n$ is convergent and the sequence (θ_n^α) defined by

$$\theta_n^\alpha = \mid t_n^\alpha \mid, \quad \alpha = 1 \tag{7}$$

$$\theta_n^\alpha = \max_{1 \leq v \leq n} \mid t_v^\alpha \mid, \quad 0 < \alpha < 1 \tag{8}$$

satisfies the condition

$$(n^\delta \theta_n^\alpha)^k = O\{(\log n)^{p+k-1}\} \ (C, 1),$$

then the series $\sum (\log n)^{-p-k+1} \, a_n \lambda_n$ is summable $\mid C, \alpha; \delta \mid_k$ for $0 < \alpha \leq 1, p \geq 0, k \geq 1, \delta \geq 0$ and $\delta k < \alpha$.

2 The Main Result

The aim of this paper is to generalize Theorem A for $\mid C, \alpha, \beta \,; \delta \mid_k$ summability. We shall prove the following theorem.

Theorem . If (λ_n) is a convex sequence such that $\sum n^{-1} \lambda_n$ is convergent and the sequence $(\theta_n^{\alpha,\beta})$ defined by

$$\theta_n^{\alpha,\beta} = \mid t_n^{\alpha,\beta} \mid, \quad \alpha = 1 \,, \, \beta > -1 \tag{9}$$

$$\theta_n^{\alpha,\beta} = \max_{1 \leq v \leq n} \mid t_v^{\alpha,\beta} \mid, \quad 0 < \alpha < 1 \,, \, \beta > -1 \tag{10}$$

satisfies the condition

$$(n^{\delta}\theta_n^{\alpha,\beta})^k = O\{(\log n)^{p+k-1}\}\ (C,1),$$

then the series $\sum(\log(n+1))^{-p+k+1}\ a_n\lambda_n$ is summable $\mid C,\alpha,\beta\ ;\delta\mid_k$ for $0<\alpha\le 1$, $\beta>-1$, $k\ge 1$, $\delta\ge 0, p\ge 0$ and $\alpha+\beta-\delta>0$.
We need the following lemmas for the proof of our theorem.
Lemma 1 ([4]). If (λ_n) is a convex sequence such that the series $\sum n^{-1}\lambda_n$ is convergent, then (λ_n) is non-negative and non-increasing, $n\Delta\lambda_n = O(1)$ and $\lambda_n \log n = o(1)$, as $n\to\infty$.
Lemma 2 ([1]). If $0<\alpha\le 1$, $\beta>-1$ and $1\le v\le n$, then

$$\mid \sum_{p=0}^{v} A_{n-p}^{\alpha-1}A_p^{\beta}a_p \mid \le \max_{1\le m\le v} \mid \sum_{p=0}^{m} A_{m-p}^{\alpha-1}A_p^{\beta}a_p \mid . \tag{11}$$

Lemma 3 ([10]). If $((\log(n+1))^{p+k-1}X_n)$ satisfies the same conditions as (λ_n) in Lemma 1, then

$$n\ (\log(n+1))^{p+k-1}\Delta X_n = O(1) \quad , as\ n\to\infty$$

and

$$\sum_{n=1}^{m} n\ (\log(n+1))^{p+k-1}\Delta^2 X_n = O(1) \quad , as\ m\to\infty.$$

Lemma 4 ([8]). If (λ_n) is a convex sequence such that the series $\sum n^{-1}\lambda_n$ is convergent, then for $p\ge 0$ and $k\ge 1$

$$\sum_{n=1}^{m} \frac{\Delta(\lambda_n)^k}{(\log(n+1))^{p(k+1)+(k-1)^2}} = O(1), \quad as\ m\to\infty.$$

3 Proof of the Theorem

We write

$$X_n = \frac{\lambda_n}{(\log(n+1))^{p+k-1}}.$$

Let $(T_n^{\alpha,\beta})$ be the n-th (C,α,β) mean of the sequence (na_nX_n). Then, by (2), we have

$$T_n^{\alpha,\beta} = \frac{1}{A_n^{\alpha+\beta}}\sum_{v=1}^{n} A_{n-v}^{\alpha-1}A_v^{\beta}va_vX_v.$$

First applying Abel's transformation and then using Lemma 2, we have that

$$T_n^{\alpha,\beta} = \frac{1}{A_n^{\alpha+\beta}}\sum_{v=1}^{n-1}\Delta X_v\sum_{i=1}^{v} A_{n-i}^{\alpha-1}A_i^{\beta}ia_i + \frac{X_n}{A_n^{\alpha+\beta}}\sum_{v=1}^{n} A_{n-v}^{\alpha-1}A_v^{\beta}va_v,$$

$$\begin{aligned} | T_n^{\alpha,\beta} | &\leq \frac{1}{A_n^{\alpha+\beta}} \sum_{v=1}^{n-1} \Delta X_v \mid \sum_{i=1}^{v} A_{n-i}^{\alpha-1} A_i^{\beta} i a_i \mid + \frac{X_n}{A_n^{\alpha+\beta}} \mid \sum_{v=1}^{n} A_{n-v}^{\alpha-1} A_v^{\beta} v a_v \mid \\ &\leq \frac{1}{A_n^{\alpha+\beta}} \sum_{v=1}^{n-1} A_v^{\alpha} A_v^{\beta} \theta_v^{\alpha,\beta} \Delta X_v + X_n \theta_n^{\alpha,\beta} \\ &= T_{n,1}^{\alpha,\beta} + T_{n,2}^{\alpha,\beta}, \quad say. \end{aligned}$$

Since

$$| T_{n,1}^{\alpha,\beta} + T_{n,2}^{\alpha,\beta} |^k \leq 2^k (| T_{n,1}^{\alpha,\beta} |^k + | T_{n,2}^{\alpha,\beta} |^k),$$

in order to complete the proof of the theorem, by (6), it is sufficient to show that

$$\sum_{n=1}^{\infty} n^{\delta k-1} \mid T_{n,r}^{\alpha,\beta} \mid^k < \infty \quad , for \quad r = 1, 2.$$

Whenever $k > 1$, we can apply Hölder's inequality with indices k and k', where $\frac{1}{k} + \frac{1}{k'} = 1$, we get that

$$\begin{aligned} \sum_{n=2}^{m+1} n^{\delta k-1} \mid T_{n,1}^{\alpha,\beta} \mid^k &\leq \sum_{n=2}^{m+1} n^{\delta k-1} \mid \frac{1}{A_n^{\alpha+\beta}} \sum_{v=1}^{n-1} A_v^{\alpha} A_v^{\beta} \theta_v^{\alpha,\beta} \Delta X_v \mid^k \\ &= O(1) \sum_{n=2}^{m+1} \frac{1}{n^{1+(\alpha+\beta-\delta)k}} \{\sum_{v=1}^{n-1} v^{\alpha k} v^{\beta k} \Delta X_v (\theta_v^{\alpha,\beta})^k\} \times \{\sum_{v=1}^{n-1} \Delta X_v\}^{k-1} \\ &= O(1) \sum_{v=1}^{m} v^{(\alpha+\beta)k} \Delta X_v (\theta_v^{\alpha,\beta})^k \sum_{n=v+1}^{m+1} \frac{1}{n^{1+(\alpha+\beta-\delta)k}} \\ &= O(1) \sum_{v=1}^{m} v^{(\alpha+\beta)k} \Delta X_v (\theta_v^{\alpha,\beta})^k \int_v^{\infty} \frac{dx}{x^{1+(\alpha+\beta-\delta)k}} \\ &= O(1) \sum_{v=1}^{m} \Delta X_v (v^{\delta} \theta_v^{\alpha,\beta})^k \\ &= O(1) \sum_{v=1}^{m-1} \Delta(\Delta X_v) \sum_{p=1}^{v} (p^{\delta} \theta_p^{\alpha,\beta})^k \\ &+ O(1) \Delta X_m \sum_{v=1}^{m} (v^{\delta} \theta_v^{\alpha,\beta})^k \\ &= O(1) \sum_{v=1}^{m-1} v (\log (v+1))^{p+k-1} \Delta^2 X_v + O(m (\log (m+1))^{p+k-1} \Delta X_m) \\ &= O(1) \quad as \quad m \to \infty, \end{aligned}$$

by the application of Lemma 3. Similarly, we have that

$$\begin{aligned}
\sum_{n=1}^{m} n^{\delta k-1} \mid X_n \theta_n^{\alpha,\beta} \mid^k &= O(1) \sum_{n=1}^{m} \frac{X_n^k}{n} (n^{\delta} \theta_n^{\alpha,\beta})^k \\
&= O(1) \sum_{n=1}^{m-1} \Delta(n^{-1} X_n^k) \sum_{v=1}^{n} (v^{\delta} \theta_v^{\alpha,\beta})^k \\
&+ O(1) \frac{X_m^k}{m} \sum_{v=1}^{m} (v^{\delta} \theta_v^{\alpha,\beta})^k \\
&= O(1) \sum_{n=1}^{m-1} n(\log{(n+1)})^{p+k-1} \Delta(n^{-1} X_n^k) + O(X_m^k (\log{(m+1)})^{p+k-1}) \\
&= O(1) \sum_{n=1}^{m-1} n^{-1} X_n^k (\log{(n+1)})^{p+k-1} + O(1) \sum_{n=1}^{m-1} (\log{(n+1)})^{p+k-1} \Delta X_n^k \\
&+ O(1)((\log{(m+1)})^{p+k-1} X_m^k) \\
&= O(1) \sum_{n=1}^{m-1} \frac{(\lambda_n \log{(n+1)})^k}{(n+1)(\log{(n+1)})^{1+p(k-1)+k(k-1)}} \\
&+ O(1) \sum_{n=1}^{m-1} \frac{\Delta \lambda_n^k}{(\log{(n+1)})^{p(k-1)+(k-1)^2}} \\
&+ O\left(\frac{(\lambda_m \log{(m+1)})^k}{(\log{(m+1)})^{p(k-1)+k(k-1)+1}} \right) \\
&= O(1) \quad as \quad m \to \infty,
\end{aligned}$$

by the application of Lemma 4 and $\lambda_n \log n = O(1)$. Therefore, by (6), we get that

$$\sum_{n=1}^{\infty} n^{\delta k-1} \mid T_{n,r}^{\alpha,\beta} \mid^k < \infty \quad , for \quad r = 1, 2.$$

This completes the proof of the theorem. It should be noted that if we take $\beta = 0$, then we obtain Theorem A. This theorem also includes as particular cases the results of Pati [9] and Prasad and Bhatt [10].

References

[1] H. Bor, On a new application of quasi power increasing sequences, Proc. Estonian Acad. Sci. Phys. Math., 57 (2008), 205-209.

[2] H. Bor, On the generalized absolute Cesàro summability, Pac. J. Appl. Math., (2009), 35-40.

[3] D. Borwein, Theorems on some methods of summability, Quart. J. Math., Oxford, Ser. 9 (1958), 310-316.

[4] H. C. Chow, On the summability factors of Fourier Series, J. London Math. Soc., 16 (1941), 215-220.

[5] G. Das, A Tauberian theorem for absolute summability, Proc. Camb. Phil. Soc., 67 (1970), 321-326.

[6] T.M. Flett, On an extension of absolute summability and some theorems of Littlewood and Paley, Proc. London Math. Soc., 7 (1957), 113-141.

[7] T.M. Flett, Some more theorems concerning the absolute summability of Fourier series and power series , Proc. London Math. Soc., 7 (1958), 357-387.

[8] S. N. Lal and S. R. Singh, On the absolute summability factors in infinite series, Bull. Acad. Polon. Sci. Math. Astronom. Phys., 17 (1969), 711-714.

[9] T. Pati, The summability factors of infinite series, Duke Math. J., 21 (1954), 271-284.

[10] B. N. Prasad and S. N. Bhatt, The summability factors of a Fourier series, Duke Math. J., 24 (1957), 103- 107.

HÜSEYİN BOR
P.O.Box 121 , 06502 Bahçelievler,
Ankara, Turkey
e-mail: hbor33@gmail.com