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SHARPNESS OF NEGOI'S INEQUALITY FOR THE EULER-MASCHERONI CONSTANT

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Abstract. We present new estimates for the Euler-Mascheroni constant, which improve a result of Negoi.

1. Introduction

The Euler-Mascheroni constant $\gamma = 0.577215664\ldots$ is defined as the limit of the sequence

$$D_n = \sum_{k=1}^{n} \frac{1}{k} - \ln n \qquad (n \in \mathbb{N} := \{1, 2, 3, \ldots\}) .$$

Several bounds for $D_n - \gamma$ have been given in the literature [3, 4, 19, 22, 23, 24, 27] (see also [6, 20, 21]). For example, the following bounds for $D_n - \gamma$ were established in [19, 27]:

$$\frac{1}{2(n+1)} < D_n - \gamma < \frac{1}{2n} \qquad (n \in \mathbb{N}) .$$

The convergence of the sequence D_n to γ is very slow. Some quicker approximations to the Euler-Mascheroni constant were established in [5, 6, 7, 9, 8, 10, 15, 16, 18, 20, 21, 25, 26]. For example, Negoi [18] proved that the sequence

$$T_n = \sum_{k=1}^{n} \frac{1}{k} - \ln\left(n + \frac{1}{2} + \frac{1}{24n}\right) \tag{1.1}$$

is strictly increasing and convergent to γ. Moreover, the author proved that

$$\frac{1}{48(n+1)^3} < \gamma - T_n < \frac{1}{48n^3}. \tag{1.2}$$

The main objective of this work is to establish closer bounds for $\gamma - T_n$.

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2. Lemmas

Before stating and proving the main theorems, we first include here some preliminary results.

The constant γ is deeply related to the gamma function $\Gamma(x)$ thanks to the Weierstrass formula [1, p. 255]:

$$\Gamma(x) = \frac{e^{-\gamma x}}{x} \prod_{k=1}^{\infty} \left\{ \left(1 + \frac{x}{k}\right)^{-1} e^{x/k} \right\}$$

for any real number x, except on the negative integers $\{0, -1, -2, \ldots\}$. The logarithmic derivative of the gamma function:

$$\psi(x) = \frac{\Gamma'(x)}{\Gamma(x)}$$

is known as the psi (or digamma) function.

The following recurrence and asymptotic formulas are well known for the psi function:

$$\psi(x+1) = \psi(x) + \frac{1}{x} \tag{2.1}$$

(see [1, p.258]), and

$$\psi(x) \sim \ln x - \frac{1}{2x} - \frac{1}{12x^2} + \frac{1}{120x^4} - \frac{1}{252x^6} + \ldots \quad (x \to \infty) \tag{2.2}$$

(see [1, p.259]). From (2.1) and (2.2), we get

$$\psi(x+1) \sim \ln x + \frac{1}{2x} - \frac{1}{12x^2} + \frac{1}{120x^4} - \frac{1}{252x^6} + \ldots \quad (x \to \infty)\ . \tag{2.3}$$

It is also known [1, p.258] that

$$\psi(n+1) = -\gamma + \sum_{k=1}^{n} \frac{1}{k}\ . \tag{2.4}$$

The following lemmas are also needed in our present investigation.

Lemma 2.1. *If the sequence $(\lambda_n)_{n\in\mathbb{N}}$ converges to zero and if there exists the following limit:*

$$\lim_{n\to\infty} n^k(\lambda_n - \lambda_{n+1}) = l \in \mathbb{R} \qquad (k > 1)\ ,$$

then

$$\lim_{n\to\infty} n^{k-1}\lambda_n = \frac{l}{k-1} \qquad (k > 1)\ .$$

This lemma is suitable for accelerating some convergences, or in constructing some asymptotic expansions. For proofs and other details, see, e.g. [11, 12, 13, 14, 15, 16, 17].

Lemma 2.2 ([2, Theorem 9])**.** *Let $k \geq 1$ and $n \geq 0$ be integers. Then for all real numbers $x > 0$:*

$$S_k(2n; x) < (-1)^{k+1}\psi^{(k)}(x) < S_k(2n+1; x), \tag{2.5}$$

where

$$S_k(p; x) = \frac{(k-1)!}{x^k} + \frac{k!}{2x^{k+1}} + \sum_{i=1}^{p} \left[B_{2i} \prod_{j=1}^{k-1} (2i+j) \right] \frac{1}{x^{2i+k}},$$

and B_i $(i = 0, 1, 2, \ldots)$ *are Bernoulli numbers, defined by*

$$\frac{t}{e^t - 1} = \sum_{i=0}^{\infty} B_i \frac{t^i}{i!}$$

(*see* [1, p. 804]).

In particular, it follows from (2.5) that

$$\frac{1}{x} + \frac{1}{2x^2} + \frac{1}{6x^3} - \frac{1}{30x^5} + \frac{1}{42x^7} - \frac{1}{30x^9} < \psi'(x) < \frac{1}{x} + \frac{1}{2x^2} + \frac{1}{6x^3} - \frac{1}{30x^5} + \frac{1}{42x^7} - \frac{1}{30x^9} + \frac{5}{66x^{11}}, \quad x > 0\ . \tag{2.6}$$

From (2.1) and (2.6), we obtain

$$\frac{1}{x} - \frac{1}{2x^2} + \frac{1}{6x^3} - \frac{1}{30x^5} + \frac{1}{42x^7} - \frac{1}{30x^9} < \psi'(x+1) < \frac{1}{x} - \frac{1}{2x^2} + \frac{1}{6x^3} - \frac{1}{30x^5} + \frac{1}{42x^7}, \quad x > 0\ . \tag{2.7}$$

3. Main results

3.1. We define the sequence $(u_n)_{n \in \mathbb{N}}$ by

$$u_n = \ln\left(n + \frac{1}{2} + \frac{1}{24n}\right) - \psi(n+1) - \frac{a}{\left(n + b + \frac{c}{n+d}\right)^3}\ . \tag{3.1}$$

We are interested in finding the values of the parameters a, b, c and d such that $(u_n)_{n \in \mathbb{N}}$ is the *fastest* sequence which would converge to zero. This provides the best approximations of the form:

$$\psi(n+1) \approx \ln\left(n + \frac{1}{2} + \frac{1}{24n}\right) - \frac{a}{\left(n + b + \frac{c}{n+d}\right)^3}\ . \tag{3.2}$$

Our study is based on the above Lemma 2.1.

Theorem 3.1. *Let the sequence* $(u_n)_{n \in \mathbb{N}}$ *be defined by* (3.1). *Then for*

$$a = \frac{1}{48}, \quad b = \frac{83}{360}\ , \quad c = \frac{4909}{64800}\ , \quad d = \frac{11976997}{37112040}\ , \tag{3.3}$$

we have

$$\lim_{n \to \infty} n^8 (u_n - u_{n+1}) = \frac{1763157528883853}{83111968235520000} \tag{3.4}$$

and

$$\lim_{n \to \infty} n^7 u_n = \frac{1763157528883853}{581783777648640000}\ . \tag{3.5}$$

The speed of convergence of the sequence $(u_n)_{n \in \mathbb{N}}$ *is given by the order estimate* $O\left(n^{-7}\right)$.

Proof. First of all, we write the difference $u_n - u_{n+1}$ as the following power series in n^{-1}:

$$\begin{aligned} u_n - u_{n+1} = & \frac{1-48a}{16n^4} + \frac{-263+8640a+17280ab}{1440n^5} \\ & + \frac{139-3840a-11520ab-11520ab^2+5760ac}{384n^6} \\ & + \Big(90720a+362880ab+362880ab^3+544320ab^2-272160ac \\ & \quad - 435456acb - 108864acd - 3685\Big)\frac{1}{6048n^7} \\ & + \Big(-193536a-967680ab+774144acbd+1935360acb^2+193536acd^2 \\ & \quad + 8663 + 2322432acb - 387072ac^2 - 1935360ab^3 - 967680ab^4 \\ & \quad + 580608acd + 967680ac - 1935360ab^2\Big)\frac{1}{9216n^8} + O\left(\frac{1}{n^9}\right) . \end{aligned} \tag{3.6}$$

The fastest sequence $(u_n)_{n\in\mathbb{N}}$ is obtained when the first four coefficients of this power series vanish. In this case

$$a = \frac{1}{48}, \quad b = \frac{83}{360}\ , \quad c = \frac{4909}{64800}\ , \quad d = \frac{11976997}{37112040}\ ,$$

we have

$$u_n - u_{n+1} = \frac{1763157528883853}{83111968235520000n^8} + O\left(\frac{1}{n^9}\right) . \tag{3.7}$$

Finally, by using Lemma 2.1, we obtain assertions (3.4) and (3.5) of Theorem 3.1. □

Solution (3.1) provides the best approximation of type (3.2):

$$\psi(n+1) \approx \ln\left(n+\frac{1}{2}+\frac{1}{24n}\right) - \frac{\frac{1}{48}}{\left(n+\frac{83}{360}+\frac{\frac{4909}{64800}}{n+\frac{11976997}{37112040}}\right)^3} . \tag{3.8}$$

Motivated by approximation (3.8), we establish Theorem 3.2 below, which provides closer bounds for $\gamma - T_n$.

Theorem 3.2. *For $n \geq 1$, then*

$$\frac{\frac{1}{48}}{\left(n+\frac{83}{360}+\frac{\frac{4909}{64800}}{n+\frac{11976997}{37112040}}\right)^3} < \gamma - T_n < \frac{\frac{1}{48}}{\left(n+\frac{83}{360}\right)^3} . \tag{3.9}$$

Proof. We only prove the right-hand inequality in (3.9). The proof of the left-hand inequality in (3.9) is similar. The inequality (3.9) can be written for $n \geq 1$ as

$$\frac{\frac{1}{48}}{\left(n+\frac{83}{360}+\frac{\frac{4909}{64800}}{n+\frac{11976997}{37112040}}\right)^3} < \ln\left(n+\frac{1}{2}+\frac{1}{24n}\right) - \psi(n+1) < \frac{\frac{1}{48}}{\left(n+\frac{83}{360}\right)^3} . \tag{3.10}$$

The upper bound of (3.9) is obtained by considering the function $f(x)$ which is defined, for $x>0$, by

$$f(x)=\ln\left(x+\frac{1}{2}+\frac{1}{24x}\right)-\psi(x+1)-\frac{\frac{1}{48}}{\left(x+\frac{83}{360}\right)^3}\ .$$

We conclude from the asymptotic formula (2.3) that

$$\lim_{x\to\infty} f(x)=0\ .$$

Differentiating $f(x)$ and applying the second inequality in (2.7) yields,

$$\begin{aligned} f'(x)&=\frac{24x^2-1}{x(24x^2+12x+1)}-\psi'(x+1)+\frac{1049760000}{(360x+83)^4}\\ &>\frac{24x^2-1}{x(24x^2+12x+1)}-\left(\frac{1}{x}-\frac{1}{2x^2}+\frac{1}{6x^3}-\frac{1}{30x^5}+\frac{1}{42x^7}\right)+\frac{1049760000}{(360x+83)^4}\\ &=\frac{p(x)}{210x^7(24x^2+12x+1)(360x+83)^4}\ , \end{aligned}$$

where

$$\begin{aligned} p(x)&=147550579398783+637562673548352(x-2)+1095096221221183(x-2)^2\\ &\quad+997896029835428(x-2)^3+528831825356263(x-2)^4\\ &\quad+164401992148725(x-2)^5+27912981996000(x-2)^6\\ &\quad+2004050160000(x-2)^7>0 \quad \text{for} \quad x\ge 2\ . \end{aligned}$$

Therefore, $f'(x)>0$ for $x\ge 2$.

Direct computation would yield

$$\begin{aligned} f(1)&=\gamma+\ln\left(\frac{37}{24}\right)-\frac{87910307}{86938307}=-0.00110059\dots\ ,\\ f(2)&=\gamma+\ln\left(\frac{121}{48}\right)-\frac{1555288881}{1035563254}=-0.000072039\dots\ . \end{aligned}$$

Consequently, the sequence $\big(f(n)\big)_{n\in\mathbb{N}}$ is strictly increasing. This leads us to

$$f(n)<\lim_{n\to\infty} f(n)=0\ ,\quad n\ge 1\ ,$$

which means that the upper bound in assertion (3.9) of Theorem 3.2 holds true for all $n\in\mathbb{N}$. The proof of Theorem 3.2 is thus completed. □

Remark 1. *In fact, the following inequality holds true:*

$$\gamma-T_n<\frac{\frac{1}{48}}{\left(n+\frac{83}{360}+\frac{\frac{4909}{64800}}{n+\frac{11976997}{37112040}+\frac{\frac{1763157528883853}{2754607025923200}}{n+\frac{2160995763710564441795}{13086874547647741578024}}}\right)^3} \tag{3.11}$$

for $n\in\mathbb{N}$.

3.2. We now define the sequence $(v_n)_{n\in\mathbb{N}}$ by

$$v_n = \ln\left(n + \frac{1}{2} + \frac{1}{24n}\right) - \psi(n+1) - \frac{1}{a_1n^3 + b_1n^2 + c_1n + d_1} \ . \tag{3.12}$$

where $a_1, b_1, c_1, d_1 \in \mathbb{R}$. Following the same method used in the proof of Theorem 3.1, we find that for

$$a_1 = 48 \ , \quad b_1 = \frac{166}{5} \ , \quad c_1 = \frac{5569}{300} \ , \quad d_1 = \frac{58741}{28000} \ , \tag{3.13}$$

we have

$$\lim_{n\to\infty} n^8(v_n - v_{n+1}) = \frac{183358033}{9953280000} \quad \textit{and} \quad \lim_{n\to\infty} n^7 v_n = \frac{183358033}{69672960000} \ . \tag{3.14}$$

The speed of convergence of the sequence $(v_n)_{n\in\mathbb{N}}$ is given by the order estimate $O\left(n^{-7}\right)$.

Theorem 3.3. *For $n \geq 1$, then*

$$\frac{1}{48n^3 + \frac{166}{5}n^2 + \frac{5569}{300}n + \frac{58741}{28000}} < \gamma - T_n \ . \tag{3.15}$$

Proof. The inequality (3.15) can be written for $n \geq 1$ as

$$\frac{1}{48n^3 + \frac{166}{5}n^2 + \frac{5569}{300}n + \frac{58741}{28000}} < \ln\left(n + \frac{1}{2} + \frac{1}{24n}\right) - \psi(n+1) \ . \tag{3.16}$$

We consider the function $F(x)$ defined for $x > 0$ by

$$F(x) = \ln\left(x + \frac{1}{2} + \frac{1}{24x}\right) - \psi(x+1) - \frac{1}{48x^3 + \frac{166}{5}x^2 + \frac{5569}{300}x + \frac{58741}{28000}} \ .$$

We conclude from the asymptotic formula (2.3) that

$$\lim_{x\to\infty} F(x) = 0 \ .$$

Differentiating $F(x)$ and applying the first inequality in (2.7) yields,

$$\begin{aligned}
F'(x) &= \frac{24x^2 - 1}{x(24x^2 + 12x + 1)} - \psi'(x+1) \\
&\quad + \frac{23520000(43200x^2 + 19920x + 5569)}{(4032000x^3 + 2788800x^2 + 1559320x + 176223)^2} \\
&< \frac{24x^2 - 1}{x(24x^2 + 12x + 1)} - \left(\frac{1}{x} - \frac{1}{2x^2} + \frac{1}{6x^3} - \frac{1}{30x^5} + \frac{1}{42x^7} - \frac{1}{30x^9}\right) \\
&\quad + \frac{23520000(43200x^2 + 19920x + 5569)}{(4032000x^3 + 2788800x^2 + 1559320x + 176223)^2} \\
&= -\frac{q(x)}{210x^9(24x^2 + 12x + 1)(4032000x^3 + 2788800x^2 + 1559320x + 176223)^2} \ ,
\end{aligned}$$

where

$$\begin{aligned}
q(x) &= 18130487257947165687 + 63552993678839537457(x-3) \\
&\quad + 94471229612034347921(x-3)^2 + 79408865830190450709(x-3)^3 \\
&\quad + 41975644888778012717(x-3)^4 + 14553520724815257633(x-3)^5 \\
&\quad + 3322393272176291138(x-3)^6 + 482867798807968875(x-3)^7 \\
&\quad + 40622141576265200(x-3)^8 + 1509403327656000(x-3)^9 > 0 \quad \text{for} \quad x \geq 3 \ .
\end{aligned}$$

Therefore, $F'(x) < 0$ for $x \geq 3$.

Direct computation would yield

$$F(1) = -\frac{8640343}{8556343} + \gamma + \ln 37 - 3\ln 2 - \ln 3 = 0.000262469\dots\ ,$$
$$F(2) = -\frac{140286189}{93412126} + \gamma + 2\ln 11 - 4\ln 2 - \ln 3 = 0.000006718\dots\ ,$$
$$F(3) = -\frac{509165071}{277634766} + \gamma + \ln 11 + \ln 23 - 3\ln 2 - 2\ln 3 = 0.000000589\dots\ .$$

Consequently, the sequence $\big(F(n)\big)_{n\in\mathbb{N}}$ is strictly decreasing. This leads us to

$$F(n) > \lim_{n\to\infty} F(n) = 0\ , \quad n \geq 1\ ,$$

which means that inequality (3.15) holds true for all $n \in \mathbb{N}$. □

Remark 2. *The lower bound in* (3.15) *is sharper than one in* (3.9).

Remark 3. *In fact, the following inequality holds true:*

$$\gamma - T_n < \frac{1}{48n^3 + \frac{166}{5}n^2 + \frac{5569}{300}n + \frac{58741}{28000} - \frac{183358033}{30240000n}}\ . \tag{3.17}$$

for $n \in \mathbb{N}$.

Remark 4. *The numerical calculations presented in this work were performed by using the Maple software for symbolic computations.*

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