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ALMOST αGS-CLOSED FUNCTIONS AND SEPARATION AXIOMS

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Abstract. We introduce a new class of functions called almost αgs-closed and use the functions to improve several preservation theorems of normality and regularity and also their generalizations. The main result of the paper is that normality and weak normality are preserved under almost αgs-closed continuous surjections.

1. Introduction

In topological spaces, it is well known that normality is preserved under closed continuous surjections. Many authors have tried to weaken the condition "closed" in this theorem. In 1978, Long and Herrington [11] used almost closedness due to Singal [25]. In 1982, Malghan [14] used g-closedness. In 1986, Greenwood and Reilly [6] used α-closedness due to Mashhour et al. [15] In 1995, Yoshimura et al. [29] used almost g-closedness which is a generalization of both almost closedness and g-closedness. In 1999, Noiri [19] introduced almost αg-closedness using αg-closed sets [13]. Recently, Rajamani and Viswanathan [22] have introduced the notion of αgs-closed sets which are strictly weaker than both α-closed sets and $\hat{g}$-closed sets. We use αgs-closed sets to define a new class of functions called almost αgs-closed functions. The purpose of the present paper is to improve preservation theorems of separation axioms, that is, normality, weak normality, mild normality, almost normality, regularity, almost regularity, quasi-regularity and strong s-regularity. The following properties are main results of the present paper.

Theorem A. Normality and weak normality are preserved under almost αgs-closed continuous surjections.

Theorem B. Regularity and strong s-regularity are preserved under almost α-open almost αgs-closed continuous surjections.

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2. PRELIMINARIES

Throughout this paper (X, τ) *and* (Y, σ)*(or* X *and* Y*) represent topological spaces on which no separation axioms are assumed unless otherwise mentioned. For a subset* A *of a space* (X, τ), $cl(A)$, $int(A)$ *and* A^c *denote the closure of* A, *the interior of* A *and the complement of* A *respectively.*

We recall the following definitions which are useful in the sequel.

Definition 2.1. *A subsetA of a space* (X, τ) *is called :*
(i) semi-open set [10] *if* $A \subseteq cl(int(A))$;
(ii) α*-open set* [16] *if* $A \subseteq int(cl(int(A)))$;
(iii) regular open set [19] *if* $A = int(cl(A))$.

The complements of the above mentioned sets are called their respective closed sets.

The family of regular open (resp. regular closed) sets of a space (X, τ) *is denoted by* $RO(X, \tau)$ *(resp.* $RC(X, \tau)$*) or simply by* $RO(X)$ *(resp.* $RC(X)$*).*

The family of α*-open sets of a space* (X, τ) *is denoted by* τ^α. *It is known* [16] *that* $\tau \subset \tau^\alpha$ *and* τ^α *is a topology for* X. *The closure (resp. interior) of a subset* A *of* X *with respect to* τ^α *is denoted by* $\alpha cl(A)$ *(resp.* $\alpha int(A)$*). It is known in* [1] *that* $\alpha cl(A) = A \cup cl(int(cl(A)))$ *and* $\alpha int(A) = A \cap int(cl(int(A)))$ *for any subset* A *of a space* (X, τ).

Definition 2.2. *A subset* A *of a space* (X, τ) *is called :*
(i) a generalized closed (briefly g-closed) set [9] *if* $cl(A) \subseteq U$ *whenever* $A \subseteq U$ *and* U *is open in* (X, τ). *The complement of g-closed set is called g-open set;*
(ii) a α*-generalized closed (briefly* αg*-closed) set* [13] *if* $\alpha cl(A) \subseteq U$ *whenever* $A \subseteq U$ *and* U *is open in* (X, τ). *The complement of* αg*-closed set is called* αg*-open set;*
(iii) a α*-generalized semi-closed (briefly* αgs*-closed) set* [22] *if* $\alpha cl(A) \subseteq U$ *whenever* $A \subseteq U$ *and* U *is semi-open in* (X, τ). *The complement of* αgs*-closed set is called* αgs*-open set;*
(iv) a $\hat{g}$*-closed set* [27] *if* $cl(A) \subseteq U$ *whenever* $A \subseteq U$ *and* U *is semi-open in* (X, τ). *The complement of* $\hat{g}$*-closed set is called* $\hat{g}$*-open set;*
(v) a $r\alpha g$*-closed* [19] *if* $\alpha cl(A) \subseteq U$ *whenever* $A \subseteq U$ *and* U *is regular open in* (X, τ). *The complement of* $r\alpha g$*-closed set is called* $r\alpha g$*-open set.*

Remark 2.3. *From the Definition* 2.1 *and* 2.2, *we have the following implications.*

$$\begin{array}{ccccc} \text{closed} & \longrightarrow & \hat{g}\text{-closed} & \longrightarrow & g\text{-closed} \\ \downarrow & & \downarrow & & \downarrow \\ \alpha\text{-closed} & \longrightarrow & \alpha gs\text{-closed} & \longrightarrow & \alpha g\text{-closed} \end{array}$$

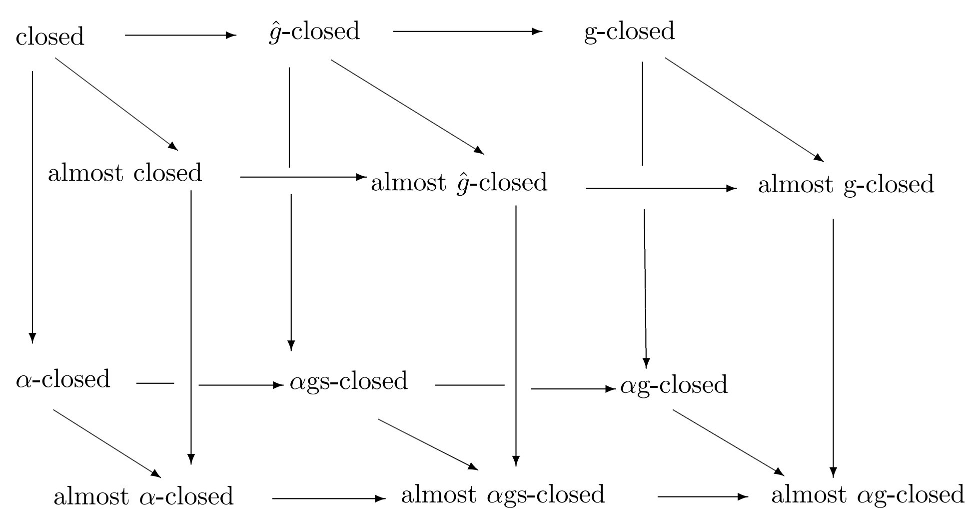
In the above remark, none of implications is reversible as the following *examples show.*

(*i*) *Let* $X = \{a, b, c\}$ *with* $\tau = \{\emptyset, \{a\}, X\}$. *Then* $\{b\}$ *is* α*-closed set but not closed.*
(*ii*) *Let* $X = \{a, b, c\}$ *with* $\tau = \{\emptyset, \{a, b\}, X\}$. *Then* $\{b, c\}$ *is* $\hat{g}$*-closed set but not closed.*
(*iii*) *Let* $X = \{a, b, c\}$ *with* $\tau = \{\emptyset, \{c\}, X\}$. *Then* $\{b, c\}$ *is g-closed set but not* $\hat{g}$*-closed.*
(*iv*) *Let* $X = \{a, b, c\}$ *with* $\tau = \{\emptyset, \{a\}, \{b, c\}, X\}$. *Then* $\{a, b\}$ *is* α*gs-closed set but not* α*-closed.*
(*v*) *Let* $X = \{a, b, c\}$ *with* $\tau = \{\emptyset, \{b\}, X\}$. *Then* $\{a\}$ *is* α*gs-closed set but not* $\hat{g}$*-closed.*
(*vi*) *Let* $X = \{a, b, c\}$ *with* $\tau = \{\emptyset, \{a\}, X\}$. *Then* $\{a, c\}$ *is* α*g-closed set but not* α*gs-closed.*
(*vii*) *Let* $X = \{a, b, c\}$ *with* $\tau = \{\emptyset, \{a\}, \{a, b\}, X\}$. *Then* $\{b\}$ *is* α*g-closed set but not g-closed.*

3. Almost αgs-closed functions

Definition 3.1. *A function* $f : (X, \tau) \to (Y, \sigma)$ *is said to be*
(*a*) α*-closed* [15] (*resp. g-closed* [14], α*g-closed* [19], $\hat{g}$*-closed,* α*gs-closed*) *if for each closed set* F *of* X, $f(F)$ *is* α*-closed* (*resp. g-closed,* α*g-closed,* $\hat{g}$*-closed,* α*gs-closed*);
(*b*) *almost closed* [25] (*resp. almost* α*-closed* [19], *almost g-closed* [18], *almost* α*g-closed* [19], *almost* $\hat{g}$*-closed, almost* α*gs-closed*) *if for each* $F \in RC(X, \tau)$, $f(F)$ *is closed* (*resp.* α*-closed, g-closed,* α*g-closed,* $\hat{g}$*-closed,* α*gs-closed*).

Remark 3.2. *We have the following diagram for properties of functions* :



The following two examples show that almost $\hat{g}$*-closedness is strictly* *weaker than almost closedness and $\hat{g}$-closedness.*

Example 3.3. *Let $X = \{a, b, c, d\}$, $\tau = \{\emptyset, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}, \{a, b, d\}, X\}$ and $\sigma = \{\emptyset, \{a\}, \{b, c\}, \{a, b, c\}, X\}$. Then the identity function $f : (X, \tau) \to (X, \sigma)$ is almost $\hat{g}$-closed. However, it is not almost closed since there exists $\{a, c, d\} \in RC(X, \tau)$ such that $f(\{a, c, d\}) = \{a, c, d\}$ is not closed in (X, σ).*

Example 3.4. *Let $X = \{a, b, c, d\}$, $\tau = \{\emptyset, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}, \{a, b, d\}, X\}$ and $\sigma = \{\emptyset, \{a\}, \{b, c\}, \{a, b, c\}, X\}$. Then the identity function $f : (X, \tau) \to (X, \sigma)$ is almost $\hat{g}$-closed. However, it is not $\hat{g}$-closed since there exists a closed set $\{c\}$ such that $f(\{c\}) = \{c\}$ is not $\hat{g}$-closed in (X, σ).*

The following two examples show that almost g-closedness is strictly weaker than almost $\hat{g}$-closedness and g-closedness.

Example 3.5. *Let $X = \{a, b, c, d\}$, $\tau = \{\emptyset, \{a\}, \{b, c\}, \{a, b, c\}, X\}$ and $\sigma = \{\emptyset, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}, X\}$. Then the identity function $f : (X, \tau) \to (X, \sigma)$ is almost g-closed. However, it is not almost $\hat{g}$-closed since there exists $\{a, d\} \in RC(X, \tau)$ such that $f(\{a, d\}) = \{a, d\}$ is not $\hat{g}$-closed in (X, σ).*

Example 3.6. *Let $X = \{a, b, c, d\}$, $\tau = \{\emptyset, \{d\}, \{b, c\}, \{b, c, d\}, X\}$ and $\sigma = \{\emptyset, \{a\}, \{b\}, \{a, b\}, X\}$. Then the identity function $f : (X, \tau) \to (X, \sigma)$ is almost g-closed. However, it is not g-closed since there exists a closed set $\{a\}$ of (X, τ) such that $f(\{a\}) = \{a\}$ is not g-closed in (X, σ).*

The following three examples show that almost αgs-closedness is strictly weaker than almost α-closedness, αgs-closedness and almost $\hat{g}$-closedness.

Example 3.7. *Let $X = \{a, b, c, d\}$, $\tau = \{\emptyset, \{a\}, \{b\}, \{a, b\}, X\}$ and $\sigma = \{\emptyset, \{a\}, \{b, c\}, \{a, b, c\}, X\}$. Then the identity function $f : (X, \tau) \to (X, \sigma)$ is almost αgs-closed. However, it is not almost α-closed since there exists $\{a, c\} \in RC(X, \tau)$ such that $f(\{a, c\}) = \{a, c\}$ is not α-closed in (X, σ).*

Example 3.8. *Let $X = \{a, b, c, d\}$, $\tau = \{\emptyset, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}, \{a, b, d\}, X\}$ and $\sigma = \{\emptyset, \{a\}, \{b, c\}, \{a, b, c\}, X\}$. Then the identity function $f : (X, \tau) \to (X, \sigma)$ is almost αgs-closed. However, it is not αgs-closed since there exists a closed set $\{c\}$ of (X, τ) such that $f(\{c\}) = \{c\}$ is not αgs-closed in (X, σ).*

Example 3.9. *Let $X = \{a, b, c, d\}$, $\tau = \{\emptyset, \{d\}, \{a, b, c\}, X\}$ and $\sigma = \{\emptyset, \{a\}, \{b\}, \{a, b\}, X\}$. Then the identity function $f : (X, \tau) \to (X, \sigma)$ is almost αgs-closed. However, it is not almost $\hat{g}$-closed since there exists $\{d\} \in RC(X, \tau)$ such that $f(\{d\}) = \{d\}$ is not $\hat{g}$-closed in (X, σ).*

The following three examples show that almost αg-closedness is strictly weaker than almost g-closedness, αg-closedness and almost αgs-closedness.

Example 3.10. *Let $X = \{a, b, c, d\}$, $\tau = \{\emptyset, \{a\}, \{b\}, \{a, b\}, \{b, c\}, \{a, b, c\}, X\}$ and $\sigma = \{\emptyset, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}, X\}$. Then the identity function $f : (X, \tau) \to (X, \sigma)$ is almost αg-closed. However, it is not almost αgs-closed since there exists $\{a, d\} \in RC(X, \tau)$ such that $f(\{a, d\}) = \{a, d\}$ is not αgs-closed in (X, σ).*

Example 3.11. *Let $X = \{a, b, c, d\}$, $\tau = \{\emptyset, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}, \{a, b, d\}, X\}$ and $\sigma = \{\emptyset, \{a\}, \{b, c\}, \{a, b, c\}, X\}$. Then the identity function $f : (X, \tau) \to (X, \sigma)$ is almost αg-closed. However, it is not αg-closed since there exists a closed set $\{c\}$ of (X, τ) such that $f(\{c\}) = \{c\}$ is not αg-closed in (X, σ).*

Example 3.12. *Let $X = \{a, b, c, d\}$, $\tau = \{\emptyset, \{d\}, \{a, b, c\}, X\}$ and $\sigma = \{\emptyset, \{a\}, \{b\}, \{a, b\}, \{a, d\}, \{a, b, d\}, X\}$. Then the identity function $f : (X, \tau) \to (X, \sigma)$ is almost αg-closed. However, it is not almost g-closed since there exists $\{d\} \in RC(X, \tau)$ such that $f(\{d\}) = \{d\}$ is not g-closed in (X, σ).*

Theorem 3.13. *A surjection $f : X \to Y$ is almost αgs-closed if and only if for each subset S of Y and each $U \in RO(X)$ containing $f^{-1}(S)$ there exists an αgs-open set V of Y such that $S \subset V$ and $f^{-1}(V) \subset U$.*

Proof. *Necessity. Suppose that f is almost αgs-closed. Let S be a subset of Y and $U \in RO(X)$ containing $f^{-1}(S)$. Put $V = Y - f(X - U)$, then V is an αgs-open set of Y such that $S \subset V$ and $f^{-1}(V) \subset U$.*
Sufficiency. Let F be any regular closed set of X. Then $f^{-1}(Y - f(F)) \subset X - F$ and $X - F \in RO(X)$. There exists an αgs-open set V of Y such that $Y - f(F) \subset V$ and $f^{-1}(V) \subset X - F$. Therefore, we have $f(F) \supset Y - V$ and $F \subset f^{-1}(Y-V)$. Hence, we obtain $f(F) = Y-V$ and $f(F)$ is αgs-closed in Y. This shows that f is almost αgs-closed.

Corollary 3.14. *If $f : X \to Y$ is an almost αgs-closed surjection, then for each semi-closed set F of Y and each $U \in RO(X)$ containing $f^{-1}(F)$ there exists an α-open set V of Y such that $F \subset V$ and $f^{-1}(V) \subset U$.*

Proof. *Let F be a semi-closed set of Y and $U \in RO(X)$ containing $f^{-1}(F)$. By Theorem 3.13, there exists an αgs-open set W of Y such that $F \subset W$ and $f^{-1}(W) \subset U$. Since W is αgs-open, we have $F \subset \alpha int(W)$. Put $V = \alpha int(W)$, then V is α-open in Y and $f^{-1}(V) \subset U$.*

4. Normal spaces

In this section, we make use of αgs-closed sets to obtain further characterizations and preservation theorems of normal spaces.

Theorem 4.1. *The following are equivalent for a space X :*
(a) X is normal;
(b) For any disjoint closed sets A and B, there exist disjoint αgs-open sets

U, V such that $A \subset U$ and $B \subset V$;
(c) For any closed set A and any open set V containing A, there exists an αgs-open set U of X such that $A \subset U \subset \alpha cl(U) \subset V$.

Proof. (a) $\Rightarrow$ (b). This is obvious since every open set is αgs-open.
(b) $\Rightarrow$ (c). Let A be a closed set and V an open set containing A. Then A and $X - V$ are disjoint closed sets. There exist disjoint αgs-open sets U and W such that $A \subset U$ and $X - V \subset W$. Since $X - V$ is closed and hence semi-closed, we have $X - V \subset \alpha int(W)$ and $U \cap \alpha int(W) = \emptyset$. Therefore, we obtain $\alpha cl(U) \cap \alpha int(W) = \emptyset$ and hence $A \subset U \subset \alpha cl(U) \subset X - \alpha int(W) \subset V$.
(c) $\Rightarrow$ (a). Let A, B be disjoint closed sets of X. Then $A \subset X - B$ and $X - B$ is open. There exists an αgs-open set G of X such that $A \subset G \subset \alpha cl(G) \subset X - B$. Since A is closed, we have $A \subset \alpha int(G)$. Put $U = int(cl(int(\alpha int(G))))$ and $V = int(cl(int(X - \alpha cl(G))))$. Then U and V are disjoint open sets of X such that $A \subset U$ and $B \subset V$. Therefore, X is normal.

Theorem 4.2. If $f : X \to Y$ is a continuous almost αgs-closed surjection and X is a normal space, then Y is normal.

Proof. Let A and B be any disjoint closed sets of Y. Then $f^{-1}(A)$ and $f^{-1}(B)$ are disjoint closed sets of X. Since X is normal, there exist disjoint open sets U and V such that $f^{-1}(A) \subset U$ and $f^{-1}(B) \subset V$. Let $G = int(cl(U))$ and $H = int(cl(V))$, then G and H are disjoint regular open sets of X such that $f^{-1}(A) \subset G$ and $f^{-1}(B) \subset H$. By Theorem 3.13, there exists αgs-open sets K and L of Y such that $A \subset K$, $B \subset L$, $f^{-1}(K) \subset G$ and $f^{-1}(L) \subset H$. Since G and H are disjoint, so are K and L. It follows from Theorem 4.1 that Y is normal.

The following two corollaries are immediate consequences of Theorem 4.2.

Corollary 4.3 [11]. If $f : X \to Y$ is a continuous almost closed surjection and X is a normal space, then Y is normal.

Corollary 4.4 [6]. If $f : X \to Y$ is a continuous α-closed surjection and X is a normal space, then Y is normal.

Definition 4.5. A space X is said to be
(a) weakly normal [30] if for each decreasing sequence $\{F_n\}$ of closed sets of X such that $\cap\{F_n : n \in N\} = \emptyset$ and each closed set H of X with $H \cap F_1 = \emptyset$, there exist $n \in N$ and an open set U of X such that $F_n \subset U$ and $cl(U) \cap H = \emptyset$;
(b) mildly normal [26] if for any disjoint regular closed sets A and B, there exist disjoint open sets U and V such that $A \subset U$ and $B \subset V$;
(c) almost normal [24] if for every pair of disjoint sets A and B, one of which is closed and the other is regular closed, there exist disjoint open sets U and V such that $A \subset U$ and $B \subset V$.

Lemma 4.6 [19]. If A is an α-open set of a space X, then the following hold: $\alpha cl(A) = cl(A) = cl(int(A))$.

Lemma 4.7 [**17**] *A space X is weakly normal if and only if for each decreasing sequence $\{F_n\}$ of closed sets of X such that $\cap\{F_n : n \in N\} = \emptyset$ and each open set U of X such that $F_1 \subset U$, there exist $n \in N$ and open set G of X such that $F_n \subset G \subset cl(G) \subset U$.*

Theorem 4.8. *If $f : X \to Y$ is an almost αgs-closed continuous surjection and X is a weakly normal space, then Y is weakly normal.*

Proof. *Let $\{F_n\}$ be any decreasing sequence of closed sets of Y with no common point and any open set V of Y such that $F_1 \subset V$. Then $\{f^{-1}(F_n)\}$ is a decreasing sequence of closed sets of X with no common point and $f^{-1}(V)$ is an open sets of X such that $f^{-1}(F_1) \subset f^{-1}(V)$. Since X is weakly normal, by Lemma 4.7, there exist $n \in N$ and an open set U of X such that $f^{-1}(F_n) \subset U \subset cl(U) \subset f^{-1}(V)$. Therefore, $f^{-1}(F_n) \subset int(cl(U))$ and by Corollary 3.14, there exists an α-open set G of Y such that $F_n \subset G$ and $f^{-1}(G) \subset int(cl(U))$. Since $cl(U)$ is regular closed and f is almost αgs-closed, $f(cl(U))$ is αgs-closed in Y. Thus, we obtain $F_n \subset G \subset \alpha cl(G) \subset \alpha cl(f(cl(U))) \subset V$. Let $H = int(cl(int(G)))$, then by Lemma 4.6 we have $F_n \subset H \subset cl(H) = \alpha cl(G) \subset V$. It follows from Lemma 4.7 that Y is weakly normal.*

Corollary 4.9 [**17**]. *Weak normality is preserved under almost closed continuous surjections.*

Lemma 4.10 [**19**].
(i) A subset A of a space X is rαg-open if and only if $F \subset \alpha int(A)$ whenever $F \in RC(X)$ and $F \subset A$.
(ii) Every αg-closed set is rαg-closed but not conversely.

Theorem 4.11. *The following are equivalent for a space X :*
(a) X is mildly normal;
(b) for any disjoint $H,\ K \in RC(X)$, there exist disjoint αgs-open sets U, V such that $H \subset U$ and $K \subset V$;
(c) for any disjoint $H,\ K \in RC(X)$, there exist disjoint αg-open sets U, V such that $H \subset U$ and $K \subset V$;
(d) for any disjoint $H,\ K \in RC(X)$, there exist disjoint rαg-open sets U, V such that $H \subset U$ and $K \subset V$;
(e) for any $H \in RC(X)$ and any $V \in RO(X)$ containing H, there exists an αg-open set U of X such that $H \subset U \subset \alpha cl(U) \subset V$;
(f) for any $H \in RC(X)$ and any $V \in RO(X)$ containing H, there exists an α-open set U of X such that $H \subset U \subset \alpha cl(U) \subset V$;
(g) for any disjoint $H,\ K \in RC(X)$, there exist disjoint α-open sets U, V such that $H \subset U$ and $K \subset V$.

Proof. *It is obvious that (a) $\Rightarrow$ (b), (b) $\Rightarrow$ (c) and (c) $\Rightarrow$ (d).*
(d) $\Rightarrow$ (e). Let $H \in RC(X)$ and $V \in RO(X)$ containing H. There exist disjoint αg-open sets U, W such that $H \subset U$ and $X - V \subset W$. By Lemma 4.10, we have $X - V \subset \alpha int(W)$ and $U \cap \alpha int(W) = \emptyset$. Therefore, we obtain

$\alpha cl(U) \cap \alpha int(W) = \emptyset$ *and hence* $H \subset U \subset \alpha cl(U) \subset X - \alpha int(W) \subset V$.
$(e) \Rightarrow (f)$. *Let* $H \in RC(X)$ *and* $V \in RO(X)$ *containing* H. *There exists an* $r\alpha g$-*open set* G *of* X *such that* $H \subset G \subset \alpha cl(G) \subset V$. *Since* $H \in RC(X)$, *By Lemma* 4.10, *we have* $H \subset \alpha int(G)$. *Put* $U = \alpha int(G)$, *then* U *is* α-*open in* X *and* $H \subset U \subset \alpha cl(U) \subset V$.
$(f) \Rightarrow (g)$. *Let* H *and* K *be any disjoint regular closed sets of* X. *Then, since* $H \subset X - K$ *and* $X - K \in RO(X)$, *there exists an* α-*open set* U *of* X *such that* $H \subset U \subset \alpha cl(U) \subset X - K$. *Put* $V = X - \alpha cl(U)$, *then* U *and* V *are disjoint* α-*open sets of* X *such that* $H \subset U$ *and* $K \subset V$.
$(g) \Rightarrow (a)$. *Let* H *and* K *be any disjoint regular closed sets of* X. *Then there exist disjoint* α-*open sets* A *and* B *of* X *such that* $H \subset A$ *and* $K \subset B$. *Since* A *and* B *are disjoint, we have* $int(cl(int(A))) \cap int(cl(int(B))) = \emptyset$. *Now, Put* $U = int(cl(int(A)))$ *and* $V = int(cl(int(B)))$, *then* U *and* V *are disjoint open sets of* X *such that* $H \subset U$ *and* $K \subset V$. *Therefore,* X *is mildly normal.*

Definition 4.12. *A function* $f : X \to Y$ *is said to be*
(*a*) *R-map* [2] (*resp. almost-continuous* [25]) *if* $f^{-1}(V)$ *is regular open* (*resp. open*) *in* X *for every* $V \in RO(Y)$;
(*b*) *almost open* [25] (*resp. almost* α-*open* [19]) *if* $f(U)$ *is open* (*resp.* α-*open*) *in* Y *for every regular open set* U *of* X;
(*c*) α-*open* [15] *if* $f(U)$ *is* α-*open in* Y *for every open set* U *of* X.

Remark 4.13 [**19**]. *Both almost-openness and* α-*openness imply almost* α-*openness but not conversely as the following examples shows.*

Example 4.14 [**19**]. *Let* $X = \{a, b, c, d\}$ *and* $\tau = \{\emptyset, \{c\}, \{d\}, \{a, c\}, \{c, d\}, \{a, c, d\}, X\}$. *Let* $Y = \{a, b, c\}$ *and* $\sigma = \{\emptyset, Y, \{a\}, \{a, b\}\}$. *Then a function* $f : (X, \tau) \to (Y, \sigma)$, *defined as* $f(a) = f(d) = a$, $f(b) = b$ *and* $f(c) = c$, *is almost* α-*open. However, it is neither almost open nor* α-*open.*

Theorem 4.15. *Let* $f : X \to Y$ *be an R-map and an almost* αg-*closed surjection. If* X *is a mildly normal space, then* Y *is mildly normal.*

Proof. *Let* A *and* B *be any disjoint regular closed sets of* Y. *Then* $f^{-1}(A)$ *and* $f^{-1}(B)$ *are disjoint regular closed sets of* X. *Since* X *is mildly normal, there exist disjoint open sets* U *and* V *of* X *such that* $f^{-1}(A) \subset U$ *and* $f^{-1}(B) \subset V$. *Put* $G = int(cl(U))$ *and* $H = int(cl(V))$, *then* G *and* H *are disjoint regular open sets of* X *such that* $f^{-1}(A) \subset G$ *and* $f^{-1}(B) \subset H$. *By Theorem* 3.8 [19], *there exist* αg-*open sets* K *and* L *of* Y *such that* $A \subset K$, $B \subset L$, $f^{-1}(K) \subset G$ *and* $f^{-1}(L) \subset H$. *Since* G *and* H *are disjoint, so are* K *and* L. *It follows from Theorem* 4.11 *that* Y *is mildly normal.*

Corollary 4.16. *Let* $f : X \to Y$ *is an R-map and an almost* αgs-*closed surjection and* X *is mildly normal then* Y *is mildly normal.*

Lemma 4.17 [**19**]. *If a function* $f : X \to Y$ *is almost continuous almost* α-*open and* V *is regular open in* Y, *then* $f^{-1}(V)$ *is regular open in* X.

Theorem 4.18. *If $f : X \to Y$ is an almost α-open almost αg-closed continuous surjection and X is an almost normal spave, then Y is almost normal.*

Proof. *Let B be any closed set of Y and $V \in RO(Y)$ containing B. Since f is continuous and almost α-open, $f^{-1}(B)$ is closed and $f^{-1}(V) \in RO(X)$ by Lemma 4.17. Since X is almost normal and $f^{-1}(B) \subset f^{-1}(V)$, there exists $U \in RO(X)$ such that $f^{-1}(B) \subset U \subset cl(U) \subset f^{-1}(V)$ [24, Theorem 2.1]. Since f is almost α-open and almost αg-closed, $f(U)$ is α-open and $f(cl(U))$ is αg-closed in Y. Therefore, we obtain $B \subset f(U) \subset \alpha cl(f(U)) \subset \alpha cl(f(cl(U))) \subset V$. Put $G = int(cl(int(f(U))))$. Then G is open in Y and $\alpha cl(f(U)) = cl(int(f(U))) = cl(G)$ by Lemma 4.6. Therefore, we obtain $B \subset f(U) \subset G \subset cl(G) \subset V$. It follows from [24, Theorem 2.1] that Y is almost normal.*

Corollary 4.19 [29]. *Almost normality is preserved under almost open almost g-closed continuous surjections.*

5. Regular spaces

In this section, we improve preservation theorems of regularity almost regularity, quasi-regularity.

Definition 5.1. *A space X is said to be*
(a) *almost regular [23] if for each $F \in RC(X)$ and each $x \in X - F$, there exist disjoint open sets U and V of X such that $x \in U$ and $F \subset V$;*
(b) *quasi-regular [21] if for every nonempty open set V of X, there exists a nonempty open set U in X such that $cl(U) \subset V$;*
(c) *strongly s-regular [5] if for any closed set A of X and any point $x \in X - A$ there exists an $F \in RC(X)$ such that $x \in F$ and $F \cap A = \emptyset$.*

It is shown in [5, Theorem 1] that a space X is strongly s-regular if and only if every open set of X is the union of regular closed sets. Strongly s-regular spaces are called $P\sum$-spaces by Wang [28]. Ganster [5] showed that strong s-regularity is strictly weaker than regularity and is independent of almost regularity.

Theorem 5.2 [19]. *The following are equivalent for a space (X, τ) :*
(a) *(X, τ) is regular (resp. almost regular);*
(b) *for each closed (resp. regular closed) set F and each $x \in X - F$, there exist disjoint U, $V \in \tau^{\alpha}$ such that $x \in U$ and $F \subset V$;*
(c) *for each open (resp. regular open) set V and $x \in V$, there exists $U \in \tau^{\alpha}$ such that $x \in U \subset \alpha cl(U) \subset V$.*

Theorem 5.3. *If $f : X \to Y$ is an almost α-open almost αgs-closed continuous surjection and X is a regular space, then Y is regular.*

Proof. *Let y be any point of Y and V any open neighbourhood of y. There exists a point $x \in X$ with $f(x) = y$. Since X is regular and f is continuous,* *there exists an open set U of X such that $x \in U \subset cl(U) \subset f^{-1}(V)$. Therefore, we have $y \in f(U) \subset f(int(cl(U))) \subset f(cl(U)) \subset V$ and $f(int(cl(U)))$ is α-open because $int(cl(U)) \in RO(X)$ and f is almost α-open. Since $cl(U) \in RC(X)$ and f is almost αgs-closed, $f(cl(U))$ is αgs-closed and hence $y \in f(int(cl(U))) \subset \alpha cl(f(int(cl(U)))) \subset \alpha cl(f(cl(U))) \subset V$. It follows from Theorem 5.2 that Y is regular.*

Corollary 5.4 [**19**]. *Regularity is preserved under almost α-open almost αg-closed continuous surjections.*

Theorem 5.5. *If $f : X \to Y$ is an almost α-open almost αg-closed almost continuous surjection and X is an almost regular space, then Y is almost regular.*

Proof. *Let y be any point of Y and $V \in RO(Y)$ containing y. Since f is almost α-open almost continuous, $f^{-1}(V) \in RO(Y)$ by Lemma 4.17. Take a point $x \in f^{-1}(y)$. Since X is almost regular, there exists $U \in RO(X)$ such that $x \in U \subset cl(U) \subset f^{-1}(V)$ [23, Theorem 2.2]. Hence $y \in f(U) \subset f(cl(U)) \subset V$. Since f is almost α-open almost αg-closed, $f(U)$ is α-open in Y and $f(cl(U))$ is αg-closed in Y and hence we have $y \in f(U) \subset \alpha cl(f(U)) \subset \alpha cl(f(cl(U))) \subset V$. It follows from Theorem 5.2 that Y is almost regular.*

Definition 5.6. *A function $f : X \to Y$ is said to be*
(a) feebly continuous [4] if $int(f^{-1}(V)) \neq \emptyset$ for every nonempty open set V of Y;
(b) feebly open [4] if $int(f(U)) \neq \emptyset$ for every nonempty open set U of X;
(c) almost feebly open [19] if $int(f(U)) \neq \emptyset$ for every nonempty $U \in RO(X)$.

Remark 5.7 [**19**]. *It is obvious that every feebly open function is almost feebly open. However, the converse is not true in general as the following example shows.*

Example 5.8 [**19**]. *Let $X = Y = \{a, b, c\}$, $\tau = \{\emptyset, \{a\}, \{b\}, \{a, b\}, \{a, c\}, X\}$ and $\sigma = \{\emptyset, \{a\}, \{b\}, \{a, b\}, Y\}$. Let $f : (X, \tau) \to (Y, \sigma)$ be a function defined as follows : $f(a) = c$, $f(b) = a$ and $f(c) = b$. Then f is almost feebly open but it is not feebly open since we have $RO(X, \tau) = \{\emptyset, \{b\}, \{a, c\}, X\}$ and $int(f(\{a\})) = \emptyset$.*

Theorem 5.9. *If $f : X \to Y$ is an almost feebly open feebly continuous almost αgs-closed surjection and X is a quasi-regular space, then Y is quasi-regular.*

Proof. *Let V be any nonempty open set of Y. Since f is feebly continuous, $int(f^{-1}(V)) \neq \emptyset$ and by the quasi-regularity of X there exists a nonempty open set U of X such that $U \subset cl(U) \subset int(f^{-1}(V))$. We have $f(int(cl(U))) \subset f(cl(U)) \subset V$. Since f is almost feebly open, $int(f(int(cl(U)))) \neq \emptyset$. Since f is almost αgs-closed, $f(cl(U))$ is αgs-closed and hence $\alpha cl(f(cl$*

$(U))) \subset V$. *Now, put* $G = int(f(int(cl(U))))$, *then by Lemma* 4.6 *we obtain* $\emptyset \neq G \subset cl(G) = \alpha cl(G) \subset \alpha cl(f(cl(U))) \subset V$. *This shows that* Y *is quasi-regular.*

Corollary 5.10 [**8**]. *Quasi regularity is preserved under feebly open feebly continuous closed surjections.*

We shall conclude the section with a preservation theorem of strongly s-regular spaces.

Theorem 5.11. *If* $f : X \to Y$ *is an almost* α*-open almost* αgs*-closed continuous surjection and* X *is a strongly s-regular space, then* Y *is strongly s-regular.*

Proof. *Let* V *be any open set of* Y *and* y *any point of* V. *Since* f *is continuous,* $f^{-1}(V)$ *is open in* X. *For a point* $x \in f^{-1}(y)$, *there exists* $F \in RC(X)$ *such that* $x \in F \subset f^{-1}(V)$; *hence* $y = f(x) \in f(F) \subset V$. *Since* f *is continuous, we have* $f(F) = f(cl(int(F))) \subset cl(f(int(F)))$. *Since* f *is almost* αgs*-closed,* $f(F)$ *is* αgs*-closed and* $\alpha cl(f(F)) \subset V$. *Moreover,* f *is almost* α*-open,* $f(int(F))$ *is* α*-open in* Y *and by Lemma* 4.6 *we have* $cl(f(int(F))) = cl(int(f(int(F)))) = \alpha cl(f(int(F))) \subset \alpha cl(f(F))$. *Therefore, we obtain* $cl(int(f(int(F)))) \in RC(Y)$ *and* $y \in f(F) \subset cl(f(int(F))) = cl(int(f(int(F))))$ $\subset \alpha cl(f(F)) \subset V$. *It follows from* [5, *Theorem* 1] *that* Y *is strongly s-regular.*

References

1. *D. Andrijevic, Some properties of the topology of* α*-sets, Mat. Vesnik,* 36(1984), 1-10.
2. *D. Carnahan, Some properties related to compactness in topological spaces, Ph.D. Thesis, Univ. of Arkansas,* 1973.
3. *R. Devi, K. Balachandran and H. Maki, On generalized* α*-continuous maps and* α*-generalized continuous maps, Far East J. Math. Sci., Special Volume* (1997), *Part I,* 1-15.
4. *Z. Frolik, Remarks concerning the invariance of Baire spaces under mapping, Czechoslovak Math. J.,* 11(86)(1961), 381-385.
5. *M. Ganster, On strongly s-regular spaces, Glasnik Mat.,* 25(45)(1990), 195-201.
6. *S. Greenwood and I. L. Reilly, On feebly closed mappings, Indian J. Pure Appl. Math.,* 17(1986), 1101-1105.
7. *S. Jafari, T. Noiri, N. Rajesh and M. L. Thivagar, Another generalization of closed sets. Kochi J. Math. Vol* 3(2008), 25-38.
8. *D. S. Jankovic and Ch. Konstadilaki-Savvopoulou, On* α*-continuous functions, Math., Bohemica,* 117(1992), 259-270.
9. *N. Levine, Generalized closed sets in topology, Rend. Circ. Mat. Palermo* (2), 19(1970), 89-96.
10. *N. Levine, Semi-open sets and semi-continuity in topological spaces, Amer. Math. Monthly* 70(1963), 36-41.
11. *P. E. Long and L. L. Herrington, Basic properties of regular-closed functions, Rend Circ. Mat. Palermo* (2), 27(1978), 20-28.
12. *H. Maki, R. Devi and K. Balachandran, Generalized* α*-closed sets in topology, Bull. Fukuoka Univ. Ed. III,* 42(1993), 13-21.

13. *H. Maki, R. Devi and K. Balachandran, Associated topologies of generalized α-closed sets and α-generalized closed sets, Mem. Fac. Sci. Kochi Univ. Math.*, 15(1994), 51-63.
14. *S. R. Malghan, Generalized closed maps, J. Karnatak Univ. Sci.*, 27 (1982), 82-88.
15. *A. S. Mashhour, I. A. Hasanein and S. N. El-Deeb, α-continuous and α-open mappings, Acta Math. Hungar.*, 41(1983), 213-218.
16. *O. Njastad, On some classes of nearly open sets, Pacific J. Math.*, 15(1965), 961-970.
17. *T. Noiri, Almost - closed images of countably paracompact spaces, Prace Mat.*, 20(1978), 423-426.
18. *T. Noiri, Mildly normal spaces and some functions, Kyungpook Math. J.*, 36(1996), 183-190.
19. *T. Noiri, Almost αg-closed functions and separation axioms, Acta Math. Hungar*, 82(3)(1999), 193-205.
20. *N. Palaniappan and K. C. Rao, Regular generalized closed sets, Kyungpook Math. J.*, 33(1993), 211-219.
21. *J. Porter and R. G. Woods, Extensions and Absolute of Hausdorff spaces, Springer Verlag*, 1988.
22. *M. Rajamani and K. Viswanathan, On αgs-closed sets in topological spaces, Acta Ciencia Indica, XXXM* (3)(2004), 21-25.
23. *M. K. Singal and S. P. Arya, On almost-regular spaces, Glasnik Mat.*, 4(24)(1969), 89-99.
24. *M. K. Singal and S. P. Arya, Almost normal and almost completely regular spaces, Glasnik Mat.*, 5(25)(1970), 141-152.
25. *M. K. Singal and A. R. Singal, Almost-continuous mappings, Yokohama Math. J.*, 16(1968), 63-73.
26. *M. K. Singal and A. R. Singal, Mildly normal spaces, Kyungpook Math. J.*, 13(1973), 27-31.
27. *M. K. R. S. Veera Kumar, $\hat{g}$-closed sets in Topological spaces, Bull. Allahabad Math. Soc.* 18(2003), 99-112.
28. *Guo Jun Wang, On S-closed spaces, Acta Math. Sinica*, 24(1981), 55- 63.
29. *M. Yoshimura, T. Miwa and T. Noiri, A generalization of regular closed and g-closed functions, Stud. Cerc. Mat.*, 47(1995), 353-358.
30. *P. Zenor, On countable paracompactness and normality, Prace Mat.*, 13(1969), 23-32.

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