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SOME GLOBAL RESULTS ON HOLOMORPHIC LAGRANGIAN FIBRATIONS

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Abstract. The globalization of some local structures as the complex Liouville vector field, complex Liouville 1-form, totally singular complex Hamiltonians and complex nonlinear connection on holomorphic Lagrangian fibrations is studied. Also, we give a new characterization of equivalence of two holomorphic Lagrangian foliations. The notions are introduced here by analogy with the real case, see [16, 17, 18].

1. Introduction

In the smooth category the cohomological obstructions for the globalization of some local structures as Liouville vector fields or locally Lagrangians on Lagrangian foliations was intensively studied in [16, 17]. Also, in [10] are given some extensions of this results on affine complex foliated manifolds endowed with a complex tangent structure.

The aim of this paper is to obtain similar results in the complex-analytic category for some local structures on holomorphic Lagrangian fibrations. Firstly, following [5, 6], we recall the notion of holomorphic symplectic fibrations, we present some examples and by analogy with the real case [11], we consider affine holomorphic symplectic fibrations. In the second section, with respect to the natural holomorphic vertical foliation, we find the cohomological obstructions for the globalization of the complex Liouville vector field and of the totally singular complex Hamiltonians defined on a local chart and we give a new characterization of equivalence of two holomorphic Lagrangian foliations. We also consider transversal distributions and we find cohomological obstructions for the globalization of a complex nonlinear connection and for the existence of an affine transversal distribution.

We work in the category of complex analytic sets. A complex manifold $\mathcal{M}$ of dimension $2n$ equipped with a holomorphic symplectic form $\omega \in H^0(\mathcal{M}, \Omega^2(\mathcal{M}))$ is called a *holomorphic symplectic manifold*. A submanifold $\mathcal{V}$ of $\mathcal{M}$ is said to be *Lagrangian* if $\mathcal{V}$ has dimension n and the restriction of ω on the smooth part

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of $\mathcal{V}$ is identically zero. Let $(\mathcal{M},\omega)$ be a holomorphic symplectic manifold and M a complex manifold of dimension n. A proper surjective holomorphic map $\pi:(\mathcal{M},\omega)\to M$ is a *holomorphic symplectic fibration.* If the underlying submanifold of every fiber of π is Lagrangian, then $\pi:(\mathcal{M},\omega)\to M$ is called a *holomorphic Lagragian fibration.*

Example 1.1. *Let us consider the morphism $\pi:\mathbb{C}^4\to\mathbb{C}^2$ which is defined by $(x,y,z,w)\to(xy,y)$. If we define a symplectic form on $\mathbb{C}^4$ by $dx\wedge dz+dy\wedge dw$, then π is a holomorphic symplectic fibration.*

Example 1.2. *Let $\mathcal{M}:=A\times\mathbb{C}^3$, where A is a three-dimensional torus. We define the action of $\mathbb{Z}_2$ on $\mathcal{M}$ by*

$$(x,y,z,u,v,w)\to(-x,-y,z+\tau,-u,-v,w),$$

where (x,y,z) are global coordinates of A and τ is a 2-torsion element of A. If we define a holomorphic symplectic form on $\mathcal{M}$ by $dx\wedge du+dy\wedge dv+dz\wedge dw$, then the morphism $\mathcal{M}/\mathbb{Z}_2\to\mathbb{C}^3/\mathbb{Z}_2$ is a holomorphic symplectic fibration.

Example 1.3. *Let M be an arbitrary n-dimensional complex manifold and $\pi:T^{'*}M\to M$ its holomorphic cotangent bundle. If $(U_\alpha,(z^i))$, $i=1,\ldots,n$ is a local chart on M and $\omega=dz^i\wedge d\zeta_i$ is the holomorphic symplectic 2-form, where (ζ_i) should be regarded as the components of a point $\zeta\in T^{'*}M$ with respect to the canonical base $\{dz^i\}$, then $\pi:T^{'*}M\to M$ is a holomorphic symplectic fibration.*

A morphism of two holomorphic symplectic fibrations $\pi^{'}:\mathcal{M}^{'}\to M^{'}$ and $\pi:\mathcal{M}\to M$ is a couple (f_0,f_1), where $f_0:M^{'}\to M$ and $f_1:\mathcal{M}^{'}\to\mathcal{M}$ are holomorphic such that $\pi\circ f_1=f_0\circ\pi^{'}$, i.e. f_1 sends fibers to fibers; we also say that f_1 is a f_0-morphism of holomorphic symplectic fibrations.

2. Affine holomorphic symplectic fibrations

Let $(U_\alpha,\varphi_\alpha)$ be a local chart on M with the complex coordinates (z^k), $k=1,\ldots,n$ and (V_α,ψ_α) be a local chart on $\mathcal{M}$ with the complex coordinates $u=(z^k,\zeta_k)$, $k=1,\ldots,n$ such that $\pi(V_\alpha)=U_\alpha$.

Definition 2.1. *A holomorphic symplectic fibration $\pi:\mathcal{M}\to M$ is said to be affine if at local change maps $(V_\alpha,\psi_\alpha)\to(V_\beta,\psi_\beta)$ on $\mathcal{M}$, the change rules of the local complex coordinates have the form*

$$z^{'j}=z^{'j}(z^i)\,,\,\zeta^{'}_j=\frac{\partial z^i}{\partial z^{'j}}\zeta_i+\varphi^{'}_j(z^i), \tag{2.1}$$

where $z^{'j}$ and $\varphi^{'}_j$ are holomorphic functions on (z^i) variables and $\det(\frac{\partial z^{'j}}{\partial z^i})\neq 0$.

Definition 2.2. *An affine local section in the affine holomorphic symplectic fibration $\pi:\mathcal{M}\to M$ is a holomorphic map $s:U_\alpha\to\mathcal{M}$ such that $\pi\circ s=\mathrm{Id}|_{U_\alpha}$ and its local components change according to the rule*

$$s^{'}_j(z^{'})=\frac{\partial z^i}{\partial z^{'j}}s_i(z)+\varphi^{'}_j(z). \tag{2.2}$$

The set of all affine sections of $\mathcal{M}$ is denoted by $\Gamma(\mathcal{M})$ and it is an affine module over $\mathcal{F}(M)$, i.e. for every $f_1,\ldots,f_p\in\mathcal{F}(M)$ such that $f_1+\ldots+f_p=1$ and $s^1,\ldots,s^p\in\Gamma(\mathcal{M})$ then $f_1s^1+\ldots+f_ps^p\in\Gamma(\mathcal{M})$, where the affine combination is taken at every point $z\in M$. We notice that a partition of the unity can be smooth, but not holomorphic, see for instance [13] p. 7. Thus, using a smooth partition of the unity on the base M it can easily proved that every affine holomorphic symplectic fibration allows an affine section.

Let $\ker \pi_* := V^{'}\mathcal{M} \to \mathcal{M}$ be the holomorphic vertical bundle of $\mathcal{M}$ and $\Gamma(V^{'}\mathcal{M})$ be the module of its sections. The local complex coordinates on $V^{'}\mathcal{M}$ have the form (z^i, ζ_i, η_i) and the change rule of these coordinates are given by

$$z^{'j} = z^{'j}(z)\,,\ \zeta^{'}_j = \frac{\partial z^i}{\partial z^{'j}}\zeta_i + \varphi^{'}_j(z)\,,\ \eta^{'}_j = \frac{\partial z^i}{\partial z^{'j}}\eta_i. \tag{2.3}$$

Definition 2.3. *A Liouville type section is a vertical section $S \in \Gamma(V^{'}\mathcal{M})$ which has the local form*

$$S_i(z,\zeta) = \zeta_i + C_i(z). \tag{2.4}$$

Proposition 2.1. *Every Liouville type section in $\Gamma(V^{'}\mathcal{M})$ defines an affine section in $\Gamma(\mathcal{M})$ and conversely.*

Proof. Using (2.3) at local charts change, we have $S^{'}_j = \frac{\partial z^i}{\partial z^{'j}}S_i$. Now, taking into account the local forms of $S^{'}_j$ and S_i from (2.4), it follows $\zeta^{'}_j + C^{'}_j = \frac{\partial z^i}{\partial z^{'j}}(\zeta_i + C_i)$. Using (2.1), it follows

$$C^{'}_j = \frac{\partial z^i}{\partial z^{'j}}C_i - \varphi^{'}_j.$$

Thus, the local functions $\{-C_i(z)\}$ are the local components of a global affine section in $\Gamma(\mathcal{M})$. Conversely, for a global affine section $s \in \Gamma(\mathcal{M})$ having the local components $s_i(z)$, the local functions $\zeta_i - s_i$ on $\mathcal{M}$ verify the change rule (2.4). □

Note that inside of (2.1) we can take into account the particular case $\varphi^{'}_j = 0$, when $\mathcal{M}$ is identified with the holomorphic cotangent bundle of a complex manifold M, namely $\mathcal{M} = T^{'*}M$. Fore more details about the geometry of $T^{'*}M$ complex manifold see the Ch. VI from [8].

Throughout this paper we consider $\pi : (\mathcal{M}, \omega) \to M$ to be an affine holomorphic symplectic fibration.

Let us consider $\mathcal{V}$ be the leafs of the vertical foliation (the foliation by fibers), characterized by $z^i = const.$ and $\omega = dz^i \wedge d\zeta_i$ the holomorphic symplectic $(2,0)$-form on $\mathcal{M}$. Then $\omega_{|\mathcal{V}} = 0$ and in this case we call $(\mathcal{M}, \omega, \mathcal{V})$ a *holomorphic Lagrangian foliation*.

Let J be the natural complex structure of the manifold $\mathcal{M}$ and $T^{'}\mathcal{M}$ and $T^{''}\mathcal{M} = \overline{T^{'}\mathcal{M}}$ be its holomorphic and antiholomorphic subbundles, respectively. By $T_{\mathbb{C}}\mathcal{M} = T^{'}\mathcal{M} \oplus T^{''}\mathcal{M}$ we denote the complexified tangent bundle of the real tangent bundle $T_{\mathbb{R}}\mathcal{M}$. From (2.1) it results the following changes for the natural local frames of $T^{'}_u\mathcal{M}$

$$\frac{\partial}{\partial z^i} = \frac{\partial z^{'j}}{\partial z^i}\frac{\partial}{\partial z^{'j}} + (\frac{\partial z^{'k}}{\partial z^i}\frac{\partial^2 z^h}{\partial z^{'k}\partial z^{'j}}\zeta_h + \frac{\partial \varphi^{'}_j}{\partial z^i})\frac{\partial}{\partial \zeta^{'}_j}\,;\ \frac{\partial}{\partial \zeta_i} = \frac{\partial z^i}{\partial z^{'j}}\frac{\partial}{\partial \zeta^{'}_j}. \tag{2.5}$$

By conjugation over all in (2.5) we obtain the change rules of the natural local frames on $T^{''}_u\mathcal{M}$, and then, the behaviour of the J complex structures is

$$J(\frac{\partial}{\partial z^k}) = i\frac{\partial}{\partial z^k}\,;\ J(\frac{\partial}{\partial \overline{z}^k}) = -i\frac{\partial}{\partial \overline{z}^k}\,;\ J(\frac{\partial}{\partial \zeta_k}) = i\frac{\partial}{\partial \zeta_k}\,;\ J(\frac{\partial}{\partial \overline{\zeta}_k}) = -i\frac{\partial}{\partial \overline{\zeta}_k}. \tag{2.6}$$

The natural dual bases on $T^{'*}_u\mathcal{M}$ change according to the rule

$$dz^{'j}=\frac{\partial z^{'j}}{\partial z^k}dz^k\ ;\ d\zeta^{'}_j=(\frac{\partial z^{'k}}{\partial z^i}\frac{\partial^2 z^h}{\partial z^{'k}\partial z^{'j}}\zeta_h+\frac{\partial\varphi^{'}_j}{\partial z^i})dz^i+\frac{\partial z^i}{\partial z^{'j}}d\zeta_i \tag{2.7}$$

and by conjugation we obtain the change rules of the natural dual bases on $T^{''*}_u\mathcal{M}$.

Thus, the coordinates of the vectors $Z=Z^i\frac{\partial}{\partial z^i}+Z_i\frac{\partial}{\partial\zeta_i}\in T^{'}\mathcal{M}$ have the following change rules

$$Z^{'j}=\frac{\partial z^{'j}}{\partial z^i}Z^i\ ;\ Z^{'}_j=(\frac{\partial z^{'k}}{\partial z^i}\frac{\partial^2 z^h}{\partial z^{'k}\partial z^{'j}}\zeta_h+\frac{\partial\varphi^{'}_j}{\partial z^i})Z^i+\frac{\partial z^i}{\partial z^{'j}}Z_i, \tag{2.8}$$

and the coordinates of the co-vectors $\theta=\theta_i dz^i+\theta^i d\zeta_i\in T^{'*}\mathcal{M}$ change according to the rules

$$\theta^{'}_j=\frac{\partial z^i}{\partial z^{'j}}\theta_i+(\frac{\partial z^{'k}}{\partial z^i}\frac{\partial^2 z^h}{\partial z^{'k}\partial z^{'j}}\zeta_h+\frac{\partial\varphi^{'}_j}{\partial z^i})\theta^i\ ;\ \theta^{'j}=\frac{\partial z^{'j}}{\partial z^i}\theta^i. \tag{2.9}$$

By conjugation over all in (2.8) and (2.9) we get the change rules of the coordinates of the vectors from $T^{''}\mathcal{M}$ and of the co-vectors from $T^{''*}\mathcal{M}$, respectively.

3. Global results

In this section, by analogy with the real case, [16], [17], [18], we find some cohomological obstructions for the globalization of the complex Liouville vector field, the complex Liouville 1-form, the totally singular complex Hamiltonians and of a complex nonlinear connection on $\mathcal{M}$. Also, using the relative cohomology, we give in the second subsection a new characterization of equivalence of two holomorphic Lagrangian foliations.

3.1. **Complex Liouville vector field.** The tangent vectors of the leaves $\mathcal{V}$ define the structural subbundle $T^{'}\mathcal{V}$ of $T^{'}\mathcal{M}$ with local bases $\{\frac{\partial}{\partial\zeta_i}\}$ and with the transition functions $(\frac{\partial z^i}{\partial z^{'j}})$ called *vertical distribution* which in view of (2.5) is an integrable and holomorphic one.

For the holomorphic vertical foliation $\mathcal{V}$, we denote by $\Omega^0_{pr}(\mathcal{M})$ the sheaf of germs of holomorphic projectable (foliated) functions on $\mathcal{M}$ and by $\mathcal{A}^0_{pr}(\mathcal{M},\mathcal{V})$ the sheaf of germs of *affine leafwise holomorphic vertical functions*, locally given by

$$f=a^i(z)\zeta_i+b(z), \tag{3.1}$$

where a^i , $b\in\Omega^0_{pr}(\mathcal{M})$.

For the sheaf of corresponding germs we have the following exact sequence

$$0\to\Omega^0_{pr}(\mathcal{M})\stackrel{i}{\to}\mathcal{A}^0_{pr}(\mathcal{M},\mathcal{V})\stackrel{p^{'}}{\to}\Omega^0_{pr}(\mathcal{M})\otimes T^{'*}\mathcal{V}\to 0, \tag{3.2}$$

explicitly given by $b\stackrel{i}{\to}a^i\zeta_i+b\stackrel{p^{'}}{\to}a^i d\zeta_i$.

Now, let us consider the complex Liouville vector field on $\mathcal{M}$, locally given in the chart (V_α,ψ_α) by

$$\Gamma_\alpha=\zeta_i\frac{\partial}{\partial\zeta_i}. \tag{3.3}$$

Then, on the intersection $V_\alpha\cap V_\beta\neq\phi$, by (2.1) and (2.5) we have

$$\Gamma_\beta-\Gamma_\alpha=\zeta^{'}_j\frac{\partial}{\partial\zeta^{'}_j}-\zeta_i\frac{\partial}{\partial\zeta_i}=\varphi^{'}_j(z)\frac{\partial}{\partial\zeta^{'}_j} \tag{3.4}$$

and we see that the right-hand side of (3.4) defines a vertical complex vector field with coefficients in $\Omega^0_{pr}(\mathcal{M})$. Thus, the difference $\Gamma_{\alpha\beta} = \Gamma_\beta - \Gamma_\alpha$ yields a cocycle $(\delta\Gamma)_{\alpha\beta\gamma} = \Gamma_{\beta\gamma} - \Gamma_{\alpha\gamma} + \Gamma_{\alpha\beta} = 0$. This cocycle defines a Cèch cohomology class

$$[\Gamma_\alpha] \in H^1(\mathcal{M}, \Omega^0_{pr}(\mathcal{M}) \otimes T^{'}\mathcal{V}), \tag{3.5}$$

which will be called *complex linear obstruction* of $\mathcal{V}$, and its vanish leads to Γ_α is globally defined. By the same considerations as in [17], we have

Proposition 3.1. *The affine holomorphic symplectic fibration $\pi : (\mathcal{M}, \omega) \to M$ is equivalent to the holomorphic cotangent bundle of M, if and only if $[\Gamma_\alpha] = 0$.*

Proof. The necessity is obvious. Conversely, if $[\Gamma_\alpha] = 0$ then, there is an adapted atlas where

$$\varphi^{'}_j \frac{\partial}{\partial \zeta^{'}_j} = \psi^{'}_j(z)\frac{\partial}{\partial \zeta^{'}_j} - \psi_i(z)\frac{\partial}{\partial \zeta_i} \tag{3.6}$$

with ψ_i holomorphic functions on z. Then, in the new coordinates $\widetilde{z}^i = z^i$; $\widetilde{\zeta}_i = \zeta_i - \psi_i$ we obtain $\varphi^{'}_j(\widetilde{z}) = 0$. □

We notice that we can make the same considerations for the globalization of the complex Liouville 1-form locally defined by $\tau_\alpha = \zeta_i dz^i$.

3.2. **Equivalence of holomorphic Lagrangian foliations.** Let $(\mathcal{M}_a, \omega_a, \mathcal{V}_a)$, $a = 1, 2$, be two general holomorphic Lagrangian foliations and $F : (\mathcal{M}_1, \omega_1, \mathcal{V}_1) \to (\mathcal{M}_2, \omega_2, \mathcal{V}_2)$ be a morphism of the holomorphic symplectic fibrations $\pi_1 : \mathcal{M}_1 \to M_1$ and $\pi_2 : \mathcal{M}_2 \to M_2$. Thus, F sends every leaf V_1 of $\mathcal{V}_1$ into a leaf V_2 of $\mathcal{V}_2$ such that the restriction map $F : V_1 \to V_2$ is holomorphic.

Definition 3.1. *The holomorphic Lagrangian foliations $(\mathcal{M}_a, \omega_a, \mathcal{V}_a)$, $a = 1, 2$, are equivalent if*

$$F^*\mathcal{V}_2 = \mathcal{V}_1 \text{ and } F^*\omega_2 = \omega_1, \tag{3.7}$$

where $F^*\mathcal{V}_2$ is the holomorphic foliation of $\mathcal{M}_1$ whose leaves are inverse images under F of leaves of $\mathcal{V}_2$.

In the real case it is known a characterization of equivalence of two Lagrangian foliations, see Theorem 2.2. from [17].

Here, we purpose a new approach of equivalence of two general (holomorphic) Lagrangian foliations, using the relative cohomology introduced in [1] p. 78.

Let us consider $\Omega^p(\mathcal{M}_1)$ and $\Omega^p(\mathcal{M}_2)$ be the sets of all differential p-forms on $\mathcal{M}_1$ and $\mathcal{M}_2$, respectively. Define the differential complex

$$0 \longrightarrow \Omega^0(F) \xrightarrow{\widetilde{d}} \Omega^1(F) \xrightarrow{\widetilde{d}} \ldots,$$

where $\Omega^p(F) = \Omega^p(\mathcal{M}_2) \oplus \Omega^{p-1}(\mathcal{M}_1)$ and

$$\widetilde{d}(\varphi, \psi) = (d_2\varphi, F^*\varphi - d_1\psi) \text{ for any } \varphi \in \Omega^p(\mathcal{M}_2) \text{ and } \psi \in \Omega^{p-1}(\mathcal{M}_1), \tag{3.8}$$

where d_1 and d_2 denotes the exterior derivatives on $\mathcal{M}_1$ and $\mathcal{M}_2$, respectively.

Taking into acount $d_1^2 = d_2^2 = 0$ and the known relation $d_1 F^* = F^* d_2$, it is easy to see that $\widetilde{d}^2 = 0$. Denote the cohomology groups of this complex by $H^p(F)$ which are called the *relative de Rham cohomology groups* associated to the map F. We notice that the operator $\widetilde{d}$ satisfy a Poincaré type Lemma, easily obtained by using the classical Poincaré Lemma for the operators d_1 and d_2.

Now we consider that $\mathcal{M}_1$ and $\mathcal{M}_2$ satisfy Proposition 3.1. In this case the complex Liouville 1-forms locally given by $\tau_1^\alpha = \zeta_i^1 dz_1^i$ and $\tau_2^\alpha = \zeta_i^2 dz_2^i$ glue up to some global 1-forms τ_1 and τ_2, respectively.

For $(\omega_2, \tau_1) \in \Omega^2(F)$ it results

$$\widetilde{d}(\omega_2, \tau_1) = (d_2\omega_2, F^*\omega_2 - d_1\tau_1) = (0, F^*\omega_2 - \omega_1). \tag{3.9}$$

We have

Theorem 3.2. *If $(\mathcal{M}_a, \omega_a, \mathcal{V}_a)$, $a = 1, 2$, are equivalent to the holomorphic cotangent bundles of M_1 and M_2, respectively, then $F : (\mathcal{M}_1, \omega_1, \mathcal{V}_1) \to (\mathcal{M}_2, \omega_2, \mathcal{V}_2)$ is a foliation equivalence iff $\widetilde{d}(\omega_2, \tau_1) = (0, 0)$. In this case the relative cohomology class $[(\omega_2, \tau_1)] \in H^2(F)$ will be called obstruction to the existence of a equivalence of two (holomorphic) Lagrangian foliations.*

Proof. If $F : (\mathcal{M}_1, \omega_1, \mathcal{V}_1) \to (\mathcal{M}_2, \omega_2, \mathcal{V}_2)$ is a foliation equivalence of two (holomorphic) Lagrangian foliations then by (3.9) it follows that $\widetilde{d}(\omega_2, \tau_1) = (0, 0)$. Conversely, from $\widetilde{d}(\omega_2, \tau_1) = (0, 0)$ it follows that there exists $(\varphi, \psi) \in \Omega^1(F)$ such that

$$(\omega_2, \tau_1) = \widetilde{d}(\varphi, \psi) = (d_2\varphi, F^*\varphi - d_1\psi).$$

By the above relation we get $\omega_2 = d_2\varphi$ and by applying the Poincaré Lemma for operator d_2 it follows that there exists a function $\mu \in \mathcal{F}(\mathcal{M}_2)$ such that $\varphi = \tau_2 - d_2\mu$. Now we replace it in the relation $\tau_1 = F^*\varphi - d_1\psi$ and we obtain

$$\tau_1 = F^*(\tau_2 - d_2\mu) - d_1\psi = F^*\tau_2 - d_1(F^*\mu + \psi).$$

Applying d_1 in the above relation it results $\omega_1 = F^*\omega_2$, so $(\mathcal{M}_1, \omega_1, \mathcal{V}_1)$ and $(\mathcal{M}_2, \omega_2, \mathcal{V}_2)$ are equivalent. □

3.3. **Totally singular complex Hamiltonians.** As in the real case [12], we can consider the *totally singular Hamiltonian* notion on the affine holomorphic symplectic fibration $\pi : (\mathcal{M}, \omega) \to M$, that it is a real-valued function $\mathcal{H} : \mathcal{M} \to \mathbb{R}$ which is affine in the fibers coordinates, or equivalently it has a null vertical complex hessian. A such complex Hamiltonian is locally given in the chart (V_α, ψ_α) by

$$\mathcal{H}_\alpha(z, \zeta) = \alpha^i(z)(\zeta_i + \overline{\zeta}_i) + \beta(z), \tag{3.10}$$

where $\alpha = \alpha^i(z) d\zeta_i \in \Gamma(T^{'*}\mathcal{V})$ and $\alpha^i(z)$, $\beta(z) \in \Omega^{\mathbb{R}}_{pr}(\mathcal{M})$, where $\Omega^{\mathbb{R}}_{pr}(\mathcal{M})$ is the sheaf of germs of real-valued projectable functions on $\mathcal{M}$.

If we denote by $\mathcal{A}^{\mathbb{R}}_{pr}(\mathcal{M}, \mathcal{V} \oplus \overline{\mathcal{V}})$ the sheaf of germs of functions locally given by (3.10), then similarly to (3.2) we can construct the following exact sequence

$$0 \to \Omega^{\mathbb{R}}_{pr}(\mathcal{M}) \xrightarrow{i} \mathcal{A}^{\mathbb{R}}_{pr}(\mathcal{M}, \mathcal{V} \oplus \overline{\mathcal{V}}) \xrightarrow{\widetilde{p}} \Omega^{\mathbb{R}}_{pr}(\mathcal{M}) \otimes (T^{'*}\mathcal{V} \oplus T^{''*}\mathcal{V}) \to 0, \tag{3.11}$$

explicitly given by $\beta \xrightarrow{i} \alpha^i(\zeta_i + \overline{\zeta}_i) + \beta \xrightarrow{\widetilde{p}} \alpha^i(d\zeta_i + d\overline{\zeta}_i)$.

Then, on the intersection $V_\alpha \cap V_\beta \neq \phi$, from (2.1) and (2.9) we have

$$\mathcal{H}_{\alpha\beta} := \mathcal{H}_\beta - \mathcal{H}_\alpha = \alpha^{'j}(\varphi'_j + \overline{\varphi'_j}), \tag{3.12}$$

which yields a cocycle $(\delta\mathcal{H})_{\alpha\beta\gamma} = \mathcal{H}_{\beta\gamma} - \mathcal{H}_{\alpha\gamma} + \mathcal{H}_{\alpha\beta} = 0$. This cocycle defines a Cèch cohomology class

$$[\mathcal{H}_\alpha] \in H^1(\mathcal{M}, \Omega^{\mathbb{R}}_{pr}(\mathcal{M})). \tag{3.13}$$

Thus, we obtain

Proposition 3.3. $[\mathcal{H}_\alpha] = 0$ *yields $\mathcal{H}_\alpha$ is globally defined.*

3.4. **Complex nonlinear connections.** Let $N^{'}\mathcal{V} = T^{'}\mathcal{M}/T^{'}\mathcal{V}$ be the normal bundle of $\mathcal{V}$ with the holomorphic projection $p : T^{'}\mathcal{M} \to N^{'}\mathcal{V}$ which has the local bases defined by the equivalence classes $p(\frac{\partial}{\partial z^i}) = [\frac{\partial}{\partial z^i}]$ with the transition functions $(\frac{\partial z^{'j}}{\partial z^i})$.

As in the real case for vector (affine) bundles [2, 7] or the general case of holomorphic foliations [15], we can consider the following exact sequence of holomorphic vector bundles over $\mathcal{M}$

$$0 \to T^{'}\mathcal{V} \xrightarrow{i} T^{'}\mathcal{M} \xrightarrow{p} N^{'}\mathcal{V} \to 0, \tag{3.14}$$

where i and p are the canonical injection and the canonical projection, respectively.

A normalization of the vertical distribution $T^{'}\mathcal{V}$ is a distribution $T^{'}\mathcal{H}$ on $\mathcal{M}$ which is supplementary to $T^{'}\mathcal{V}$ in $T^{'}\mathcal{M}$. The distribution $T^{'}\mathcal{H}$ is called *horizontal distribution* (or complex nonlinear connection on $\mathcal{M}$, briefly c.n.c.). We denote also by $T^{'}\mathcal{H}$ the horizontal subbundle. A such normalization can be defined by a right spliting of the exact sequence (3.14), i.e. by a map $\sigma : N^{'}\mathcal{V} \to T^{'}\mathcal{M}$ which satisfies the conditions that σ is a $\mathcal{M}$-morphism of holomorphic fibrations and $p \circ \sigma = \mathrm{Id}|_{N^{'}\mathcal{V}}$.

Denoting as $T^{'}\mathcal{H} = \sigma(N^{'}\mathcal{V})$, it is a subbundle of $T^{'}\mathcal{M}$ which is supplementary to $T^{'}\mathcal{V}$, thus we obtain a normalization of $T^{'}\mathcal{V}$ with $T^{'}\mathcal{H}$ suitable horizontal bundle. In local coordinates, we can consider

$$\sigma(p(\frac{\partial}{\partial z^i})) = \frac{\partial}{\partial z^i} + N_{ki}\frac{\partial}{\partial \zeta_k} =: \frac{\delta}{\delta z^i}, \tag{3.15}$$

and $\{\frac{\delta}{\delta z^i}\}$ is a local basis of the sections of $T^{'}\mathcal{H}$. The local functions $N_{ki}(z,\zeta)$ on $\mathcal{M}$ are called the coefficients of the c.n.c.

The change rule on $V_\alpha \cap V_\beta \neq 0$ is given by

$$\frac{\delta}{\delta z^i} = \frac{\partial z^{'j}}{\partial z^i}\frac{\delta}{\delta z^{'j}}. \tag{3.16}$$

It follows that the change rule for the coefficients N_{ki} of the c.n.c. is

$$N^{'}_{jk} = \frac{\partial z^i}{\partial z^{'j}}\frac{\partial z^h}{\partial z^{'k}}N_{ih} + \frac{\partial^2 z^i}{\partial z^{'j}\partial z^{'k}}\zeta_i + \frac{\partial z^i}{\partial z^{'k}}\frac{\partial \varphi^{'}_j}{\partial z^i}. \tag{3.17}$$

Conversely, if we assume that on the domain of every local chart (V_α, ψ_α) on $\mathcal{M}$ adapted to the foliated structure on $\mathcal{M}$, the local functions $N_{ki}(z,\zeta)$ are given such that the change rule (3.17) on the intersection of two domains holds, then the map σ given by (3.15) is a normalization of $T^{'}\mathcal{V}$. The normalization σ gives an embedding of $N^{'}\mathcal{V}$ in $T^{'}\mathcal{M}$ and a decomposition of $T^{'}\mathcal{M}$ in the direct sum, $T^{'}\mathcal{M} = T^{'}\mathcal{H} \oplus T^{'}\mathcal{V}$.

By conjugation we obtain a decomposition of the complexified tangent bundle of $\mathcal{M}$, namely $T_{\mathbb{C}}\mathcal{M} = T^{'}\mathcal{H} \oplus T^{'}\mathcal{V} \oplus T^{''}\mathcal{H} \oplus T^{''}\mathcal{V}$ where $T^{''}\mathcal{H} = \mathrm{span}\{\frac{\delta}{\delta \bar{z}^i}\}$ and $T^{''}\mathcal{V} = \mathrm{span}\{\frac{\partial}{\partial \bar{\zeta}_i}\}$. The dual adapted basis are given by $\{dz^i\}$, $\{\delta\zeta_i = d\zeta_i - N_{ik}dz^k\}$, $\{d\bar{z}^i\}$ and $\{\delta\bar{\zeta}_i = d\bar{\zeta}_i - N_{\bar{i}\bar{k}}d\bar{z}^k\}$.

We notice that as in the case of holomorphic cotangent bundle $T^{'*}M$, see [9], a normalization of $T^{'}\mathcal{V}$ can be derived from a *regular complex Hamiltonian* on $\mathcal{M}$, that it is a real valued function $H : \mathcal{M} \to \mathbb{R}$ such that $h^{\bar{j}i} = \partial^2 H/\partial\zeta_i\partial\bar{\zeta}_j$ defines a hermitian metric tensor on the fibers of the vertical bundle. If we denote by $(h_{i\bar{j}})$

the inverse of $(h^{\bar{j}i})$, then by using (2.5), we obtain that the following local functions

$$\overset{CH}{N}_{jk} = -h_{j\overline{m}}\frac{\partial^2 H}{\partial z^k \partial\bar{\zeta}_m} \tag{3.18}$$

verify the change rule (3.17) and we call this normalization *the Chern-Hamilton c.n.c.* on the affine holomorphic symplectic fibration $\pi : (\mathcal{M}, \omega) \to M$.

Now, we can consider the complex tensor field locally given in the chart (V_α, ψ_α) by

$$N_\alpha = N_{jk}\frac{\partial}{\partial\zeta_j} \otimes [dz^k] \in \Gamma(T^{'}\mathcal{V} \otimes N^{'*}\mathcal{V}).$$

Then, on the intersection $V_\alpha \cap V_\beta \neq \phi$, from (2.5), (2.7) and (3.17) we have

$$N_\beta - N_\alpha = \frac{\partial z^{'j}}{\partial z^l}(\frac{\partial z^{'k}}{\partial z^i}\frac{\partial^2 z^h}{\partial z^{'k}\partial z^{'j}}\zeta_h - \frac{\partial\varphi^{'}_j}{\partial z^i})\frac{\partial}{\partial\zeta_l} \otimes [dz^i] \tag{3.19}$$

and we see that the right-hand side of (3.19) defines a complex tensor field with coefficients in $\mathcal{A}^0_{pr}(\mathcal{M}, \mathcal{V})$. Thus, the difference $N_{\alpha\beta} := N_\beta - N_\alpha$ yields a cocycle $(\delta N)_{\alpha\beta\gamma} = N_{\beta\gamma} - N_{\alpha\gamma} + N_{\alpha\beta} = 0$. This cocycle defines a Cèch cohomology class

$$[N_\alpha] \in H^1(\mathcal{M}, \mathcal{A}^0_{pr}(\mathcal{M}, \mathcal{V}) \otimes T^{'}\mathcal{V} \otimes N^{'*}\mathcal{V}), \tag{3.20}$$

which will be called *obstruction to globalization of a c.n.c.*

Proposition 3.4. $[N_\alpha] = 0$ *yields* N_α *is globally defined.*

However, $T^{'}\mathcal{H}$ is smoothly isomorphic to $N^{'}\mathcal{V}$ which is holomorphic as $T^{'}\mathcal{V}$, generally $T^{'}\mathcal{H}$ is not holomorphic subbundle of $T^{'}\mathcal{M}$. The existence of a holomorphic supplementary distribution $T^{'}\mathcal{H}$ is characterized in the general case of holomorphic foliations, see [15], by the vanishing of a certain cohomological obstruction, as follows:

By the change rule (3.17), the following 1-form

$$\Phi_{ki} = d^{''}N_{ki} \tag{3.21}$$

defines a global 1 -form Φ on $\mathcal{M}$ with values in $\mathrm{Hom}(N^{'}\mathcal{V}, T^{'}\mathcal{V})$, which is $d^{''}$-closed, hence it gives a cohomological class $[\Phi] \in H^1(\mathcal{M}, \mathrm{Hom}(N^{'}\mathcal{V}, T^{'}\mathcal{V}))$, (in view of the Dolbeault-Serre theorem). Thus, we have

Theorem 3.5. *(*[15]*) The foliation* $\mathcal{V}$ *admits a supplementary holomorphic distribution if and only if* $[\Phi] = 0$.

Finally, following [16, 17], we give the main obstructions for the existence of an affine transversal distribution of the holomorphic vertical foliation $\mathcal{V}$.

Definition 3.2. *We say that* $T^{'}\mathcal{H}$ *is an affine transversal distribution of* $T^{'}\mathcal{V}$ *if the local functions* N_{jk} *are locally given by*

$$N_{jk}(z, \zeta) = \Gamma^i_{jk}(z)\zeta_i + \beta_{jk}(z), \tag{3.22}$$

where $\Gamma^i_{jk}(z)$ *and* $\beta_{jk}(z)$ *are projectable functions on* $\mathcal{M}$, *not necessarily holomorphic.*

The relations (3.17) and (2.1) show that $\theta = d_{\mathcal{V}}(\partial N_{jk}/\partial\zeta_i)$ glue up to a global $d_{\mathcal{V}}$-closed form which yields a cohomology class

$$[\theta] \in H^1(\mathcal{M}, \underline{T^{'}\mathcal{V} \otimes T^{'*}\mathcal{V} \otimes T^{'*}\mathcal{H}}), \tag{3.23}$$

where $\underline{E}$ denotes the sheaf of germs of foliated sections of a foliated complex bundle E and $d_{\mathcal{V}}$ is the exterior derivative along the leaves of foliation $\mathcal{V}$.

By the same considerations as in the real case [16], we notice that $[\theta]$ does not depend on the choice of the affine transversal distribution from (3.22). Indeed, if we chose another affine transversal distribution $T^{'}\widetilde{\mathcal{H}}$ with the local coefficients $\widetilde{N}_{jk}$, then $P_{jk} = \widetilde{N}_{jk} - N_{jk}$ defines a global section of $T^{'}\mathcal{V} \otimes T^{'*}\mathcal{H}$. Clearly, if an affine transversal distribution exists, then $[\theta] = 0$. Conversely, if $[\theta] = 0$, we have

$$d_{\mathcal{V}}(\partial N_{jk}/\partial \zeta_i) = -d_{\mathcal{V}}(\gamma^i_{jk})\ ;\ \gamma^i_{jk} \in \Gamma(\underline{T^{'}\mathcal{V} \otimes T^{'*}\mathcal{V} \otimes T^{'*}\mathcal{H}}). \tag{3.24}$$

The local forms $\gamma^i_{jk}\delta\zeta_i$ are $d_{\mathcal{V}}$-closed, and provide some

$$[\gamma] \in H^1(\mathcal{M}, \underline{T^{'}\mathcal{V} \otimes T^{'*}\mathcal{H}}), \tag{3.25}$$

which does not depend on the choice of γ^i_{jk}. Finally, if $[\gamma] = 0$, we shall obtain $P_{jk} \in \Gamma(T^{'}\mathcal{V} \otimes T^{'*}\mathcal{H})$ such that $\gamma^i_{jk} = \partial P_{jk}/\partial \zeta_i$, and

$$\widetilde{\delta}\zeta_i = d\zeta_i - (N_{ik} + P_{ik})dz^k = 0$$

defines an affine transversal distribution $T^{'}\widetilde{\mathcal{H}}$. Hence, we have

Proposition 3.6. *The holomorphic vertical distribution $T^{'}\mathcal{V}$ has an affine transversal distribution if and only if $[\theta] = 0$ and $[\gamma] = 0$.*

Example 3.1. *If $\mathcal{M} = T^{'*}M$ is the holomorphic cotangent bundle of a complex Cartan space (M, C) where the complex Cartan structure C is purely hermitian* [8]*, with the fundamental metric tensor $h^{\bar{j}i}(z) = \partial^2 C^2/\partial\zeta_i\partial\bar{\zeta}_j$ and $\Gamma^i_{jk}(z) = -h_{j\bar{l}}\partial h^{\bar{l}i}/\partial z^k$ the local coefficients of the Chern-Cartan linear connection on $T^{'*}M$, then the Chern-Cartan c.n.c. $\overset{CC}{N}_{jk} = \Gamma^i_{jk}(z)\zeta_i = 0$ defines an affine transversal distribution on $T^{'*}M$.*

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