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PROPER EXTENSIONS OF A CERTAIN CLASS OF CARLEMAN OPERATORS

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Abstract. In this paper, we discribe all proper (in particular, selfadjoint and dissipative) extensions of a certain type of Carleman operators defined in the Hilbert space $L^2(\Omega,\mu)$. For such operators, general forms of proper extensions are given and their spectral properties investigated. Generalized resolvents are also given by means of Nevanlinna families and associated Nevanlinna pairs.

1. Introduction

In this paper, we consider in the Hilbert space $L^2(\Omega,\mu)$ Carleman operators with kernels of the form;

$$K(x,y) =_{p=0}^{+\infty} a_p \Psi_p(x) \overline{\Psi_p(y)}, \tag{1.1}$$

where $\{\Psi_p\}_{p=0}^{+\infty}$ and $\{a_p\}_{p=0}^{+\infty}$ are respectively an orthonormal sequence in $L^2(\Omega,\mu)$ and a real number sequence verifying some convergence conditions. Such Carleman operators have been studied in [2, 3], . In [2], necessary and suffisent conditions are given in order to have equal deficiency indices. In [3], quasi-selfadjoint extensions (see also [1, 17]) of operator A are investigated in the case when deficiency indices equal 1. Furthermore, general forms of corresponding generalized resolvents and generalized spectral functions are given by means of the Stieljes inversion formula.

The principal aim of this present paper is to explore the class of proper extensions of A. That is, extensions $\widetilde{A}$ satisfying the condition $A \subset \widetilde{A} \subset A^*$. Other than quasi-selfadjoint extensions, it contains also symmetric, selfadjoint and dissipative extensions of A. Our goal is to describe all elements of this class and characterize their spectral properties. But in stead of the classical approach of extension theory of symmetric operators, we will use the new approach based on the consepts of linear relations in Hilbert spaces, boundary triplets (also called boundary value spaces) and Weyl functions. This new approach was very widely developed and used during the two last decades (see for exemple [5, 6, 7, 8, 15, 16]).

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This paper is organised as follows ; in the first section, basic concepts and results concerning linear relations and boundary triplets associated to the adjoint of symmetric operator are exposed. We also give the definition of the Weyl function and remind without proofs its fundamental properties that will be used all along this work. At the end, Carleman operators (subjet of this present study) are introduced. In the second section, we express the main results. Firstly, we construct a boundary triplet for the adjoint of the studied operator. This is then used to describe all proper extensions and to find the explicit form of the Weyl function. Secondly, we characterize the spectrum of extensions. Some auxilary results are also derived. We also give the explicit form of the resolvent of any proper extension. In the last section we use the Krein-Naïmark formula to give the explicit form of generalized resolvents by means of Nevanlinna families and associated Nevanlinna pairs.

Throughout this work, all considered spaces are complex and Hilbertian. Moreover, if $\mathcal{X}$ is a such space then, $\mathcal{B}(\mathcal{X})$ will denote the space of all bounded linear operators in $\mathcal{X}$. $I_{\mathcal{X}}$ (resp. $0_{\mathcal{X}}$) will stand for the identity in $\mathcal{X}$.(resp. the null operator in $\mathcal{X}$). The symbol $\mathbb{C}_+$ (resp. $\mathbb{C}_-$) will be used for the subset $\{z \in \mathbb{C} : Imz > 0\}$ (resp. $\{z \in \mathbb{C} : Imz < 0\}$) of the complex plane. If A is an operator then, $\sigma(A)$ (resp. $\sigma_p(A)$, $\sigma_c(A)$, $\sigma_r(A)$) designs the spectrum (resp. punctual, continuous, residual spectrum) of A and $\rho(A)$ its resolvent set.

2. Preliminaries

In this section, essential results concerning the extension theory of closed densely defined symmetric operators with equal deficiency indices are exposed. An approach based on the consept of boundary triplets (also called boundary value spaces) is used. These results are of general level and can be found in details in [5, 6, 10, 11, 12]

2.1. **Linear relations.** A linear relation in $\mathcal{H}$ is a linear subset of the product $\mathcal{H} \times \mathcal{H}$. To every linear relation Θ in $\mathcal{H}$ correspond the following linear subspaces:

$$dom\Theta = \{x \in \mathcal{H} : (x, y) \in \Theta; y \in \mathcal{H}\}; \; ran\Theta = \{y \in \mathcal{H} : (x, y) \in \Theta; x \in \mathcal{H}\},$$

$$\ker\Theta = \{x \in \mathcal{H} : (x, 0) \in \Theta\} \qquad ; \qquad mul\Theta = \{y \in \mathcal{H} : (0, y) \in \Theta\}.$$

The set of all closed linear relations will be noted $\widetilde{\mathcal{C}}(\mathcal{H})$. The identification of every bounded linear operator with its graph allows to regard $\mathcal{B}(\mathcal{H})$ as a subset of $\widetilde{\mathcal{C}}(\mathcal{H})$. If Θ belongs to $\widetilde{\mathcal{C}}(\mathcal{H})$ then the inverse and the adjoint relations of Θ are respectively defined by:

$$(y, x) \in \Theta^{-1} \Longleftrightarrow (x, y) \in \Theta \tag{2.1}$$

$$(y^*, x^*) \in \Theta^* \Longleftrightarrow \langle y, y^* \rangle = \langle x, x^* \rangle ; \forall (x, y) \in \Theta \tag{2.2}$$

Let us designate by J the operator acting in $\mathcal{H} \times \mathcal{H}$ by the rule $J(x, y) = (y, -x)$. It is not difficult to see that $\Theta^* = J\Theta^{\perp}$ where $\Theta^{\perp}$ designates the orthogonal in $\mathcal{H} \times \mathcal{H}$ of Θ.

Definition 1. *An element* Θ *of* $\widetilde{\mathcal{C}}(\mathcal{H})$ *is called:*

a) *symmetric if* $\Theta \subseteq \Theta^*$,
b) *selfadjoint if* $\Theta = \Theta^*$,
c) *dissipative if* $Im\langle y, x\rangle$ *is nonnegative for all* $(x, y) \in \Theta$,
d) *dissipative maximal if it is dissipative and doesn't admit any proper dissipative extension.*

Let $\Theta \in \widetilde{\mathcal{C}}(\mathcal{H})$ and λ a complex number. Consider the relation

$$\Theta - \lambda = \{(x, y - \lambda x) ; (x, y) \in \Theta\} \tag{2.3}$$

λ belongs to the resolvent set $\rho(\Theta)$ of Θ if $(\Theta - \lambda)^{-1}$ is an element of $\mathcal{B}(\mathcal{H})$. If it is not the case, then λ is an element of the spectrum $\sigma(\Theta)$ of Θ. The punctual, continuous and residual parts of the spectrum are defined as for an operator. Sometimes (as in the present case), it is useful to define the resolvent set as a subset of the extended complex plane by saying that $\infty \in \rho(\Theta)$ if $mul\Theta = \{0\}$. In the contrary case, ∞ is said to be an eigenvalue of Θ.

Since in next sections, we shall be dealing with operators having deficiency indices equal 1, let us give a general form of a relation in this case. As $\dim \mathcal{X} \times \mathcal{X} = 2$, there are only three possibilities for any linear relation Θ in $\mathcal{X}$:

a) $\dim \Theta = 0 \Longleftrightarrow \Theta = \{(0, 0)\}$,
b) $\dim \Theta = 2 \Longleftrightarrow \Theta = \mathcal{X} \times \mathcal{X}$,
c) $\dim \Theta = 1 \Longleftrightarrow \exists (a, b) \in \left(\mathbb{C}^2\right)^* : \Theta = \Theta_{(a,b)} = \{(x, y) \in \mathcal{X} \times \mathcal{X} : ax + by = 0\}$.

In the first case, we have a symmmetric, dissipative but nonselfadjoint relation ($\Theta^* = \mathcal{X} \times \mathcal{X}$), in the second case, the relation $\Theta = \mathcal{X} \times \mathcal{X}$ is a non symmetric and non dissipative relation ($\Theta^* = \{(0, 0)\}$). For relations of the third form, the following result can be easily established;

Proposition 1. *Assume that* $\dim \mathcal{X} = 1$ *and let*

$$\Theta_{(a,b)} = \{(x, y) \in \mathcal{X} \times \mathcal{X} : ax + by = 0\}, (a, b) \in \left(\mathbb{C}^2\right)^*$$

be a linear relation in $\mathcal{X}$. *Then,*

(1) $\Theta_{(a,0)}$ *and* $\Theta_{(0,b)}$ *are both selfadjoints* $\forall a, b \in \mathbb{C}^*$,
(2) *If* $ab \neq 0$ *then* $\Theta_{(a,b)}$ *is selfadjoint if and only if,* $Im\left(\frac{a}{b}\right) = 0$,
(3) *If* $ab \neq 0$ *then* $\Theta_{(a,b)}$ *is dissipative if and only if,* $Im\left(\frac{a}{b}\right) \leq 0$,
(4) $\sigma\left(\Theta_{(a,0)}\right) = \{\infty\}$ *and* $\sigma\left(\Theta_{(0,b)}\right) = \{0\}, \forall a, b \neq 0$,
(5) *If* $ab \neq 0$ *then* $\sigma\left(\Theta_{(a,b)}\right) = \left\{-\frac{a}{b}\right\}$.

2.2. **Boundary triplets.** Let A be a closed densely defined symmetric operator with domain $\mathcal{D}(A) \subseteq \mathcal{H}$ and equal deficiency indices (e.g.$n_{\pm}(A) = \dim\ker(A^* \pm iI_{\mathcal{H}}) < +\infty$). A triplet $\Pi = (\mathcal{X}, \Gamma_0, \Gamma_1)$ constituted by a Hilbert space $\mathcal{X}$ and linear mappings $\Gamma_0, \Gamma_1 : \mathcal{D}(A^*) \longrightarrow \mathcal{X}$ is called a boundary triplet (or also boundary value space) for the adjoint A^* if,

(1) formula $\Gamma(x) = (\Gamma_0(x), \Gamma_1(x))$ defines a linear surjection from $\mathcal{D}(A^*)$ into $\mathcal{X} \times \mathcal{X}$,
(2) the abstract second Green formula

$$\langle A^*(x), y\rangle - \langle x, A^*(y)\rangle = \langle \Gamma_1(x), \Gamma_0(y)\rangle - \langle \Gamma_0(x), \Gamma_1(y)\rangle,$$

holds for all $x, y \in \mathcal{D}(A^*)$.

Note that in case of equal deficiency indices, boundary triplets always exist. Moreover, if $\Pi = (\mathcal{X}, \Gamma_0, \Gamma_1)$ and $\Pi' = (\mathcal{X}', \Gamma_0', \Gamma_1')$ are two boundary triplets for A^* then, there exists a bounded invertible operator $W = (W_{ij})_{i,j=1}^2 : \mathcal{X} \times \mathcal{X} \longrightarrow \mathcal{X}' \times \mathcal{X}'$ such that

$$W^* \begin{pmatrix} 0 & -iI_{\mathcal{X}'} \\ iI_{\mathcal{X}'} & 0 \end{pmatrix} W = \begin{pmatrix} 0 & -iI_{\mathcal{X}} \\ iI_{\mathcal{X}} & 0 \end{pmatrix}$$

and

$$\begin{pmatrix}\Gamma_0'\\ \Gamma_1'\end{pmatrix} = \begin{pmatrix} W_{11} & W_{12}\\ W_{21} & W_{22}\end{pmatrix}\begin{pmatrix}\Gamma_0\\ \Gamma_1\end{pmatrix}$$

Definition 2. *Let A and $\widetilde{A}$ be two linear operators in $\mathcal{H}$. Assume that A is symmetric, closed and densely defined. $\widetilde{A}$ is called* a proper extension of A *if,* $A \subset \widetilde{A} \subset A^*$

We have the following fundamental result,

Theorem 1. *Let A be a closed densely defined symmetric operator with equal deficiency indices and $\Pi = (\mathcal{X}, \Gamma_0, \Gamma_1)$ be a boundary triplet for A^*. Then $\widetilde{A}$ is a proper extension of A if and only if there exists $\Theta \in \widetilde{\mathcal{C}}(\mathcal{H})$ such that*

$$\mathcal{D}(\widetilde{A}) = \Gamma^{-1}(\Theta) = \{x \in \mathcal{D}(A^*) : (\Gamma_0(x), \Gamma_1(x)) \in \Theta\} \tag{2.4}$$

Moreover,

a) *$\widetilde{A}$ is selfadjoint (resp. symmetric)$\Longleftrightarrow \Theta$ is selfadjoint (resp. symmetric),*
b) *$\widetilde{A}$ is dissipative(resp. maximal dissipative)$\Longleftrightarrow \Theta$ is dissipative(resp. maximal dissipative).*

Remark 1.

a) *Extension generated by $\Theta \in \widetilde{\mathcal{C}}(\mathcal{H})$ in the sens of the preceding theorem will be noted A_Θ,*
b) *If Θ is an element of $\mathcal{B}(\mathcal{X})$ then, $\mathcal{D}(A_\Theta) = \ker(\Gamma_1 - \Theta\Gamma_0)$,*
c) *Operators A_0 and A_∞ with respective domains ;*
$\mathcal{D}(A_0) = \{x \in \mathcal{D}(A^*) : \Gamma_1(x) = 0\}$ and $\mathcal{D}(A_\infty) = \{x \in \mathcal{D}(A^*) : \Gamma_0(x) = 0\}$ are selfadjoint extensions of A. Moreover, it can be easily verified that if $n_\pm(A) = 1$ then $A_0 = A_{\Theta_{(0,b)}}$ and $A_\infty = A_{\Theta_{(a,0)}}$.

Let A be a closed densely defined symmetric operator with equal deficiency indices n and $\Pi = (\mathcal{X}, \Gamma_0, \Gamma_1)$ be a boundary triplet for A^*. According to [12], the mapping Γ_0 establishes a bijection from $\ker(A^* - zI_{\mathcal{H}})$ to $\mathbb{C}^n$ $(n = n_\pm(A))$ for every complex number $z \in \rho(A_\infty)$. Functions;

$$\gamma(z) = \left[\Gamma_0\big|_{\ker(A^* - zI_{\mathcal{H}})}\right]^{-1}, M(z) = \Gamma_1\gamma(z)\,; z \in \rho(A_\infty), \tag{2.5}$$

are respectively called $\gamma-$field and Weyl function associated to boundary triplet $\Pi = (\mathcal{X}, \Gamma_0, \Gamma_1)$. It is not difficult to see that

$$M(z)\Gamma_0(x_z) = \Gamma_1(x_z)\,; \forall x_z \in \ker(A^* - zI_{\mathcal{H}}).$$

Fundamental properties of the Weyl function are resumed in the next result [5, 6]

Theorem 2. *Let A be a closed densely defined symmetric operator wih equal deficiency indices n, $\Pi = (\mathcal{X}, \Gamma_0, \Gamma_1)$ a boundary triplet for A^*. Then,*

a) *$M(z)$ is analytic in the domain $z \in \rho(A_\infty)$,*
b) *$ImM(z)\,Imz > 0, z \in \rho(A_\infty)$,*
c) *$[M(z)]^* = M(\bar{z}), z \in \rho(A_\infty)$,*
d) *$M(z) - M(\xi) = (z - \xi)\gamma^*(\bar{\xi})\gamma(z)\,; z, \xi \in \rho(A_\infty)$,*
e) *$z \in \rho(A_\Theta) \Longleftrightarrow (\Theta - M(z))^{-1} \in \mathcal{B}(\mathcal{X}), z \in \rho(A_\infty)$,*
f) *If $\dim \mathcal{X} = 1$, then z belongs to the punctual (resp. continuous or residual) spectrum of A_Θ if and only if $M(z)$ belongs to the punctual (resp. continuous or residual) spectrum of Θ,*

g) $(A_\Theta - zI_\mathcal{H})^{-1} - (A_\infty - zI_\mathcal{H})^{-1} = \gamma(z)(\Theta - M(z))^{-1}\gamma^*(\bar{z})$;
$z \in \rho(A_\Theta) \cap \rho(A_\infty)$.

2.3. **Carleman operators.** Let Ω be an arbitrary set, μ a $\sigma-$finite measure defined on a $\sigma-$algebra of subsets of Ω and $L^2(\Omega,\mu)$ the corresponding Hilbert space of square integrable functions with respect to μ. According to the general theory of integral operators [1, 4, 13, 18], a Carleman operator in $L^2(\Omega,\mu)$ is a closed symmetric and densely defined operator of the form;

$$Af(x) = \int_\Omega K(x,y) f(y)\, d\mu(y), \tag{2.6}$$

where the kernel $K(x,y)$ satisfies the following conditions:

$$\int_\Omega |K(x,y)|^2 d\mu(y) < +\infty \text{ almost everywhere in } \Omega. \tag{2.7}$$

The domain of A is defined as follows

$$\mathcal{D}(A) = \left\{ f \in L^2(\Omega,\mu) : \int_\Omega |f(x)|\, k(x)\, dx < +\infty \right\}, k^2(x) = \int_\Omega |K(x,y)|^2 dy$$

In this paper, we consider the case,

$$K(x,y) = {}_{p=0}^{+\infty} a_p \Psi_p(x) \overline{\Psi_p(y)}, \tag{2.8}$$

where $\{\Psi_p\}_{p=0}^{+\infty}$ is an orthonormal sequence in $L^2(\Omega,\mu)$ such that

$${}_{p=0}^{+\infty} |\Psi_p(x)|^2 < +\infty \text{ almost everywhere in } \Omega, \tag{2.9}$$

and $\{a_p\}_{p=0}^{+\infty}$ a real number sequence verifying

$${}_{p=0}^{+\infty} a_p^2 |\Psi_p(x)|^2 < +\infty \text{ almost everywhere in } \Omega. \tag{2.10}$$

Note that the sequence $\{\Psi_p\}_{p=0}^{+\infty}$ is not total and ${}_{p=0}^{+\infty} a_p^2 = +\infty$. Otherwise, the corresponding operator would be either selfadjoint or Hilbert-Schmidt. This Carleman operator has been studied in [2, 3]. From [2], one can retain the following fundamental result.

Theorem 3. *If the operator A possesses equal deficiency indices $n_\pm(A) = 1$ then, for all numbers $\lambda \in \mathbb{C}\backslash\mathbb{R}$ there exists a sequence $\{\delta_p(\lambda)\}_{p=0}^{+\infty}$ of complex numbers such that;*

a) ${}_{p=0}^{+\infty} |\delta_p(\lambda)|^2 = +\infty$; ${}_{p=0}^{+\infty} \dfrac{|\delta_p(\lambda)|^2}{|a_p - \lambda|^2} < +\infty$,

b) *Function $\vartheta = {}_{p=0}^{+\infty} \delta_p(\lambda)\Psi_p$ satisfies conditions:*

$$\int_\Omega \vartheta(x)\overline{\Psi_q(x)} d\mu(x) = 0; q = 0, 1, 2, ...$$

and the function

$$\varphi_\lambda(x) = \frac{\delta_p(\lambda)}{a_p - \lambda}\Psi_p(x), \tag{2.11}$$

is a solution of the equation $A^* f = \lambda f$.

In [3], quasi-selfadjoint extensions (see [1, 17]) of operator A with deficiency indices equal to1 are considered. That is nonselfadjoint proper extensions $\widetilde{A}$ of A satisfying the additionnal condition $\dim \mathcal{D}(\widetilde{A}) = 1(mod\mathcal{D}\,(A))$. General forms of the corresponding generalized resolvents and generalized spectral functions are given using Stieljes inversion formula. In particular, one has the following result [3]

Theorem 4. *Following formulas:*

$$R_\lambda f = \mathring{R}_\lambda f + \frac{1-\omega\,(\lambda)}{\omega\,(\lambda)\,s\,(\lambda)-1}.\frac{\langle f,\varphi_{\bar{\lambda}}\rangle}{(\lambda+i)\,\langle\varphi_\lambda,\varphi_i\rangle}\varphi_\lambda \quad (Im\lambda>0), \tag{2.12}$$

$$R_{\bar{\lambda}} f = \mathring{R}_{\bar{\lambda}} f + \frac{1-\overline{\omega\,(\lambda)}}{\overline{\omega\,(\lambda)s\,(\lambda)}-1}.\frac{\langle f,\varphi_\lambda\rangle}{\left(\bar{\lambda}-i\right)\langle\varphi_{\bar{\lambda}},\varphi_{-i}\rangle}\varphi_{\bar{\lambda}} \quad (Im\lambda>0), \tag{2.13}$$

establish a bijective correspondance between the set of generalized resolvents of operator A and the set of analytic functions $\omega\,(\lambda)$ $(|\omega\,(\lambda)| \le 1, Im\lambda > 0)$. $\mathring{R}_\lambda$ is the resolvent of a certain selfadjoint extension of A;

$$s\,(\lambda) = \frac{\lambda-i}{\lambda+i} \times \frac{\sum_{p=0}^{+\infty}\dfrac{|\delta_p\,(\lambda)|^2}{(a_p-\lambda)\,(a_p-i)}}{\sum_{p=0}^{+\infty}\dfrac{|\delta_p\,(\lambda)|^2}{(a_p-\lambda)\,(a_p+i)}}. \tag{2.14}$$

In the present work, we propose a new approach based on the concept of boundary triplet. This allows to study not only quasi-selfadjoint extensions but all proper extensions of operator A. Similar formulas for generalized resolvents are obtained and corresponding spectra investigated by means of Weyl and $\gamma-$field functions.

3. Spectral properties of proper extensions

3.1. **Proper extensions and Weyl functions.** In the following, we will always suppose that deficiency indices of operator A are equal to 1.

Let us put

$$\mathcal{N}_\lambda = \ker\left(A^* - \lambda I_{\mathcal{H}}\right), e_\lambda\,(x) = \left(\sum_{p=0}^{+\infty}\frac{|\delta_p\,(\lambda)|^2}{|a_p-\lambda|^2}\right)^{-\frac{1}{2}}\varphi_\lambda\,(x)\,. \tag{3.1}$$

it is clear that $e_\lambda \in \mathcal{N}_\lambda$ and $\|e_\lambda\| = 1$. According to Von Neumann Formula, for every complex z such that $Imz \neq 0$ one has the direct sum

$$\mathcal{D}\,(A^*) = \mathcal{D}\,(A)\dotplus\mathcal{N}_z\dotplus\mathcal{N}_{\bar{z}} \tag{3.2}$$

Consider now in $\mathcal{D}\,(A^*)\times\mathcal{D}\,(A^*)$ the sesquilinear form defined by

$$\Phi\,(f,g) = \langle f,g\rangle + \langle A^*\,(f)\,, A^*\,(g)\rangle \tag{3.3}$$

Lemma 1. *The sesquilinear form Φ satisfies the following properties:*

a) $\forall f \in \mathcal{D}\,(A) : \Phi\,(f,e_i) = \Phi\,(f,e_{-i}) = \Phi\,(e_i,e_{-i}) = 0,$
b) $\Phi\,(e_i,e_i) = \Phi\,(e_{-i},e_{-i}) = 2,$
c) $\forall f \in \mathcal{D}\,(A^*) : f = f_0 + \frac{1}{2}\Phi\,(f,e_i)\,e_i + \frac{1}{2}\Phi\,(f,e_{-i})\,e_{-i},$
d) *for every complex z $(Imz \neq 0) : \Phi\,(e_z,e_i) + \Phi\,(e_z,e_{-i}) \neq 0.$*

Proof. The first two properties are directly verifiable. Let now f be an element of $\mathcal{D}(A^*)$. It follows from (3.2) that

$$f = f_0 + \alpha(i)\, e_i + \alpha(-i)\, e_{-i}, f_0 \in \mathcal{D}(A), \alpha(i), \alpha(-i) \in \mathbb{C}.$$

Thus, by using the first property, one can easily obtain that,

$$2\alpha(i) = \Phi(f, e_i) \quad ; \quad 2\alpha(-i) = \Phi(f, e_{-i}).$$

In order to prove the last assertion, let us suppose that $\Phi(e_z, e_i) + \Phi(e_z, e_{-i}) = 0$, then by c) $\Phi(e_z, e_i) - \Phi(e_z, e_{-i}) = \Phi^2(e_z, e_i) - \Phi^2(e_z, e_{-i}) = 0$ and consequently, $e_z \in \mathcal{D}(A) \cap \mathcal{N}_z = \{0\}$; so $e_z = 0$ which is impossible. □

Proposition 2. *Let f be an element of $\mathcal{D}(A^*)$ and z, $Imz \neq 0$ a complex number. Assume that*

$$f = f_0 + \alpha(z)\, e_z + \alpha(\bar{z})\, e_{\bar{z}}, f_0 \in \mathcal{D}(A), \alpha(z), \alpha(\bar{z}) \in \mathbb{C}.$$

is the canonical decomposition of f according to direct sum (3.2) then,

$$\alpha(z) = \frac{\Phi(f, e_i)\,\Phi(e_{\bar{z}}, e_{-i}) - \Phi(f, e_{-i})\,\Phi(e_{\bar{z}}, e_i)}{\Phi(e_z, e_i)\,\Phi(e_{\bar{z}}, e_{-i}) - \Phi(e_z, e_{-i})\,\Phi(e_{\bar{z}}, e_i)}$$

$$\alpha(\bar{z}) = \frac{\Phi(f, e_{-i})\,\Phi(e_z, e_i) - \Phi(f, e_i)\,\Phi(e_z, e_{-i})}{\Phi(e_z, e_i)\,\Phi(e_{\bar{z}}, e_{-i}) - \Phi(e_z, e_{-i})\,\Phi(e_{\bar{z}}, e_i)} \tag{3.4}$$

Proof. By precedent lemma,

$$f = f_0 + \frac{1}{2}\Phi(f, e_i)\, e_i + \frac{1}{2}\Phi(f, e_{-i})\, e_{-i}$$

where,

$$\begin{pmatrix} \Phi(f, e_i) \\ \Phi(f, e_{-i}) \end{pmatrix} = \begin{pmatrix} \Phi(e_z, e_i) & \Phi(e_{\bar{z}}, e_i) \\ \Phi(e_z, e_{-i}) & \Phi(e_{\bar{z}}, e_{-i}) \end{pmatrix} \begin{pmatrix} \alpha(z) \\ \alpha(\bar{z}) \end{pmatrix}$$

Uniqueness of $\alpha(z)$ and $\alpha(\bar{z})$ implies that the matrix

$$\begin{pmatrix} \Phi(e_z, e_i) & \Phi(e_{\bar{z}}, e_i) \\ \Phi(e_z, e_{-i}) & \Phi(e_{\bar{z}}, e_{-i}) \end{pmatrix}$$

is invertible and

$$\begin{pmatrix} \alpha(z) \\ \alpha(\bar{z}) \end{pmatrix} = \begin{pmatrix} \Phi(e_z, e_i) & \Phi(e_{\bar{z}}, e_i) \\ \Phi(e_z, e_{-i}) & \Phi(e_{\bar{z}}, e_{-i}) \end{pmatrix}^{-1} \begin{pmatrix} \Phi(f, e_i) \\ \Phi(f, e_{-i}) \end{pmatrix}.$$

Thus, (3.4) follows from direct computations. □

Proposition 3. *The triplet $\Pi = (\mathbb{C}, \Gamma_0, \Gamma_1)$ where Γ_0, Γ_1 are two linear mappings from $\mathcal{D}(A^*)$ to $\mathbb{C}$ given by*

$$\Gamma_0(f) = \frac{1}{2}\left[\Phi(f, e_i) + \Phi(f, e_{-i})\right]; \Gamma_1(f) = \frac{i}{2}\left[\Phi(f, e_i) - \Phi(f, e_{-i})\right], \tag{3.5}$$

is a boundary triplet for A^*.

Proof. follows from the more general formula for canonical boundary triplets presented by A. N. Kochubeï [12] for symmetric operators with arbitrary defect numbers $(n, n)\,; n < +\infty$. □

We shall now discribe all proper extensions of operator A by means of boundary triplet $\Pi = (\mathbb{C}, \Gamma_0, \Gamma_1)$, defined in the preceding theorem. Let $\widetilde{A}$ be a proper extension of A. It has been shown in preceding sections that there exists $(a, b) \in (\mathbb{C}^2)^*$ such that,

$$\begin{aligned} \mathcal{D}(\widetilde{A}) &= \{f \in \mathcal{D}(A^*) : a\Gamma_0(f) + b\Gamma_1(f) = 0\} \\ &= \{f \in \mathcal{D}(A^*) : (a + ib)\Phi(f, e_i) + (a - ib)\Phi(f, e_{-i}) = 0\}. \end{aligned} \tag{3.6}$$

Note that for $a = b = 0$ the corresponding extension is A^*. Because of that, we will always suppose that $(a, b) \neq (0, 0)$.

Theorem 5. *Let $\widetilde{A}$ be a proper extension of A. Then, there exists a complex number c such that $\mathcal{D}(\widetilde{A})$ admits one of two following representations:*

$$\mathcal{D}(\widetilde{A}) = \{f_0 + \alpha(i)(ce_i + e_{-i}); f_0 \in \mathcal{D}(A), \alpha(i) \in \mathbb{C}\}, \tag{3.7}$$

where,

$$\widetilde{A}(f) = A(f_0) + i\alpha(i)(ce_i - e_{-i}) \tag{3.8}$$

or,

$$\mathcal{D}(\widetilde{A}) = \{f_0 + \alpha(i)(e_i + ce_{-i}); f_0 \in \mathcal{D}(A), \alpha(i) \in \mathbb{C}\}, \tag{3.9}$$

where,

$$\widetilde{A}(f) = A(f_0) + i\alpha(i)(e_i - ce_{-i}) \tag{3.10}$$

Moreover, $\widetilde{A}$ is selfadjoint if and only if $|c| = 1$.

Proof. It follows from J. Von Neumann's formula written for the case of defect numbers $(1, 1)$. □

Remark 2. *It follows from theorem 5 that there exists a bijective correspondance between the set of all selfadjoint (resp. dissipative) extensions of operator A and the field of real numbers (resp. the upper half plane). Moreover, selfadjoint extension A_0 (resp.A_∞) corresponds to $c = 1$ in (3.9) (resp. $c = -1$ in (3.7)).*

Theorem 6. *The γ−field and Weyl functions associated to boundary triplet defined in proposition 3 admit the following representations:*

$$[\gamma(z)](\lambda) = \frac{2\lambda}{\Phi(e_z, e_i) + \Phi(e_z, e_{-i})}.e_z; z \in \mathbb{C} \cap \rho(A_\infty), \tag{3.11}$$

$$[M(z)](\lambda) = i\lambda\frac{\Phi(e_z, e_i) - \Phi(e_z, e_{-i})}{\Phi(e_z, e_i) + \Phi(e_z, e_{-i})}; z \in \mathbb{C} \cap \rho(A_\infty), \tag{3.12}$$

Proof. The first relation follows immediately from the fact that $\frac{2\lambda}{\Phi(e_z,e_i)+\Phi(e_z,e_{-i})}.e_z$ is an element of $\mathcal{N}_z$ and the easily verifyable equalities

$$\Gamma_0[\gamma(z)](\lambda) = \lambda\ ; \quad [\gamma(z)]\Gamma_0(e_z) = e_z$$

For the second one,

$$\begin{aligned}[M(z)](\lambda) &= \Gamma_1[\gamma(z)](\lambda)\\ &= \Gamma_1\left(\frac{2\lambda}{\Phi(e_z,e_i)+\Phi(e_z,e_{-i})}.e_z\right)\\ &= \frac{2\lambda}{\Phi(e_z,e_i)+\Phi(e_z,e_{-i})}\Gamma_1(e_z)\\ &\quad \frac{2\lambda}{\Phi(e_z,e_i)+\Phi(e_z,e_{-i})}\times\frac{i}{2}[\Phi(e_z,e_i)-\Phi(e_z,e_{-i})]\\ &= i\lambda\frac{\Phi(e_z,e_i)-\Phi(e_z,e_{-i})}{\Phi(e_z,e_i)+\Phi(e_z,e_{-i})}.\end{aligned}$$

□

3.2. **Spectral properties.** In the following, $A_{(a,b)}$ will denote the proper extension of A corresponding to $(a,b)\in\mathbb{C}^2$ and $\mathcal{D}_{(a,b)}$ it's associated domain .

Proposition 4. *Let* $(a,b)\in\mathbb{C}^2$. *Then;*

a) $0\in\sigma_p\left(A_{(a,b)}\right)$,

b) $\Psi_k\in\mathcal{D}_{(a,b)}$,

$$\iff b\left[\|\varphi_{-i}\|\,\overline{\delta_k(i)}+\|\varphi_i\|\,\overline{\delta_k(-i)}\right]=ia\left[\|\varphi_{-i}\|\,\overline{\delta_k(i)}-\|\varphi_i\|\,\overline{\delta_k(-i)}\right].$$

In this case, Ψ_k *is an eigenvector of* $A_{(a,b)}$ *corresponding to the eigenvalue* a_k.

c) *If* $\Psi_k\notin\mathcal{D}_{(a,b)}$ *then* $a_k\in\sigma_r\left(A_{(a,b)}\right)$

Proof. a) Direct computations show that for all $f\in\mathcal{D}(A^*)$:

$$\Phi(f,e_i)=-i\,\|\varphi_i\|^{-1}{}_{p=0}^{+\infty}\overline{\delta_p(i)}\,\langle f,\Psi_p\rangle, \tag{3.13}$$

$$\Phi(f,e_{-i})=i\,\|\varphi_{-i}\|^{-1}{}_{p=0}^{+\infty}\overline{\delta_p(-i)}\,\langle f,\Psi_p\rangle, \tag{3.14}$$

Then,

$$\Gamma_0(f)=\frac{i}{2}{}_{p=0}^{+\infty}\left[-\|\varphi_i\|^{-1}\,\overline{\delta_p(i)}+\|\varphi_{-i}\|^{-1}\,\overline{\delta_p(-i)}\right]\langle f,\Psi_p\rangle, \tag{3.15}$$

$$\Gamma_1(f)=\frac{1}{2}{}_{p=0}^{+\infty}\left[\|\varphi_i\|^{-1}\,\overline{\delta_p(i)}+\|\varphi_{-i}\|^{-1}\,\overline{\delta_p(-i)}\right]\langle f,\Psi_p\rangle, \tag{3.16}$$

Thus for all $f\in\mathcal{D}(A^*)$:

$$\begin{aligned} f\in\mathcal{D}_{(a,b)} &\iff a\Gamma_0(f)+b\Gamma_1(f)=0\\ &\iff ia_{p=0}^{+\infty}\left[-\|\varphi_i\|^{-1}\,\overline{\delta_p(i)}+\|\varphi_{-i}\|^{-1}\,\overline{\delta_p(-i)}\right]\langle f,\Psi_p\rangle\\ &\qquad + b_{p=0}^{+\infty}\left[\|\varphi_i\|^{-1}\,\overline{\delta_p(i)}+\|\varphi_{-i}\|^{-1}\,\overline{\delta_p(-i)}\right]\langle f,\Psi_p\rangle=0.\end{aligned}$$

Let L_Ψ be a closed subspace generated by vectors $\Psi_p, p=0,1,2,...$ It is clear that $L_\Psi^\perp\subset\mathcal{D}_{(a,b)}$ for all $(a,b)\in\mathbb{C}^2$. Since $\{\Psi_p,p=0,1,2,...\}$ is not total, then $L_\Psi^\perp\neq\{0\}$. Finally, it follows from the relation

$$\forall f\in L_\Psi^\perp; A_{(a,b)}(f)=A^*(f)=0$$

that 0 belongs to the punctual spectrum of $A_{(a,b)}$.

b) follows from precedent characterization of elements of $\mathcal{D}_{(a,b)}$ and the relation $A^*(\Psi_k)=a_k\Psi_k; k=0,1,2,...$

c) Suppose now that $\Psi_k \notin \mathcal{D}_{(a,b)}$. Let us firstly prove that $\left(A_{(a,b)} - a_k\right)$ is injective. Indeed, suppose that there exists in $\mathcal{D}_{(a,b)}$ a nonnul vector f such that $A_{(a,b)}(f) = A^*(f) = a_k f$. Setting

$$f = {}_{p=0}^{+\infty} \langle f, \Psi_p \rangle \Psi_p + f_\Psi^\perp \ ; \quad f_\Psi^\perp \perp L_\Psi,$$

it is not difficult to see that f is necessary of the form $f = \alpha_k \Psi_k; \alpha_k \in \mathbb{C}$ and then, $f \notin \mathcal{D}_{(a,b)}$. Consequently, operator $\left(A_{(a,b)} - a_k\right)$ is injective.

On the other hand, one has for all $k \in \{0, 1, 2, ...\}$,

$$\begin{aligned} \left(A_{(a,b)} - a_k\right)(f) &= (A^* - a_k)(f) = {}_{p=0}^{+\infty} (a_p - a_k) \langle f, \Psi_p \rangle \Psi_p - a_k f_\Psi^\perp \\ &= {}_{p \neq k} (a_p - a_k) \langle f, \Psi_p \rangle \Psi_p - a_k f_\Psi^\perp. \end{aligned}$$

It is clear that

$$\left(A_{(a,b)} - a_k\right)(f) \perp \Psi_k$$

and thus, a_k belongs to the residual spectrum of the operator $A_{(a,b)}$.

□

Lemma 2. *Let $z \in \rho(A_\infty)$. Then for every $g \in L^2(\Omega, \mu)$ there exists $g_\Psi^\perp \in L_\Psi^\perp$ and complex numbers C, C_0, C_1...such that;*

$$g = {}_{p=0}^{+\infty} C_p \Psi_p + C(i - z) e_i + C(i + z) e_{-i} + g_\Psi^\perp; {}_{p=0}^{+\infty} |C_p|^2 < +\infty, \tag{3.17}$$

$$C_p = \langle g, \Psi_p \rangle - C(i - z) \frac{\delta_p(i)}{\|\varphi_i\| (a_p - i)} - C(i + z) \frac{\delta_p(-i)}{\|\varphi_{-i}\| (a_p + i)}. \tag{3.18}$$

Proof. Since $z \in \rho(A_\infty)$ one has $L^2(\Omega, \mu) = (A_\infty - z)(\mathcal{D}(A_\infty))$. Hence, for $g \in L^2(\Omega, \mu)$ there exists $f_0 \in \mathcal{D}(A)$ and a complex number C such that,

$$g = (A_\infty - z)(f_0 + Ce_i - Ce_{-i})$$

Setting

$$f_0 = {}_{p=0}^{+\infty} \langle f_0, \Psi_p \rangle \Psi_p + f_{0\Psi}^\perp \ ; \ f_{0\Psi}^\perp \in L_\Psi^\perp,$$

One has

$$\begin{aligned} g &= (A_\infty - z)(f_0) + C(i - z) e_i + C(i + z) e_{-i} - z f_{0\Psi}^\perp \\ &= {}_{p=0}^{+\infty} (a_p - z) \langle f_0, \Psi_p \rangle \Psi_p + C(i - z) e_i + C(i + z) e_{-i} - z f_{0\Psi}^\perp. \end{aligned}$$

Let $g_\Psi^\perp = -z f_{0\Psi}^\perp$ and $C_p = (a_p - z) \langle f_0, \Psi_p \rangle$. Then,

$$g_\Psi^\perp \in L_\Psi^\perp \ ; {}_{p=0}^{+\infty} |C_p|^2 < +\infty,$$

and for all $p = 0, 1, 2, ...$,

$$\begin{aligned} \langle g, \Psi_p \rangle &= C_p + C(i - z) \langle e_i, \Psi_p \rangle + C(i + z) \langle e_{-i}, \Psi_p \rangle \\ &= C_p + C(i - z) \frac{\delta_p(i)}{\|\varphi_i\| (a_p - i)} + C(i + z) \frac{\delta_p(-i)}{\|\varphi_{-i}\| (a_p + i)}. \end{aligned}$$

□

Corollary 1. *Keeping notations of the preceding lemma, operator $(A_\infty - z)^{-1}$ applies $L^2(\Omega, \mu)$ into $\mathcal{D}(A_\infty)$ according to the rule,*

$$(A_\infty - z)^{-1}(g) = {}_{p=0}^{+\infty} \frac{C_p}{a_p - z} \Psi_p + Ce_i - Ce_{-i} - \frac{1}{z} g_\Psi^\perp.$$

Corollary 2. $\sigma(A_\infty) = \{0, a_0, a_1, a_2, ...\}$.

Theorem 7. *Let $(a, b) \in \mathbb{C} \times \mathbb{C}^*$ and $A_{(a,b)}$. Then,*

a) $\{0, a_0, a_1, a_2, ...\} \subset \sigma\left(A_{(a,b)}\right)$,
b) $\forall z \in \rho\left(A_{\infty}\right)$,

$$z \in \sigma\left(A_{(a,b)}\right) \Longleftrightarrow ib\left(\Phi\left(e_z, e_{-i}\right) - \Phi\left(e_z, e_i\right)\right) = a\left(\Phi\left(e_z, e_i\right) + \Phi\left(e_z, e_{-i}\right)\right).$$

In this last case, z is an eigenvalue of $A_{(a,b)}$.

Proof. Point a) has already been treated above. Let now $z \in \rho\left(A_{\infty}\right)$. Taking into account that $\sigma\left(\Theta_{(a,b)}\right) = \frac{-a}{b}$ (see proposion 1) and theorem 2, one has

$$\begin{aligned} z \quad &\in \quad \sigma\left(A_{(a,b)}\right) \Longleftrightarrow M(z) \in \sigma\left(\Theta_{(a,b)}\right) \Longleftrightarrow M(z) = \frac{-a}{b} \\ &\Longleftrightarrow \quad i\frac{\Phi\left(e_z, e_i\right) - \Phi\left(e_z, e_{-i}\right)}{\Phi\left(e_z, e_i\right) + \Phi\left(e_z, e_{-i}\right)} = \frac{-a}{b} \\ &\Longleftrightarrow \quad ib\left(\Phi\left(e_z, e_{-i}\right) - \Phi\left(e_z, e_i\right)\right) = a\left(\Phi\left(e_z, e_i\right) + \Phi\left(e_z, e_{-i}\right)\right), \end{aligned}$$

It remains to prove that z belongs to the ponctual spectrum. This follows immediately from point e) of the theorem 2 and and the fact that the spectrum of $\Theta_{(a,b)}$ is ponctual □

To end this section, let us give the general form for the resolvents of proper extensions of A.

Lemma 3. *Let $(a,b) \in \mathbb{C} \times \mathbb{C}^*$ and let $\Theta_{(a,b)}$ be its corresponding linear relation. Then, for all $z \in \rho\left(A_{\infty}\right) \cap \rho\left(A_{(a,b)}\right)$,*

$$\left(\Theta_{(a,b)-}M(z)\right)^{-1} = \frac{-b}{a + bM(z)}.I_{\mathbb{C}}. \tag{3.19}$$

Proof. Note firstly that $z \in \rho\left(A_{\infty}\right) \cap \rho\left(A_{(a,b)}\right) \Longrightarrow M(z) \neq \frac{-a}{b}$. Moreover,

$$\begin{aligned} \left(\Theta_{(a,b)-}M(z)\right)^{-1} &= \{(x, y - M(z)x) : ax + by = 0; x, y \in \mathbb{C}\}^{-1} \\ &= \{(y - M(z)x, x) : ax + by = 0; x, y \in \mathbb{C}\} \\ &= \left\{\left(-\frac{a + bM(z)}{b}x, x\right), x \in \mathbb{C}\right\} \\ &= \left\{\left(x, \frac{-b}{a + bM(z)}x\right), x \in \mathbb{C}\right\} \\ &= \frac{-b}{a + bM(z)}.I_{\mathbb{C}}. \end{aligned}$$

□

Proposition 5. *Let $(a,b) \in \mathbb{C} \times \mathbb{C}^*$. Then, for all $z \in \rho\left(A_{\infty}\right) \cap \rho\left(A_{(a,b)}\right)$ and all $g \in L^2(\Omega, \mu)$,*

$$\begin{aligned} \left(A_{(a,b)} - z\right)^{-1}(g) &=_{p=0}^{+\infty} \frac{C_p}{a_p - z}\Psi_p + Ce_i - Ce_{-i} - \frac{1}{z}g_{\Psi}^{\perp} \\ &- \frac{b}{a + bM(z)} \times \frac{4\langle g, e_{\bar{z}}\rangle}{\left(\Phi\left(e_i, e_{\bar{z}}\right) + \Phi\left(e_{-i}, e_{\bar{z}}\right)\right)\left(\Phi\left(e_z, e_i\right) + \Phi\left(e_z, e_{-i}\right)\right)}e_z, \end{aligned}$$

where constants $g_{\Psi}^{\perp}, C, C_p, p = 0, 1, 2, ...$are defined from lemma 2.

Proof. Note firstly that $\gamma^*(\bar{z})$ applies $\ker(A^*-z)$ into $\mathbb{C}$ according to the rule,

$$\gamma^*(\bar{z})(h) = \frac{2\langle h, e_{\bar{z}}\rangle}{\Phi(e_i, e_{\bar{z}}) + \Phi(e_{-i}, e_{\bar{z}})}$$

On the other hand, by theorem 2, one has for every $g \in L^2(\Omega,\mu)$,

$$\begin{aligned}\left(A_{(a,b)} - z\right)^{-1}(g) &= (A_\infty - z)^{-1}(g) + \gamma(z)\left(\Theta_{(a,b)} - M(z)\right)^{-1}\gamma^*(\bar{z})\left(\langle g, e_{\bar{z}}\rangle e_{\bar{z}}\right) \\ &= (A_\infty - z)^{-1}(g) + \\ &+ \langle g, e_{\bar{z}}\rangle \gamma(z)\left(\Theta_{(a,b)} - M(z)\right)^{-1}\left(\frac{2}{\Phi(e_i, e_{\bar{z}}) + \Phi(e_{-i}, e_{\bar{z}})}\right) \\ &= (A_\infty - z)^{-1}(g) + \\ &+ \langle g, e_{\bar{z}}\rangle \gamma(z)\left(\frac{-b}{a + bM(z)} \times \frac{2}{\Phi(e_i, e_{\bar{z}}) + \Phi(e_{-i}, e_{\bar{z}})}\right) \\ &= (A_\infty - z)^{-1}(g) + \\ &- \frac{b}{a + bM(z)}\langle g, e_{\bar{z}}\rangle \gamma(z)\left(\frac{2}{\Phi(e_i, e_{\bar{z}}) + \Phi(e_{-i}, e_{\bar{z}})}\right) \\ &= (A_\infty - z)^{-1}(g) + \\ &- \frac{b}{a + bM(z)} \times \frac{4\langle g, e_{\bar{z}}\rangle}{\left(\Phi(e_i, e_{\bar{z}}) + \Phi(e_{-i}, e_{\bar{z}})\right)\left(\Phi(e_z, e_i) + \Phi(e_z, e_{-i})\right)} e_z.\end{aligned}$$

hence, the result follows now from lemma 2 and corollary 1. □

4. Generalized resolvents

4.1. **Nevanlinna families and exit space extensions.** So far, all considered extensions are canonical (e.g. act in the same space that operator A). In this section, investigations concern selfadjoint (exit space) extensions, that is selfadjoint extensions acting in a wider space. Except theorem 8, all following definitions and statements are taken from [7, 8].

Definition 3. *A family of linear relations* $\tau(\lambda), \lambda \in \mathbb{C}\backslash\mathbb{R}$, *on Hilbert space* $\mathcal{X}$ *is called a* Nevanlinna family *if;*

(1) *for every* $\lambda \in \mathbb{C}_+(\mathbb{C}_-)$, *the relation* $\tau(\lambda)$ *is maximal dissipative (accumulative respectively),*
(2) $\tau(\lambda)^* = \tau(\bar{\lambda})\,; \lambda \in \mathbb{C}\backslash\mathbb{R}$,
(3) *for some (and hence all)* $\mu \in \mathbb{C}_+(\mathbb{C}_-)$, *the operator family* $(\tau(\lambda)+\mu)^{-1} \in (\mathcal{B}(\mathcal{X}))$ *is holomorphic for all* $\lambda \in \mathbb{C}_+(\mathbb{C}_-)$.

By the maximality condition, each relation $\tau(\lambda)$,$\lambda \in \mathbb{C}\backslash\mathbb{R}$ is necessarily closed.

The class of Nevanlinna families in $\mathcal{X}$ is denoted by $\tilde{R}(\mathcal{X})$.

Definition 4. *A pair* $\{\Phi_1, \Phi_2\}$ *of holomorphic* $\mathcal{B}(\mathcal{X})$*−valued functions on* $\mathbb{C}_+\cup\mathbb{C}_-$ *is said to be a* Nevanlinna pair *if,*

a) $\dfrac{Im\left(\Phi_1(\lambda), \Phi_2^*(\lambda)\right)}{Im\lambda} \geq 0; \lambda \in \mathbb{C}_+ \cup \mathbb{C}_-$,
b) $\Phi_2^*\left(\bar{\lambda}\right)\Phi_1(\lambda) - \Phi_1^*\left(\bar{\lambda}\right)\Phi_2(\lambda) = 0; \lambda \in \mathbb{C}_+ \cup \mathbb{C}_-$,
c) $0 \in \rho\left(\Phi_2(\lambda) \pm i\Phi_1(\lambda)\right), \lambda \in \mathbb{C}_\pm$.

Note that Nevanlinna families and Nevanlinna pairs are connected via the formula

$$\tau(\lambda) = \{(\Phi_1(\lambda)(x), \Phi_2(\lambda)(x)) : x \in \mathcal{X}\} \tag{4.1}$$

Let A be a symmetric operator on a Hilbert space $\mathcal{H}$ with equal defect numbers. Let $\widetilde{A}$ be a selfadjoint extension of A acting on a Hilbert space $\widetilde{\mathcal{H}}$ which contains $\mathcal{H}$ as a closed subspace. Denote by $P_{\mathcal{H}}$ the orthoprojector of $\widetilde{\mathcal{H}}$ onto $\mathcal{H}$. The compression $R_\lambda = P_{\mathcal{H}}\left.(A-\lambda)^{-1}\right|_{\mathcal{H}}$ of the resolvent of $\widetilde{A}$ to $\mathcal{H}$ is said to be *a generalized resolvent of* A.

For the studied Carleman operator, one has the following result,

Theorem 8. *Let $\widetilde{A}$ be an exit space selfadjoint extension of A. Then, there exists a Nevanlinna pair $\{\Phi_1, \Phi_2\}$ of holomorphic scalar functions such that for every $\lambda \in \rho(\widetilde{A}) \cap \rho(A_\infty)$; $\Phi_2(\lambda) + M(\lambda)\Phi_1(\lambda) \neq 0$ and*

$$\begin{aligned} R_\lambda =& {}_{p=0}^{+\infty} \frac{C_p}{a_p - z}\Psi_p + C(e_i - e_{-i}) - \frac{1}{z} g_\Psi^\perp \\ &- \frac{\Phi_1(\lambda)}{\Phi_2(\lambda) + M(\lambda)\Phi_1(\lambda)} \times \frac{4\langle g, e_{\bar{z}}\rangle}{(\Phi(e_i, e_{\bar{z}}) + \Phi(e_{-i}, e_{\bar{z}}))(\Phi(e_z, e_i) + \Phi(e_z, e_{-i}))} e_z, \end{aligned}$$

where constants $g_\Psi^\perp, C, C_p, p = 0, 1, 2, \ldots$ are defined from lemma 2.

Proof. Suppose firstly that $R_\lambda = (A_\infty - \lambda)^{-1}$. In this case, one can take as a Nevanlinna pair $\{\Phi_1, \Phi_2\} = \{0, \Phi_2\}$; where Φ_2 is a holomorphic function on $\mathbb{C}_+ \cup \mathbb{C}_-$ such that $\Phi_2(\lambda) \neq 0$.

Suppose now that $R_\lambda \neq (A_\infty - \lambda)^{-1}$. According to [5, 14], there exists a Nevanlinna function $\tau(.) \in \tilde{R}(\mathcal{X})$ such that the Krein-Naïmark formula

$$R_\lambda = (A_\infty - \lambda)^{-1} - \gamma(\lambda)(\tau(\lambda) + M(\lambda))^{-1}\gamma^*(\bar{\lambda}) \tag{4.2}$$

holds. $M(\lambda)$ and $\gamma(\lambda)$ are the Weyl function and the $\gamma-$field defined in preceding section. Let $\{\Phi_1, \Phi_2\}$ be a Nevanlinna pair satisfying (4.1). Then,

$$\begin{aligned} (\tau(\lambda) + M(\lambda))^{-1} &= \{(\Phi_1(\lambda)x, (\Phi_2(\lambda) + M(\lambda)\Phi_1(\lambda))x) : x \in \mathbb{C}\}^{-1} \\ &= \{((\Phi_2(\lambda) + M(\lambda)\Phi_1(\lambda))x, \Phi_1(\lambda)x) : x \in \mathbb{C}\} \end{aligned}$$

Remark that $\Phi_2(\lambda) + M(\lambda)\Phi_1(\lambda) \neq 0$. Indeed;

$$\begin{aligned} \Phi_2(\lambda) + M(\lambda)\Phi_1(\lambda) &= 0 \wedge (\tau(\lambda) + M(\lambda))^{-1} (\in \mathcal{B}(\mathbb{C})) \\ &\Longrightarrow \Phi_1(\lambda) = 0 \\ &\Longrightarrow \Phi_2(\lambda) = (\tau(\lambda) + M(\lambda))^{-1} = 0 \\ &\Longrightarrow R_\lambda = (A_\infty - \lambda)^{-1}. \end{aligned}$$

Consequently,

$$\begin{aligned} (\tau(\lambda) + M(\lambda))^{-1} &= \left\{\left(x, \frac{\Phi_1(\lambda)}{\Phi_2(\lambda) + M(\lambda)\Phi_1(\lambda)}x\right) : x \in \mathbb{C}\right\} \\ &= \frac{\Phi_1(\lambda)}{\Phi_2(\lambda) + M(\lambda)\Phi_1(\lambda)}.I_{\mathbb{C}}. \end{aligned}$$

Replacing in formula (4.2) and using the same reasonning as in proposition 5, one can easily obtain the expected result. □

Remark 3. *The canonical selfadjoint extension case* (*proposition 5*) *can be derived from the precedent theorem by setting* $\{\Phi_1, \Phi_2\} = \{b, a\}$

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