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COMPACT MULTIPLICATION OPERATORS ON NONLOCALLY CONVEX WEIGHTED SPACES OF CONTINUOUS FUNCTIONS

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Abstract. Let V be a system of weights on a completely regular Hausdorff space and let $B(E)$ be the topological vector space of all continuous linear operators on a Hausdorff topological vector space E. Let $CV_0(X,E)$ and $CV_b(X,E)$ be the nonlocally convex weighted spaces of continuous functions. In the present paper, we characterize compact multiplication operators M_ψ on CV_0 (X,E) (or $CV_b(X,E)$) induced by the operator-valued mappings $\psi : X \to B(E)$ (or the vector-valued mappings $\psi : X \to E$, where E is a topological algebra).

1. Introduction

The theory of multiplication operators has extensively been studied during the last three decades on different function spaces. Many authors like Abrahamse [1] , Axler [6] , Halmos [12] , Singh and Kumar [35] , Takagi and Yokouchi [45] have studied these operators on $L^P - space$s, whereas Arazy [4] , Axler [5] , Bonet, Domanski and Lindström [9] , Shields and Williams [34] , Feldman [10] , Ghatage and Sun [11] , Stegenga [42] , and Vukotic [46] have explored these operators on spaces of analytic functions. Also, a study of these operators on weighted spaces of continuous functions has been made by Singh and Manhas [36, 37, 38, 39, 40] , Manhas and Singh [24] , Manhas [21, 22, 23] , Khan and Thaheem [17, 18] , Alsulami and Khan [2, 3] , and Oubbi [29] , In this paper, we have made efforts to characterize compact multiplication operators on the nonlocally convex weighted spaces of continuous functions generalizing some of the results of the author [22, 23] and Alsulami and Khan [2].

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2. Preliminaries

Throughout this paper, we shall assume, unless stated otherwise, that X is a completely regular Hausdorff space and E is a non-trivial Hausdorff topological vector space with a base $\mathcal{N}$ of closed balanced shrinkable neighbourhoods of zero. A neighbourhood G of zero in E is called shrinkable [19] if $t\bar{G} \subseteq intG$, for $0 \leq t < 1$. It is proved by Klee [19, Theorem 4 and Theorem 5] that every Hausdorff topological vector space has a base of shrinkable neighbourhoods of zero, and also the Minkowski functional ρ_G of any such neighbourhood G is continuous and satisfies

$$\bar{G} = \{y \in E : \rho_G(y) \leq 1\}, \ \text{int}G = \{y \in E : \rho_G(y) < 1\}.$$

Let $C(X,E)$ be the vector space of all continuous E-valued functions on X. Let V be a set of non-negative upper semicontinuous functions on X. Then V is said to be directed upward if for given $u, v \in V$ and $\alpha \geq 0$, there exists $w \in V$ such that $\alpha u, \alpha v \leq w$ (pointwise). A directed upward set V is called a system of weights if for each $x \in X$, there exists $v \in V$ such that $v(x) > 0$. Let U and V be two systems of weights on X. Then we say that $U \leq V$ if for every $u \in U$, there exists $v \in V$ such that $u \leq v$. Now, for a given system of weights V, we define

$$CV_0(X,E) = \{f \in C(X,E) : \ vf \text{ vanishes at infinity on } X \text{ for each } v \in V\},$$

and

$$CV_b(X,E) = \{f \in C(X,E) : \ vf \ (X) \text{ is bounded in } E \text{ for each } v \in V\}.$$

Clearly $CV_0(X,E) \subseteq CV_b(X,E)$. When $E\,(= \mathbb{R}$ or $\mathbb{C})$, the above spaces are denoted by $CV_0(X)$ and $CV_b(X)$. The weighted topology on $CV_b(X,E)\,(resp. CV_0(X,E))$ is defined as the linear topology which has a base of neighbourhoods of zero consisting of all sets of the form

$$N(v,G) = \{f \in CV_b(X,E) : vf(X) \subseteq G\},$$

where $v \in V$ and $G \in \mathcal{N}$.

With this topology, the vector space $CV_b(X,E)\,(resp. CV_0(X,E))$ is called the weighted space of vector-valued continuous functions which is not necessarily locally convex. For more details on these weighted spaces, we refer to $[13, 14, 15, 16, 19, 27]$. In case E is a locally convex space, a detailed information can be found in $[7, 8, 25, 26, 30, 31, 32, 33, 43, 44.]$. Let $B(E)$ be the vector space of all continuous linear operators on E. We denote by $\mathcal{B}$, the family of all bounded subsets of E. For each $B \in \mathcal{B}$ and $G \in \mathcal{N}$, we define the set

$$W(B,G) = \{T \in B(E) : T(B) \subseteq G\}.$$

Then clearly $B(E)$ is a topological vector space with a linear topology which has a base of neighbourhoods of zero consisting of all sets of the form $W(B,G)$. This topology is known as the topology of uniform convergence on bounded subsets of E.

By a topological algebra E we mean a topological vector space which is also an algebra such that multiplication in E is separately continuous. Multiplication in E is said to be left (right) hypocontinuous if for each $G \in \mathcal{N}$ and $B \in \mathcal{B}$, there exists $H \in \mathcal{N}$ such that $BH \subseteq$ $(resp.\ HB \subseteq G)$. In case E is equipped with both left and right hypocontinuous multiplication, we call E as a topological algebra with hypocontinuous multiplication. Clearly every topological algebra with joint continuous multiplication is always a topological algebra with hypocontinuous multiplication. For more details on these algebras, we refer to Mallios [20].

For the mapping $\psi : X \to B(E)$ (or $\psi : X \to E$, E as a topological algebra), we define the linear map $M_\psi : CV_0(X,E) \to F(X,E)$ by $M_\psi(f) = \psi.f$, for every $f \in CV_0(X,E)$, where $F(X,E)$ denotes the vector space of all functions from X into E and the product $\psi.f$ is defined pointwise on X as $(\psi.f)(x) = \psi_x(f(x))$ (or $(\psi.f)(x) = \psi(x)(f(x))$, for every $x \in X$. In case M_ψ takes $CV_0(X,E)$ into itself and is continuous, we call M_ψ, the multiplication operator on $CV_0(X,E)$ induced by the mapping ψ.

3. Compact Multiplication Operators

Throughout this section, we shall assume that for each $x \in X$, there exists $f \in CV_0(X)$ such $f(x) \neq 0$. In case X is locally compact Hausdorff space this condition is automatically satisfied.

In order to present the desired results, we need to record some definitions and results as follows.

Let $T \in B(E)$. Then T is said to be compact if it maps bounded subsets of E into relatively compact subsets of E. A completely regular Hausdorff space X is called a $K_{\mathbb{R}} -$ *space* if a function $f : X \to \mathbb{R}$ is continuous if and only if $f \mid K$ is continuous for each compact subset K of X. Clearly all locally compact or metrizable spaces are $K_{\mathbb{R}} -$ *spaces*. A completely regular Hausdorff space X is said to be a $V_{\mathbb{R}} -$ *space* with respect to a given system of weights V on X if a function $f : X \to \mathbb{R}$ is necessarily continuous whenever, for each $v \in V$, the restriction of f to $\{x \in X : v(x) \geq 1\}$ is continuous. Also, if $V_1 \leq V_2$ for two systems of weights on X, then of course any $(V_1)_{\mathbb{R}} -$ *space* is again a $(V_2)_{\mathbb{R}} -$ *space*. For more details on $V_{\mathbb{R}} -$ *spaces*, we refer to Bierstedt [8].

A subset $H \subseteq CV_0(X,E)$ is called equicontinuous at $x_0 \in X$ if for every neighbourhood G of zero in E, there exists a neighbourhood N of x_0 in X such that $f(x) - f(x_0) \in G$, for every $x \in N$ and $f \in H$. If H is equicontinuous at every point of X, then we say that H is equicontinuous on X. Moreover, using nets, we say that a subset $H \subseteq CV_0(X,E)$ is equicontinuous on X if and only if for every $x \in X$ and for every net $x_\alpha \to x$ in X,

$$\sup\{\rho_G(f(x_\alpha) - f(x)) : f \in H\} \to 0, \quad \text{for every } G \in \mathcal{N}.$$

The following generalized Arzela-Ascoli type theorem and related results can be found in Khan and Oubbi [16].

Theorem 3.1. *Let X be a completely regular Hausdorff $V_{\mathbb{R}} -$ space and let E be a quasi-complete Hausdorff topological vector space. Then a subset $M \subseteq CV_0(X,E)$ is relatively compact if and only if*

(i) *M is equicontinuous;*
(ii) *$M(x) = \{f(x) : f \in M\}$ is relatively compact in E, for each $x \in X$;*
(iii) *vM vanishes at infinity on X for each $v \in V$ (i,e., for each $v \in V$ and $G \in N$, there exists a compact set $K \subseteq X$ such that $v(x) f(x) \in G$, for all $f \in M$ and $x \in X \backslash K$).*

Corollary 3.2. *Let X be a locally compact Hausdorff space and let E be a quasi-complete Hausdorff topological vector space. Let V be a system of constant weights on X. Then a subset $M \subseteq CV_0(X,E)$ is relatively compact if and only if*

(i) *M is equicontinuous;*
(ii) *$M(x) = \{f(x) : f \in M\}$ is relatively compact in E, for each $x \in X$;*
(iii) *M uniformly vanishes at infinity on X (i,e., for every $G \in N$, there exists a compact set $K \subseteq X$ such that $f(x) \in G$, for all $f \in M$ and $x \in X \backslash K$).*

Remark. *Theorem 3.2 and Corollary 3.7 of [24] are proved for a completely regular Hausdorff K_R − space X.But with slight modification in the proofs both the results are still valid if we take X as a completely regular Hausdorff $V_{\mathbb{R}}$ − space.*

Now we are ready to present the characterization of compact multiplication operators on $CV_0(X,E)$.

Theorem 3.3. *Let X be a completely regular Hausdorff $V_{\mathbb{R}}$ − space and let E be a non-zero quasi-complete Hausdorff topological vector space. Let $\psi : X \to B(E)$ be an operator-valued mapping. Then $M_\psi : CV_0(X,E) \to CV_0(X,E)$ is a compact multiplication operator if the following conditions are satisfied:*

(i) *$\psi : X \to B(E)$ is continuous in the topology of uniform convergence on bounded subsets of E;*
(ii) *for every $v \in V$ and $G \in N$, there exist $u \in V$ and $H \in N$, such that $u(x)\,y \in H$ implies that $v(x)\,\psi_x(y) \in G$, for every $x \in X$ and $y \in E$;*
(iii) *for every $x \in X$, $\psi(x)$ is a compact operator on E;*
(iv) *$\psi : X \to B(E)$ vanishes at infinity uniformly on X, i,e., for each $G \in N$ and $B \in B$, there exists a compact set $K \subseteq X$ such that $\psi_x(B) \subseteq G$, for every $x \in X \backslash K$;*
(v) *for every bounded set $F \subseteq CV_0(X,E)$, the set $\{\psi_x o f : f \in F\}$ is equicontinuous for every $x \in X$.*

Proof. According to [24, Corollary 3.7] and Remark 1, conditions (i) and (ii) imply that M_ψ is a multiplication operator on $CV_0(X,E)$. Let $S \subseteq CV_0(X,E)$ be a bounded set. To prove that M_ψ is a compact operator, it is enough to show that the set $M_\psi(S)$ satisfies all the conditions of Theorem 1. Fix $x_0 \in X$. We shall verify that the set $M_\psi(S)$ is equicontinuous at x_0. Let $G \in \mathcal{N}$. Then there exists $H \in \mathcal{N}$ such that $H + H \subseteq G$. Choose $v \in V$ such that $v(x_0) \geq 1$. Let $F_v = \{x \in X : v(x) > 1\}$. Consider the set $B = \{f(x) : x \in F_v,\ f \in S\}$. Clearly the set B is bounded in E. By condition (i), there exists a neighbourhood K_1 of x_0 such that $\psi_x - \psi_{x_0} \in W(B,H)$, for every $x \in K_1$. Further, it implies that $\psi_x(f(x)) - \psi_{x_0}(f(x)) \in H$, for every $x \in K_1 \cap F_v$ and $f \in S$. Again, by condition (v), there exists a neighbourhood K_2 of x_0 such that $\psi_{x_0}(f(x) - f(x_0)) \in H$, for every $x \in K_2$ and $f \in S$. Let $N = K_2 \cap K_1 \cap F_v$. Then for every $x \in N$ and $f \in S$, we have

$$\begin{aligned} \psi_x(f(x)) - \psi_{x_0}(f(x_0)) &= \psi_x(f(x)) - \psi_{x_0}(f(x)) + \psi_{x_0}(f(x)) - \psi_{x_0}(f(x_0)) \\ &\in H + H \subseteq G. \end{aligned}$$

This proves the equicontinuity of the set $M_\psi(S)$ at x_0 and hence it is equicontinuous on X. This established the condition (i) of Theorem 1. To prove condition (ii) of Theorem 1, we shall show that the set $M_\psi(S)(x_0)$ is relatively compact in E for each $x_0 \in X$. Since the set $B = \{f(x_0) :\ f \in S\}$ is bounded in E and ψ_{x_0} is compact operator on E, by condition (iii), it follows that the set $\psi_{x_0}(B) = M_\psi(S)(x_0)$ is relatively compact in E. Finally we shall establish condition (iii) of Theorem 1 by showing that the set $vM_\psi(S)$ vanishes at infinity on X for each $v \in V$. Fix $v \in V$ and $G \in \mathcal{N}$. Since the set $B = \{v(x) f(x) : x \in X,\ f \in S\}$ is bounded in E, according to Condition (iv), there exists a compact set $K \subseteq X$ such that $\psi_x(B) \subseteq G$, for every $x \in X \backslash K$. That is, $v(x)\psi_x(f(x)) \in G$, for every $x \in X\backslash K$ and $f \in S$. This proves that the set $vM_\psi(S)$ vanishes at infinity on X. With this the proof of the theorem is complete. □

Theorem 3.4. *Let X be a completely regular Hausdorff $V_{\mathbb{R}}-$ space and let E be a non-zero quasi-complete Hausdorff topological vector space. Let U be a system of constant weights on X such that $\cup \leq V$. Let $\psi : X \to B(E)$ be an operator-valued mapping. Then conditions (i) through (v) in Theorem 3 are necessary and sufficient for M_ψ to be a compact multiplication operator on $CV_0(X,E)$.*

Proof. We suppose that M_ψ is a compact multiplication operator on $CV_0(X,E)$. To prove condition (i), we fix $x_0 \in X$, $B \in \mathcal{B}$ and $G \in \mathcal{N}$. Let $v \in V$ and $f \in CV_0(X)$ be such that $v(x_0) \geq 1$ and $f(x_0) = 1$. Let $K_1 = \{x \in X : v(x)|f(x)| \geq 1\}$. Then K_1 is a compact subset of X such that $x_0 \in K_1$. According to [26, Lemma 2, p. 69], there exists $h \in CV_0(X)$ such that $h(K_1) = 1$. For each $y \in B$, we define the function $g_y : X \to E$ as $g_y(x) = h(x)y$, for every $x \in X$. If we put $F = \{g_y : y \in B\}$, then F is clearly bounded in $CV_0(X,E)$ and hence the set $M_\psi(F)$ is relatively compact in $CV_0(X,E)$. According to Theorem 1, the set $M_\psi(F)$ is equicontinuous at x_0. This means that there exits a neighbourhood K_2 of x_0 such that $\psi_x(g_y(x)) - \psi_{x_0}(g_y(x_0)) \in G$, for every $x \in K_2$ and $y \in \mathcal{B}$. Let $K = K_1 \cap K_2$. Then we have $\psi_x(y) - \psi_{x_0}(y) \in G$, for every $x \in K$ and $y \in \mathcal{B}$. This shows that $\psi_x - \psi_{x_0} \in W(B,G)$, for every $x \in K$. This proves that $\psi : X \to B(E)$ is continuous at x_0 and hence on X. In view of Remark 1, the proof of condition (ii) follows from Corollary 3.7 of [24]. To establish condition (iii), let $x_0 \in X$. We select $f \in CV_0(X)$ such that $f(x_0) = 1$. Let $B \in \mathcal{B}$. Then for each $y \in B$, we define the function $h_y : X \to E$ as $h_y(x) = f(x)y$, for every $x \in X$. Clearly the set $S = \{h_y : y \in B\}$ is bounded in $CV_0(X,E)$ and hence the set $M_\psi(S)$ is relatively compact in $CV_0(X,E)$. Again, according to Theorem 1, it follows that the set $M_\psi(S)(x_0) = \{\psi_{x_0}(y) : y \in B\}$ is relatively compact in E. This proves that ψ_{x_0} is a compact operator on E. Now, to prove condition (iv), we suppose that $\psi : X \to B(E)$ does not vanishes at infinity on X. This implies that there exist $G \in \mathcal{N}$ and $B \in \mathcal{B}$ such that for every compact set $K \subseteq X$, there exists $x_k \in X\backslash K$ for which $\psi_{x_k}(B) \subsetneq G$. Further, it implies that there exists $y_k \in B$ such that $\psi_{x_k}(y_k) \notin G$. According to [41, Lemma 3.1], there exists an open neighbourhood N_k of x_k such that each $v \in V$ is bounded on N_k. Let $O_k = N_k \cap X\backslash K$. Then O_k is an open neighbourhood of x_k for each compact set $K \subseteq X$. Further, according to [26, Lemma 2, p.69], there exists $f_k \in CV_0(X)$ such that $0 \leq f_k \leq 1$, $f_k(x_k) = 1$ and $f_k(X\backslash O_k) = 0$. For each compact $K \subseteq X$, we define the function $h_k : X \to E$ as $h_k(x) = f_k(x)y_k$, for every $x \in X$. Clearly the set $M = \{h_k : K \subseteq X,\ K \text{ compact subset}\}$ is bounded in $CV_0(X,E)$ and hence the set $M_\psi(M)$ is relatively compact in $CV_0(X,E)$. Since $U \leq V$, we can select $v \in V$ such that $v(x) \geq 1$, for every $x \in X$. Again, Theorem 1 implies that the set $vM_\psi(M)$ vanishes at infinity on X. This implies that there exists a compact set $K_0 \subseteq X$ such that $v(x)\psi_x(h_k(x)) \in G$, for all $h_k \in M$ and for every $x \in X\backslash K_0$. Since $v(x) \geq 1$, for all x, it follows that $\psi_x(f_{k_0}(x)y_{k_0}) \in G$, for every $\mathrm{x} \in X\backslash K_0$.

For $x = x_{k_0}$, we have $\psi_{x_{k_0}}(y_{k_0}) \in G$, which is a contradiction. This proves that $\psi : X \to B(E)$ vanishes at infinity on X. Finally, we shall prove condition (v). Let $F \subseteq CV_0(X,E)$ be a bounded set. Fix $x_0 \in X$ and $G \in \mathcal{N}$. Then there exists $H \in \mathcal{N}$ such that $H+H \subseteq G$. Clearly the set $B = \{f(x) : x \in X,\ f \in F)\}$ is bounded in E. Since $\psi : X \to B(E)$ is continuous at x_0, there exists a neighbourhood N_1 of x_0 in X such that $\psi_x(f(x)) - \psi_{x_0}(f(x)) \in H$, for every $x \in N_1$ and $f \in F$. Again, since the set $M_\psi(F)$ is relatively compact in $CV_0(X,E)$, according to Theorem 1, the set $M_\psi(F)$ is equicontinuous at x_0. This implies that there exists a neighbourhood N_2 of x_0 in X such that $\psi_x(f(x)) - \psi_{x_0}(f(x_0)) \in H$, for every $x \in N_2$ and $f \in F$. Let $N = N_1 \cap N_2$. Then for every $x \in N$ and $f \in F$, we have

$$\psi_{x_0}(f(x) - f(x_0)) = \psi_{x_0}(f(x)) - \psi_x(f(x)) + \psi_x(f(x)) - \psi_{x_0}(f(x_0)) \in H+H \subseteq G.$$

This proves condition (v). This completes the proof of the theorem as the sufficient part is already proved in Theorem 3. □

Theorem 3.5. *Let X be a completely regular Hausdorff $V_{\mathbb{R}}$ − space and let E be a quasi-complete Hausdorff topological algebra with hypocontinuous multiplication containing the unit element e. Let U be a system of constant weights on X such that $U \leq V$. Then the vector-valued mapping $\psi : X \to E$ induces a compact multiplication operator M_ψ on $CV_0(X,E)$ if and only if*

(i) *$\psi : X \to E$ is continuous;*

(ii) *for every $v \in V$ and $G \in N$, there exist $u \in V$ and $H \in N$ such that $u(x)\,y \in H$ implies that $v(x)\,\psi(x)\,y \in G$, for every $x \in X$ and $y \in E$;*

(iii) *for every $x \in X$, the operator $L_{\psi(x)} : E \to E$, defined by $L_{\psi(x)}(y) = \psi(x)\,y$, for every $y \in E$, is compact;*

(iv) *$\psi : X \to E$ vanishes at infinity on X;*

(v) *for every bounded set $F \subseteq CV_0(X,E)$, the set $\left\{L_{\psi(x)} o f : f \in F\right\}$ is equicontinuous for every $x \in X$.*

Proof. In [24, Theorem 3.2], Manhas and Singh have characterized the weighted composition operators $W_{\psi,\phi}$ on $CV_0(X,E)$ induced by the mappings $\phi : X \to X$ and $\psi : X \to E$. If we take $\phi : X \to X$ as the identity map, then Theorem 3.2 of Manhas and Singh [24] and Remark 1 implies that M_ψ is a multiplication operator on $CV_0(X,E)$ if and only if condition (i)-(ii) of Theorem 5 hold. Also, using similar arguments of Theorem 4, it can be shown that M_ψ is a compact operator on $CV_0(X,E)$. □

Corollary 3.6. *Let X be a locally compact Hausdorff space and let E be a non-zero quasi-complete Hausdorff topological vector space. Let V be a system of constant weights on X.Let $\psi : X \to B(E)$ be an operator-valued mapping. Then $M_\psi : CV_0(X,E) \to CV_0(X,E)$ is a compact multiplication operator if and only if the following conditions are satisfied:*

(i) *$\psi : X \to B(E)$ is continuous in the topology of uniform convergence on bounded subsets of E;*

(ii) *for every $G \in N$, there exists $H \in N$, such that $y \in H$ implies that $\psi_x(y) \in G$, for every $x \in X$ and $y \in E$;*

(iii) *for every $x \in X$, $\psi(x)$ is a compact operator on E;*

(iv) $\psi : X \to B(E)$ *vanishes at infinity uniformly on* X, *i,e., for each* $G \in N$ *and* $B \in B$, *there exists a compact set* $K \subseteq X$ *such that* $\psi_x(B) \subseteq G$, *for every* $x \in X \backslash K$;

(v) *for every bounded set* $F \subseteq CV_0(X, E)$, *the set* $\{\psi_x o f : f \in F\}$ *is equicontinuous for every* $x \in X$.

Proof. The proof follows from Theorem 4 after using Corollary 2 instead of Theorem 1. □

Remark. *(i) In case* E *is a quasi-complete locally convex Hausdorff space and* X *is a locally compact Hausdorff space, Theorem 4 reduces to [23, Theorem 3.4].*

(ii) In Corollary 6, if E *is a quasi-complete locally convex Hausdorff space, it reduces to Theorem 2.4 of Manhas [22].*

(iii) If X *is as* $V_{\mathbb{R}}-$*space without isolated points, then it is proved in [2, Corollary 4] that there is no non-zero compact multiplication operator* M_ψ *on* $CV_0(X, E)$. *But if* X *is a* $V_{\mathbb{R}} -$ *space with isolated points, then Theorem 4 provide (e.g. see Example 1 below) non-zero compact multiplication operators* M_ψ *on some weighted spaces* $CV_0(X, E)$ *whereas it is not the case with some of the* $L^p -$ *spaces and spaces of analytic functions. In [35], Singh and Kumar have shown that the zero operator is the only compact multiplication operator on* $L^p -$ *spaces (with non-atomic measure). In [9], Bonet Domanski and Lindstrom have shown that there is no non-zero compact multiplication operator on Weighted Banach Spaces of analytic functions. Also, recently, Ohno and Zhao [28] have proved that the zero operator is the only compact multiplication operator on Bloch Spaces.*

Example 3.1. *Let* $X = \mathbb{Z}$, *the set of integers with the discrete topology and let* $V = K^+(\mathbb{Z})$, *the set of positive constant functions on* $\mathbb{Z}$. *Let* $E = C_b(\mathbb{R})$ *be the Banach space of bounded continuous complex valued functions on* $\mathbb{R}$. *For each* $t \in \mathbb{Z}$, *we define an operator* $A_t : C_b(\mathbb{R}) \to C_b(\mathbb{R})$ *as* $A_t f(s) = f(t)$, *for every* $f \in C_b(\mathbb{R})$ *and for every* $s \in \mathbb{R}$. *Clearly, for each* $t \in \mathbb{Z}$, A_t *is a compact operator. Let* $\psi : \mathbb{Z} \to B(E)$ *be defined as* $\psi(t) = e^{-|t|} A_t$, *for* $t \in \mathbb{Z}$. *Then all the conditions of Corollary 6 are satisfied by the mapping* ψ *and hence* M_ψ *is a compact multiplication operator on* $C_0(\mathbb{Z}, E)$. *In case we take* $E = C(\mathbb{R})$ *with compact-open topology, then the mapping* $\psi : \mathbb{Z} \to B(E)$ *defined as above does not induces the compact multiplication operator* M_ψ *on* $C_0(\mathbb{Z}, E)$. *But, if* $E = C(\mathbb{R})$ *with compact-open topology and we define* $\psi : \mathbb{Z} \to B(E)$ *as* $\psi(t) = e^{-|t|} A_{t_0}$, *for every* $t \in \mathbb{Z}$, *where* A_{t_0} *is a fixed compact operator on* $C(\mathbb{R})$ *defined as* $A_{t_0} f(s) = f(t_0)$, *for every* $f \in C(\mathbb{R})$ *and for every* $s \in \mathbb{R}$, *then it turns out that* M_ψ *is a compact multiplication operator on* $C_0(\mathbb{Z}, E)$.

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