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ON SPACE LIKE SUBMANIFOLDS WITH $R = aH + b$ IN DE SITTER SPACE FORM $S_p^{n+p}(c)$

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Abstract. In this paper, we investigate n-dimensional complete spacelike sumbmanifolds M^n $(n \geq 3)$ with $R = aH + b$ in de Sitter space form $S_p^{n+p}(c)$. Some rigidity theorems are obtained for these spacelike submanifolds.

1. Introduction

A de Sitter space form $S_p^{n+p}(c)$ is an $(n+p)$-dimensional connected pseudo-Riemannian manifold of index p with constant sectional curvature $c > 0$. A submanifold immersed in $S_p^{n+p}(c)$ is said to be spacelike if the induced metric in M^n from the metric of the ambient space $S_p^{n+p}(c)$ is positive definite. Since Goddard's conjecture (see [6]), several papers about spacelike hypersurfaces with constant mean curvature in de Sitter space have been published. For the study of spacelike hypersurface with constant scalar curvature in de Sitter space, there are also many results such as [2, 9, 15, 16]. There are some results about space like sumanifolds with constant scalar curvature and higher codimension in de Sitter space form $S_p^{n+p}(c)$ such as [17]. Recently, F.E.C.Camargo, et al.[5] and Chao X.L. [4] obtained some interesting characters for space like submanifolds with parallel normalized mean vector(which is much weaker than the condition to have parallel mean curvature vector) in $S_p^{n+p}(c)$. In this note, we consider complete space like submanifolds with $R = aH + b$ in de Sitter space and we get the following results:

Theorem 1.1. *Let M^n $(n \geq 3)$ be a complete space like submanifold with $R = aH + b$, $(n-1)a^2 + 4nc - 4nb \geq 0$ and $a \geq 0$ in $S_p^{n+p}(c)$. If $S < 2\sqrt{(n-1)}c$, then M^n is totally umbilical.*

Theorem 1.2. *Let M^n $(n \geq 3)$ be a complete space like submanifold with $R = aH + b$, $(n-1)a^2 + 4nc - 4nb \geq 0$ and $a \geq 0$ in de Sitter space form $S_p^{n+p}(c)$. Suppose that M^n has bounded mean curvature H :*

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(1) If $\sup(H)^2 < \frac{4(n-1)}{(n-2)^2p+4(n-1)}c$*, then* $S = nH^2$ *and* M^n *is totally umbilical.*

(2) If $\sup(H)^2 = \frac{4(n-1)}{(n-2)^2p+4(n-1)}c$*, then either* $S = nH^2$ *and* M^n *is totally umbilical, or* $\sup(S) = nc\frac{(n-2)^2p^2+4(n-1)}{(n-2)^2p+4(n-1)}$*.*

(3-a) If $\sup(H)^2 > c > \frac{4(n-1)}{(n-2)^2p+4(n-1)}c$*, then either* $S = nH^2$ *and* M^n *is totally umbilical, or* $n \sup H^2 < \sup S \leq S^+$*.*

(3-b) If $\sup(H)^2 = c > \frac{4(n-1)}{(n-2)^2p+4(n-1)}c$*, then either* $S = nH^2$ *and* M^n *is totally umbilical, or* $n \sup H^2 < \sup S \leq S^+$*.*

(3-c) If $c > \sup(H)^2 > \frac{4(n-1)}{(n-2)^2p+4(n-1)}c$*, then either* $S = nH^2$ *and* M^n *is totally umbilical, or* $S^- \leq \sup S \leq S^+$*.*

Where $S^+ = \frac{n(n-2)}{2(n-1)}[\frac{2(n-2)^2p^2+4(n-1)(p+1)}{2(n-2)}\sup H^2 + \sup|H|p^2\sqrt{\frac{[(n-2)^2p+4(n-1)]supH^2-4(n-1)c}{p}}] - npc$*, and* $S^- = \frac{n(n-2)}{2(n-1)}[\frac{2(n-2)^2p^2+4(n-1)(p+1)}{2(n-2)}\sup H^2 - \sup|H|p^2\sqrt{\frac{[(n-2)^2p+4(n-1)]supH^2-4(n-1)c}{p}}] - npc$*.*

Remark. *(i) We take the parallel normalized mean curvature vector off from our theorems. (ii)From the conclusion (1) in theorem 1.2, we obtain the theorem 1.2 in* [4] *.*

Recently, the first author in [7] obtained an intrinsic inequality for space like submanifolds in $S_p^{n+p}(c)$,

Theorem 1.3. [7] *If* M^n $(n > 1)$ *is a complete space-like submanifold of indefinite space form* $M_p^{n+p}(c)(c > 0)$ $(p \geq 1)$*, Ric and R are Ricci curvature tensor and the normalized scalar curvature of* M^n*, respectively, then*

$$|Ric|^2 \geq 2cRn(n-1)^2 - c^2n(n-1)^2. \tag{1.1}$$

Moreover, $|Ric|^2 = 2cRn(n-1)^2 - c^2n(n-1)^2$ *if and only if* M^n *is a spacelike Einstein submanifolds with* $Ric_{ij} = c(n-1)g_{ij}$*, where g is the Riemannian metric of* M^n*.*

In this note, we also obtain the following result:

Theorem 1.4. *Let* M^n $(n \geq 3)$ *be a complete space-like submanifold of de Sitter space form* $S_p^{n+p}(c)$ $(p > 1)$*. If the mean curvature satisfies the following inequality:*

$$H^2 < \frac{c}{(n-1)(1-\frac{1}{p})}, \tag{1.2}$$

then $|Ric|^2 = 2cRn(n-1)^2 - c^2n(n-1)^2$ *if and only if* M^n *is totally geodesic, where Ric and R are Ricci curvature tensor and the normalized scalar curvature of* M^n*, respectively.*

2. Preliminaries

We choose a local field of semi-Riemannian orthonormal frames $\{e_1, \cdots, e_n, e_{n+1}, \cdots, e_{n+p}\}$ in $S_p^{n+p}(c)$ such that, restricted to M^n, $e_1, \cdots, e_n$ are tangent to M^n. Let $\omega_1, \cdots, \omega_{n+p}$ be its dual frame field such that the semi-Riemannian metric of $S_p^{n+p}(c)$ is given by $ds^2 = \sum_{A=1}^{n+p} \epsilon_A(\omega_A)^2$, where $\epsilon_i = 1$, $i = 1, \cdots, n$ and $\epsilon_\alpha = -1$,

$\alpha = n+1, \cdots, n+p$. Then the structure equations of $S_p^{n+p}(c)$ are given by

$$d\omega_A = -\sum_B \epsilon_B \omega_{AB} \wedge \omega_B, \quad \omega_{AB} + \omega_{BA} = 0, \tag{2.1}$$

$$d\omega_{AB} = -\sum_C \epsilon_C \omega_{AC} \wedge \omega_{CB} + \frac{1}{2}\sum_{CD} K_{ABCD}\omega_C \wedge \omega_D, \tag{2.2}$$

$$K_{ABCD} = c\epsilon_A\epsilon_B(\delta_{AC}\delta_{BD} - \delta_{AD}\delta_{BC}). \tag{2.3}$$

We restrict these forms to M^n, then

$$\omega_\alpha = 0, \quad \alpha = n+1, \cdots, n+p, \tag{2.4}$$

and the Riemannian metric of M^n is written as $ds^2 = \sum_i \omega_i^2$. Since

$$0 = d\omega_\alpha = -\sum_i \omega_{\alpha,i} \wedge \omega_i, \tag{2.5}$$

by Cartan's lemma we may write

$$\omega_{\alpha,i} = \sum_j h_{ij}^\alpha \omega_j, \quad h_{ij}^\alpha = h_{ji}^\alpha. \tag{2.6}$$

From these formulas, we obtain the structure equations of M^n:

$$d\omega_i = -\sum_j \omega_{ij} \wedge \omega_j, \quad \omega_{ij} + \omega_{ji} = 0,$$

$$d\omega_{ij} = -\sum_k \omega_{ik} \wedge \omega_{kj} + \frac{1}{2}\sum_{k,l} R_{ijkl}\omega_k \wedge \omega_l,$$

$$R_{ijkl} = c(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}) - (h_{ik}^\alpha h_{jl}^\alpha - h_{il}^\alpha h_{jk}^\alpha), \tag{2.7}$$

where R_{ijkl} are the components of curvature tensor of M^n. We call

$$B = \sum_{i,j,\alpha} h_{ij}^\alpha \omega_i \otimes \omega_j \otimes e_\alpha \tag{2.8}$$

the second fundamental form of M^n. The mean curvature vector is $h = \frac{1}{n}\sum_{i,\alpha} h_{ii}^\alpha e_\alpha = \sum_\alpha \sigma^\alpha e_\alpha$, where $\sigma^\alpha = \frac{1}{n}\sum_i h_{ii}^\alpha$. We denote $S = \sum_{i,j,\alpha}(h_{ij}^\alpha)^2$, and $H^2 = |h|^2$. We call that M^n is maximal if its mean curvature field vanishes, i.e. $h = 0$.

Let $h_{ij,k}^\alpha$ and $h_{ij,kl}^\alpha$ denote the covariant derivative and the second covariant derivative of h_{ij}^α. Then we have $h_{ij,k}^\alpha = h_{ik,j}^\alpha$ and

$$h_{ij,kl}^\alpha - h_{ij,lk}^\alpha = -\sum_m h_{im}^\alpha R_{mjkl} - \sum_m h_{mj}^\alpha R_{mikl} - \sum_\beta h_{ij}^\beta R_{\alpha\beta kl}, \tag{2.9}$$

where $R_{\alpha\beta kl}$ are the components of the normal curvature tensor of M^n.

$$R_{\alpha\beta kl} = -\sum_i (h_{ik}^\alpha h_{il}^\beta - h_{ik}^\beta h_{il}^\alpha), \tag{2.10}$$

$$Ric_{ik} = (n-1)c\delta_{ik} - \sum_\alpha(\sum_l h_{ll}^\alpha)h_{ik}^\alpha + \sum_l h_{il}^\alpha h_{lk}^\alpha, \tag{2.11}$$

$$n(n-1)R = n(n-1)c + S - n^2H^2, \tag{2.12}$$

where R is the normalized scalar curvature.

The Laplacian $\triangle h_{ij}^{\alpha}$ of h_{ij}^{α} is defined by $\triangle h_{ij}^{\alpha} = \sum_k h_{ijkk}^{\alpha}$. From (2.9) we have

$$\triangle h_{ij}^{\alpha} = \sum_k h_{kkij}^{\alpha} - \sum h_{km}^{\alpha} R_{mijk} - \sum h_{mi}^{\alpha} R_{mkjk} - \sum h_{ki}^{\beta} R_{\alpha\beta jk}. \tag{2.13}$$

Now, we assume $H > 0$. We choose $e_{n+1} = \frac{h}{H}$. Hence

$$tr(H^{n+1}) = nH, \quad tr(H^{\alpha}) = 0, \quad \alpha \neq n+1, \tag{2.14}$$

where H^{α}denote the matrix (h_{ij}^{α}). Let us define

$$\Phi_{ij}^{n+1} = h_{ij}^{n+1} - H\delta_{ij}, \quad \Phi_{ij}^{\alpha} = h_{ij}^{\alpha}, \quad \alpha \neq n+1. \tag{2.15}$$

Therefore

$$\Phi^{n+1} = H^{n+1} - HI, \quad \Phi^{\alpha} = H^{\alpha}, \quad \alpha \neq n+1. \tag{2.16}$$

where Φ^{α} denotes the matrix (Φ_{ij}^{α}). Then

$$|\Phi^{n+1}|^2 = tr(H^{n+1})^2 - nH^2, \quad \sum_{\alpha \neq n+1} |\Phi^{\alpha}|^2 = \sum_{\alpha \neq n+1} (h_{ij}^{\alpha})^2, \quad tr(\Phi^{\beta}) = 0, \tag{2.17}$$

for $\forall \beta \in \{n+1, \cdots, n+p\}$. Thus

$$|\Phi|^2 = \sum_{\alpha=n+1}^{n+p} |\Phi^{\alpha}|^2 = S - nH^2. \tag{2.18}$$

Set $S_1 = tr(H^{n+1})^2$ and $S_2 = \sum_{\alpha \neq n+1} (h_{ij}^{\alpha})^2$, so $S = S_1 + S_2$, where S_1, S_2 are well defined on M.

By replacing (2.7) (2.10) and (2.14) into (2.13), we get the following equations:

$$\begin{aligned} \triangle h_{ij}^{n+1} \quad = \quad & nch_{ij}^{n+1} - nHc\delta_{ij} + nH_{ij} + \sum_{\beta,k,m} h_{km}^{n+1} h_{mk}^{\beta} h_{ij}^{\beta} - 2 \sum_{\beta,k,m} h_{km}^{n+1} h_{mj}^{\beta} h_{ik}^{\beta} \\ & + \sum_{\beta,k,m} h_{mi}^{n+1} h_{mk}^{\beta} h_{kj}^{\beta} - nH \sum_m h_{mi}^{n+1} h_{mj}^{n+1} + \sum_{\beta,k,m} h_{jm}^{n+1} h_{mk}^{\beta} h_{ki}^{\beta} \end{aligned} \tag{2.19}$$

and

$$\begin{aligned} \triangle h_{ij}^{\alpha} \quad = \quad & nch_{ij}^{\alpha} - nHc\delta_{ij} + nH_{ij} + \sum_{\beta,k,m} h_{km}^{\alpha} h_{mk}^{\beta} h_{ij}^{\beta} - 2 \sum_{\beta,k,m} h_{km}^{\alpha} h_{mj}^{\beta} h_{ik}^{\beta} \\ & + \sum_{\beta,k,m} h_{mi}^{\alpha} h_{mk}^{\beta} h_{kj}^{\beta} - nH \sum_m h_{mi}^{\alpha} h_{mj}^{n+1} + \sum_{\beta,k,m} h_{jm}^{\alpha} h_{mk}^{\beta} h_{ki}^{\beta}. \end{aligned} \tag{2.20}$$

Since

$$\frac{1}{2}\triangle S = \sum_{\alpha,ijk} (h_{ijk}^{\alpha})^2 + \sum_{\alpha,ij} h_{ij}^{\alpha} \triangle h_{ij}^{\alpha}, \tag{2.21}$$

from (2.19) and (2.20), we have that

$$\begin{aligned} \frac{1}{2}\triangle S \quad = \quad & \sum_{\alpha,ijk} (h_{ijk}^{\alpha})^2 + h_{ij}^{n+1}(nH)_{ij} + ncS - n^2cH^2 - nH \sum_{\alpha} tr(H^{n+1}(H^{\alpha})^2) \\ & + \sum_{\alpha,\beta} [tr(H^{\alpha}H^{\beta})]^2 + \sum_{\alpha,\beta} N(H^{\alpha}H^{\beta} - H^{\beta}H^{\alpha}), \end{aligned} \tag{2.22}$$

where $N(A) = trAA'$ for all matrix $A = (a_{ij})$.

In this note we consider the spacelike submanifolds with $R = aH + b$ in de Sitter space from $S_p^{n+p}(c)$, where a, b are real constants. Following Cheng-Yau [3], Chao X.L. in [4] introduced a modified operator acting on any C^2-function f by

$$L(f) = \sum_{ij}(nH\delta_{ij} - h_{ij}^{n+1})f_{ij} + \frac{n-1}{2}a\Delta f. \tag{2.23}$$

We need the following Lemmas.

Lemma 2.1. [4]*Let M^n be a spacelike submanifolds in $S_p^{n+p}(c)$ with $R = aH + b$, and $(n-1)a^2 + 4nc - 4nb \geq 0$. We have the following:*

(1)

$$\sum (h_{ijk}^{\alpha})^2 \geq n^2|\nabla H|^2. \tag{2.24}$$

(2)If the mean curvature H of M^n is bounded and $a \geq 0$, then there is a sequence of points $\{x_k\} \in M^n$ such that

$$\lim_{k\to\infty} nH(x_k) = \sup(nH), \quad \lim_{k\to\infty} |\nabla nH(x_k)| = 0, \quad \lim_{k\to\infty} \sup(L(nH)(x_k)) \leq 0 \tag{2.25}$$

Lemma 2.2. [1, 11] *Let $\mu_1, \cdots, \mu_n$ be real numbers such that $\sum_i \mu_i = 0$ and $\sum_i \mu_i^2 = \beta^2$, where $\beta \geq 0$ is constant. Then*

$$|\sum_i \mu_i^3| \leq \frac{n-2}{\sqrt{n(n-1)}}\beta^3, \tag{2.26}$$

and equality holds if and only if at least $n-1$ of μ_i's are equal.

Lemma 2.3. [13] *Let x_i, y_i $i = 1, \cdots, n$, be the real numbers such that $\sum_i x_i = 0 = \sum_i y_i$. Then*

$$|\sum_i x_i^2 y_i| \leq \frac{n-2}{\sqrt{n(n-1)}}(\sum_i x_i^2)(\sum_i y_i^2)^{\frac{1}{2}}. \tag{2.27}$$

Lemma 2.4. [7] *Let $a_1, \cdots, a_n$ and $b_1, \cdots, b_n$ be real numbers such that $\sum_i b_i = 0$. Then*

$$\sum_{ij} a_i a_j (b_i - b_j)^2 \leq \frac{n}{\sqrt{n-1}}(\sum_i a_i^2)(\sum_i b_i^2). \tag{2.28}$$

3. Proof of the theorems

First, we prove the following algebraic lemmas,

Lemma 3.1. *Let A, B be two real symmetric matrixes such that $trA = trB = 0$. Then*

$$|tr(A^2B)| \leq \frac{n-2}{\sqrt{n(n-1)}}[trA^2][tr(B^2)]^{\frac{1}{2}}. \tag{3.1}$$

Proof. We can find an orthogonal matrix Q such that $QAQ^{-1} = (\widetilde{a_{ij}})$ where $\widetilde{a_{ij}} = \widetilde{a_{ii}}\delta_{ij}$. Since Q is an orthogonal matrix, we have $trB = trQBQ^{-1} = tr(\widetilde{b_{ij}})$.

So we have

$$tr(A^2B) = tr(QAQ^{-1}QAQ^{-1}QBQ^{-1}) = tr((\widetilde{a_{ij}})^2(\widetilde{b_{ij}})) = \sum_i \widetilde{a_{ii}}^2\widetilde{b_{ii}} \tag{3.2}$$

Since $trA = trB = 0$, we have $\sum_i \widetilde{a_{ii}} = \sum_i \widetilde{b_{ii}} = 0$. By using Lemma 2.3, we have

$$
\begin{aligned}
|tr(A^2B)| = |\sum_i \widetilde{a_{ii}}^2 \widetilde{b_{ii}}| &\le \frac{n-2}{\sqrt{n(n-1)}} (\sum_i \widetilde{a_{ii}}^2)(\sum_i \widetilde{b_{ii}}^2)^{\frac{1}{2}} \\
&\le \frac{n-2}{\sqrt{n(n-1)}} (\sum_i \widetilde{a_{ii}}^2)(\sum_{ij} \widetilde{b_{ij}}^2)^{\frac{1}{2}} \\
&= \frac{n-2}{\sqrt{n(n-1)}} [trA^2][trB^2]^{\frac{1}{2}}.
\end{aligned}
$$

This proves this Lemma. □

Lemma 3.2. *Let A and B be real symmetric matrixes satisfying $tr(A) = 0$. Then*

$$
tr(B)tr(A^2B) - (tr(AB))^2 \le \frac{n}{2\sqrt{n-1}} tr(A^2)tr(B^2). \tag{3.3}
$$

Proof. We can find an orthogonal matrix Q such that $QAQ^{-1} = (\widetilde{a_{ij}})$ where $\widetilde{a_{ij}} = \widetilde{a_{ii}}\delta_{ij}$. Since Q is an orthogonal matrix, we have $trB = trQBQ^{-1} = tr(\widetilde{b_{ij}})$. By using $trA = 0$ and Lemma 2.4, we have

$$
\begin{aligned}
tr(B)tr(A^2B) - (tr(AB))^2 &= \sum_i \widetilde{b_{ii}} [\sum_i \widetilde{a_{ii}}^2 \widetilde{b_{ii}}] - [\sum_i \widetilde{a_{ii}} \widetilde{b_{ii}}]^2 \\
&= \frac{1}{2} \sum_{ij} \widetilde{b_{ii}} \widetilde{b_{jj}} (\widetilde{a_{ii}} - \widetilde{a_{jj}})^2 \\
&\le \frac{n}{2\sqrt{n-1}} (\sum_i \widetilde{a_{ii}}^2)(\sum_i \widetilde{b_{ii}}^2) \\
&\le \frac{n}{2\sqrt{n-1}} (\sum_i \widetilde{a_{ii}}^2)(\sum_{ij} \widetilde{b_{ij}}^2) \\
&= \frac{n}{2\sqrt{n-1}} tr(A^2)tr(B^2).
\end{aligned}
$$

This proves this Lemma. □

Proof of Theorem 1.1: Choose a local orthonormal frame field $\{e_1, \cdots, e_n\}$ such that $h_{ij}^{n+1} = \lambda_i^{n+1}\delta_{ij}$ and $\Phi_{ij}^{n+1} = \lambda_i^{n+1}\delta_{ij} - H\delta_{ij}$. Let $\mu_i = \lambda_i^{n+1} - H$ and denote $\Phi_1^2 = \sum_i \mu_i^2$. From (2.12) and (2.22) and the relation $R = aH + b$, we have

$$\begin{aligned}
L(nH) &= \sum_{ij}(nH\delta_{ij} - h_{ij}^{n+1})(nH)_{ij} + \frac{(n-1)a}{2}\triangle(nH) \\
&= nH\triangle(nH) - \sum_{ij} h_{ij}^{n+1}(nH)_{ij} + \frac{1}{2}\triangle(n(n-1)R - n(n-1)b) \\
&= \frac{1}{2}\triangle[(nH)^2 + n(n-1)R] - n^2|\nabla H|^2 - \sum_{ij} h_{ij}^{n+1}(nH)_{ij} \\
&= \frac{1}{2}\triangle[n(n-1)c + S] - n^2|\nabla H|^2 - \sum_{ij} h_{ij}^{n+1}(nH)_{ij} \\
&= \frac{1}{2}\triangle S - n^2|\nabla H|^2 - \sum_{ij} h_{ij}^{n+1}(nH)_{ij} \qquad (3.4) \\
&= \frac{1}{2}\triangle S - n^2|\nabla H|^2 - h_{ij}^{n+1}(nH_{ij}) \\
&\geq \sum_{ijk\alpha}(h_{ijk}^{\alpha})^2 - n^2|\nabla H|^2 + \underbrace{ncS_1 - n^2cH^2 + (tr(H^{n+1}H^{n+1}))^2 - nHtr(H^{n+1})^3}_{I} \\
&\quad \underbrace{ncS_2 - nH\sum_{\alpha\neq n+1} tr(H^{n+1}(H^{\alpha})^2) + \sum_{\beta\neq n+1}[tr(H^{n+1}H^{\beta})]^2 + \sum_{\alpha,\beta\neq n+1}[tr(H^{\alpha}H^{\beta})]^2}_{II}.
\end{aligned}$$

Firstly, we estimate (I):

$$\begin{aligned}
-nHtr(H^{n+1})^3 &= -nH\sum(\lambda_i^{n+1})^3 = -nH[\sum(\mu_i^3) + 3S_1H - 3nH^3 + nH^3] \\
&= -3nS_1H^2 + 2n^2H^4 - nH\sum(\mu_i^3). \qquad (3.5)
\end{aligned}$$

By applying Lemma 2.2 to real numbers $\mu_1, \cdots, \mu_n$, we get

$$-nHtr(H^{n+1})^3 \geq -3nS_1H^2 + 2n^2H^4 - \frac{n(n-2)}{\sqrt{n(n-1)}}|H||\Phi_1|^3. \qquad (3.6)$$

So

$$I \geq |\Phi_1|^2(nc - nH^2 + |\Phi_1|^2 - \frac{n(n-2)}{\sqrt{n(n-1)}}|H||\Phi_1|). \qquad (3.7)$$

Consider the quadratic form $P(x,y) = -x^2 - \frac{n-2}{\sqrt{n-1}}xy + y^2$. By the orthogonal transformation

$$\begin{aligned}
u &= \frac{1}{\sqrt{2n}}((1+\sqrt{n-1})y + (1-\sqrt{n-1})x) \\
v &= \frac{1}{\sqrt{2n}}((-1+\sqrt{n-1})y + (1+\sqrt{n-1})x)
\end{aligned}$$

$P(x,y) = \frac{n}{2\sqrt{n-1}}(u^2 - v^2)$. Take $x = \sqrt{n}H$ and $y = |\Phi_1|$; we obtain $u^2+v^2 = x^2+y^2$, and by (3.7), we have

$$\begin{aligned}
I &\geq |\Phi_1|^2(nc + \frac{n}{2\sqrt{n-1}}(u^2 - v^2)) \geq |\Phi_1|^2(nc - \frac{n}{2\sqrt{n-1}}(u^2+v^2) + \frac{n}{2\sqrt{n-1}}2u^2) \\
&\geq |\Phi_1|^2(nc - \frac{n}{2\sqrt{n-1}}(u^2+v^2)) \geq |\Phi_1|^2(nc - \frac{n}{2\sqrt{n-1}}S_1). \qquad (3.8)
\end{aligned}$$

Finally, we estimate (II):

Since $trH^\alpha = 0$, we can use Lemma 3.2 and get

$$-nHtr(H^{n+1}(H^\alpha)^2) + (trH^{n+1}H^\alpha)^2 \geq -\frac{n}{2\sqrt{(n-1)}}S_1 tr(H^\alpha)^2. \tag{3.9}$$

so we have

$$II \geq ncS_2 - nHtr(H^{n+1}(H^\alpha)^2) + (trH^{n+1}H^\alpha)^2 \geq S_2(nc - \frac{n}{2\sqrt{(n-1)}}S_1). \tag{3.10}$$

From the inequalities (2.24) (3.4) (3.8)and (3.10), we get

$$\begin{aligned} L(nH) &\geq |\Phi_1|^2(nc - \frac{n}{2\sqrt{n-1}}S_1) + S_2(nc - \frac{n}{2\sqrt{(n-1)}}S_1) \\ &\geq |\Phi|^2(nc - \frac{n}{2\sqrt{(n-1)}}S_1) \geq |\Phi|^2(nc - \frac{n}{2\sqrt{(n-1)}}S), \end{aligned}$$

that is,

$$L(nH) \geq |\Phi|^2(nc - \frac{n}{2\sqrt{(n-1)}}S). \tag{3.11}$$

From the assumption $S < 2\sqrt{n-1}c$ and Eq. (2.12), we have

$$nH^2 + n(n-1)(aH+b) - n(n-1)c = S < 2\sqrt{n-1}c, \tag{3.12}$$

So we know that H is bounded. According to Lemma 2.1 (2), there exists a sequence of points $\{x_k\}$ in M^n such that

$$\lim_{k\to\infty} nH(x_k) = \sup(nH), \quad \lim_{k\to\infty} \sup(L(nH)(x_k)) \leq 0. \tag{3.13}$$

From Eq.(2.12) and (2.18), we have

$$|\Phi|^2 = n(n-1)(H^2 - c + R) = n(n-1)(H^2 - c + aH + b). \tag{3.14}$$

Notice that $\lim_{k\to\infty} nH(x_k) = \sup(nH)$ and R is constant, so we have

$$\lim_{k\to\infty} |\Phi|^2(x_k) = \sup|\Phi|^2, \quad \lim_{k\to\infty} S(x_k) = \sup S \tag{3.15}$$

Evaluating (3.11) at the points x_k of the sequence, taking the limit and using (3.13), we obtain that

$$0 \geq \lim_{k\to\infty} \sup(L(nH)(x_k)) \geq \sup|\Phi|^2(nc - \frac{n}{2\sqrt{(n-1)}}\sup S) \tag{3.16}$$

If $S < \sqrt{2(n-1)}c$, we have $\sup|\Phi| = 0$, that is, $\Phi = 0$. Thus, we infer that $S = nH^2$ and M^n is totally umbilical. This proves Theorem 1.1.

Proof of Theorem 1.2:

$$\begin{aligned} &-nH\sum_{\alpha\neq n+1} tr(H^{n+1}(H^\alpha)^2) \\ =\ &-nH\sum_{\alpha\neq n+1} tr[(H^{n+1}-HI)(H^\alpha)^2] - nH^2\sum_{\alpha\neq n+1} tr(H^\alpha)^2 \\ =\ &-nH\sum_{\alpha\neq n+1} tr(\Phi^{n+1}(\Phi^\alpha)^2) - nH^2S_2 \end{aligned} \tag{3.17}$$

By applying Lemma 3.1 to the matrixes $\Phi^{n+1}, \cdots, \Phi^{n+p}$, we get

$$-nH\sum_{\alpha\neq n+1} tr(H^{n+1}(H^\alpha)^2) \geq -n|H|\frac{n-2}{\sqrt{n(n-1)}}|\Phi_1|S_2 - nH^2S_2. \tag{3.18}$$

So

$$\begin{aligned} II &\geq ncS_2 - nH \sum_{\alpha \neq n+1} tr(H^{n+1}(H^\alpha)^2) + \sum_{\alpha,\beta \neq n+1} [tr(H^\alpha H^\beta)]^2 \\ &\geq ncS_2 - n|H| \frac{n-2}{\sqrt{n(n-1)}} |\Phi_1| S_2 - nH^2 S_2 + \sum_{\alpha \neq n+1} [tr(\Phi^\alpha)^2]^2. \quad (3.19) \end{aligned}$$

From the inequalities (2.24) (3.4) (3.7)and (3.19), we get

$$L(nH) \geq |\Phi|^2 (nc - nH^2 + \frac{|\Phi|^2}{p} - \frac{n(n-2)}{\sqrt{n(n-1)}} |H||\Phi|). \quad (3.20)$$

According to Lemma 2.1 (2), there exists a sequence of points $\{x_k\}$ in M^n such that

$$\lim_{k\to\infty} nH(x_k) = \sup(nH), \quad \lim_{k\to\infty} \sup(L(nH)(x_k)) \leq 0. \quad (3.21)$$

Evaluating (3.20) at the points x_k of the sequence, taking the limit and using (3.21), we obtain that

$$\begin{aligned} 0 &\geq \lim_{k\to\infty} \sup(L(nH)(x_k)) \\ &\geq \sup|\Phi|^2 (nc - n\sup H^2 + \frac{\sup|\Phi|^2}{p} - \frac{n(n-2)}{\sqrt{n(n-1)}} \sup|H| \sup|\Phi|). \quad (3.22) \end{aligned}$$

Consider the following polynomial given by

$$L_{\sup H}(x) = \frac{x^2}{p} - \frac{n(n-2)}{\sqrt{n(n-1)}} \sup|H| x + nc - n\sup H^2. \quad (3.23)$$

(1) If $\sup(H)^2 < \frac{4(n-1)}{(n-2)^2 p + 4(n-1)} c$, it is easy to check that the discriminant of $L_{\sup H}(x)$ is negative. Hence, for any x, $L_{\sup H}(x) > 0$, so does $L_{\sup H}(\sup|\Phi|) > 0$. From (3.22), we know that $\sup|\Phi| = 0$, that is $|\Phi| = 0$. Thus, we infer that $S = nH^2$ and M^n is totally umbilical.

(2) If $\sup(H)^2 = \frac{4(n-1)}{(n-2)^2 p + 4(n-1)} c$, we have

$$L_{\sup H}(x) = (\sup|\Phi| - \frac{n(n-2)p}{\sqrt{n}} \sqrt{\frac{c}{p(n-2)^2 + 4(n-1)}})^2 \geq 0. \quad (3.24)$$

If $(\sup|\Phi| - \frac{n(n-2)p}{\sqrt{n}} \sqrt{\frac{c}{p(n-2)^2+4(n-1)}})^2 > 0$, from (3.22) we have $\sup|\Phi| = 0$, that is $|\Phi| = 0$. Thus, we infer that $\sup(S) = nH^2$ and M^n is totally umbilical. If $\sup|\Phi| = \frac{n(n-2)p}{\sqrt{n}} \sqrt{\frac{c}{p(n-2)^2+4(n-1)c}}$, from (2.18) we have that $\sup(S) = nc\frac{(n-2)^2p^2+4(n-1)}{(n-2)^2p+4(n-1)}$.

(3) If $\sup(H)^2 > \frac{4(n-1)}{(n-2)^2 p + 4(n-1)} c$, we know that $L_{\sup H}(x)$ has two real roots $x^-_{\sup H}$ and $x^+_{\sup H}$ given by

$$x^-_{\sup H} = p\sqrt{\frac{n}{4(n-1)}} \{(n-2)\sup|H| - \sqrt{\frac{[(n-2)^2 p + 4(n-1)]\sup H^2 - 4(n-1)c}{p}}\}$$

$$x^+_{\sup H} = p\sqrt{\frac{n}{4(n-1)}} \{(n-2)\sup|H| + \sqrt{\frac{[(n-2)^2 p + 4(n-1)]\sup H^2 - 4(n-1)c}{p}}\}$$

it is easy to say that $x^+_{\sup H}$ is always positive; on the other hand, $x^-_{\sup H} < 0$ if and only if $\sup H^2 > c$, $x^-_{\sup H} = 0$ if and only if $\sup H^2 = c$, and $x^-_{\sup H} > 0$ if and only if $\sup H^2 < c$.

In this case, we also have that

$$L_{\sup H}(x) = \frac{1}{p}(\sup|\Phi| - x^-_{\sup H})(\sup|\Phi| - x^+_{\sup H}) \tag{3.25}$$

From (3.22) and (3.25), we have that

$$0 \geq \sup|\Phi|^2 \frac{1}{p}(\sup|\Phi| - x^-_{\sup H})(\sup|\Phi| - x^+_{\sup H}) \tag{3.26}$$

(3-a) If $\sup(H)^2 > c > \frac{4(n-1)}{(n-2)^2p+4(n-1)}c$, then we have $x^-_{\sup H} < 0$. Therefore, from (3.26), we have $\sup|\Phi|^2 = 0$, i.e. M^n is totally umbilical or $0 < \sup|\Phi| \leq x^+_{\sup H}$, i.e.

$$n \sup H^2 < \sup S \leq S^+,$$

where $S^+ = \frac{n(n-2)}{2(n-1)}[\frac{2(n-2)^2p^2+4(n-1)(p+1)}{2(n-2)} \sup H^2 + \sup|H|p^2\sqrt{\frac{[(n-2)^2p+4(n-1)]\sup H^2 - 4(n-1)c}{p}}] - npc$.

(3-b) If $\sup(H)^2 = c > \frac{4(n-1)}{(n-2)^2p+4(n-1)}c$, then we have $x^-_{\sup H} = 0$. Therefore, from (3.26), we have $\sup|\Phi|^2 = 0$, i.e. M^n is totally umbilical or $0 < \sup|\Phi| \leq x^+_{\sup H}$, i.e.

$$n \sup H^2 < \sup S \leq S^+.$$

(3-c) If $c > \sup(H)^2 > \frac{4(n-1)}{(n-2)^2p+4(n-1)}c$, then we have $x^-_{\sup H} > 0$. Therefore, from (3.26), we have $\sup|\Phi|^2 = 0$, i.e. M^n is totally umbilical or $x^-_{\sup H} \leq \sup|\Phi| \leq x^+_{\sup H}$, i.e.

$$S^- \leq \sup S \leq S^+,$$

where $S^- = \frac{n(n-2)}{2(n-1)}[\frac{2(n-2)^2p^2+4(n-1)(p+1)}{2(n-2)} \sup H^2 - \sup|H|p^2\sqrt{\frac{[(n-2)^2p+4(n-1)]\sup H^2 - 4(n-1)c}{p}}] - npc$.

This proves theorem 1.2.

Proof of Theorem 1.4: From theorem 1.3, we only need to prove that M^n is a spacelike Einstein submanifolds with $Ric_{ij} = c(n-1)g_{ij}$ if and only if M^n is totally geodesic.

If M^n is totally geodesic in $S^{n+p}_p(c)$, we have $Ric_{ij} = c(n-1)g_{ij}$ by the equation (2.11).

Conversely, if M^n is a spacelike Einstein submanifolds with $Ric_{ij} = c(n-1)g_{ij}$, we have $R = c = 0H + c$ and $S = n^2H^2$. From inequality (3.20), we have the following:

$$L(nH) \geq |\Phi|^2(nc - nH^2 + \frac{|\Phi|^2}{p} - \frac{n(n-2)}{\sqrt{n(n-1)}}|H||\Phi|).$$

Because $Ric_{ij} = c(n-1)\delta_{ij}$, we see by the Bonnet-Myers theorem that M^n is bounded and hence compact. Since L is self-adjoint, we have

$$0 \geq \int_{M^n} |\Phi|^2(nc - nH^2 + \frac{|\Phi|^2}{p} - \frac{n(n-2)}{\sqrt{n(n-1)}}|H||\Phi|). \tag{3.27}$$

Since $n^2|H|^2 = S$ and $|\Phi|^2 = S - nH^2 = n(n-1)H^2$, we have

$$\begin{aligned} & nc - nH^2 + \frac{|\Phi|^2}{p} - \frac{n(n-2)}{\sqrt{n(n-1)}}|H||\Phi| \\ = \; & nc - nH^2 + \frac{n(n-1)H^2}{p} - n(n-2)H^2 \\ = \; & nc - n(n-1)(1-\frac{1}{p})H^2. \end{aligned}$$

If $H^2 < \frac{c}{(n-1)(1-\frac{1}{p})}$, we have $(nc - nH^2 + \frac{|\Phi|^2}{p} - \frac{n(n-2)}{\sqrt{n(n-1)}}|H||\Phi|) > 0$, which together with (3.27) yields $|\Phi|^2 = 0$. That is, $S = nH^2$, so we know that $n^2H^2 = nH^2$, so we have $H = 0$, i.e. $S = nH^2 = 0$, so M^n is totally geodesic. This proves Theorem 1.4.

Remark. *When $p = 1$, since $n^2H^2 = S$, we have $nc - nH^2 + \frac{|\Phi|^2}{p} - \frac{n(n-2)}{\sqrt{n(n-1)}}|H||\Phi| = nc > 0$, so we know that $|\Phi| = 0$, i.e. $S = nH^2$, so M^n is totally geodesic.*

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