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A STUDY ON $\left|\overline{N}, p, q\right|_k$ SUMMABILITY FACTORS OF INFINITE SERIES

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Abstract. New result concerning $\left|\overline{N}, p, q\right|_k$ summability of the infinite series $\sum a_n \lambda_n$ is presented.

1. Introduction

Let $\sum a_n$ be a given infinite series with sequence of partial sums (s_n). Let (T_n) denote the sequence of (N, p, q) means of (s_n). The (N, p, q) transforms of (s_n) is defined by

$$T_n = \frac{1}{R_n} \sum_{v=0}^{n} p_{n-v} q_v s_v \quad , \tag{1.1}$$

where

$$R_n = \sum_{v=0}^{n} p_{n-v} q_v \neq 0, \text{ for any } n (p_{-1} = q_{-1} = R_{-1} = 0). \tag{1.2}$$

Necessary and sufficient conditions for the (N, p, q) method to be regular are
(i) $\lim_{n\to\infty} p_{n-v} q_v / R_n = 0$ for each v, and
(ii) $\left|\sum_{v=0}^{n} p_{n-v} q_v\right| < K \left|R_n\right|$, where K is a positive constant independent of n. The series $\sum a_n$ is said to be summable $\left|R, p_n\right|_k$, $k \geq 1$, if

$$\sum_{n=1}^{\infty} n^{k-1} \left|\varphi_n - \varphi_{n-1}\right|^k < \infty \quad , \tag{1.3}$$

where

$$\varphi_n = \frac{1}{P_n} \sum_{v=0}^{n} p_v s_v \ . \tag{1.4}$$

and $P_n = p_0 + p_1 + \ldots + p_n \to \infty$, as $n \to \infty$,

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The series $\sum a_n$ is said to be summable $|N,p_n|$, if

$$\sum_{n=1}^{\infty} |\sigma_n - \sigma_{n-1}| < \infty \ , \tag{1.5}$$

where

$$\sigma_n = \frac{1}{P_n} \sum_{v=0}^{n} p_{n-v} s_v \ , \tag{1.6}$$

and it is said to be summable $|N,p,q|_k$, $k \geq 1$, if

$$\sum_{n=1}^{\infty} n^{k-1} |T_n - T_{n-1}|^k < \infty \ , \tag{1.7}$$

where T_n as defined by (1.1).

For $k = 1$, $|N,p,q|_k$ summability reduces to $|N,p,q|$ summability.

The series $\sum a_n$ is said to be (N,p,q) bounded or $\sum a_n = O(1)(N,p,q)$ if

$$t_n = \sum_{v=1}^{n} p_{n-v} q_v s_v = O(R_n) \ \text{ as } \ n \to \infty \ . \tag{1.8}$$

By M, we denote the set of sequences $p = (p_n)$ satisfying

$$\frac{p_{n+1}}{p_n} \leq \frac{p_{n+2}}{p_{n+1}} \leq 1, \ \ p_n > 0, \ \ n = 0, 1, \dots \ .$$

It is known (Das [1]) that for $p \in M$, (1.5) holds iff

$$\sum_{n=1}^{\infty} \frac{1}{nP_n} \left| \sum_{v=1}^{n} p_{n-v} v a_v \right| < \infty \ . \tag{1.9}$$

For $p \in M$, the series $\sum a_n$ is said to be $|N,p_n|_k$ –summable $k \geq 1$ (Sulaiman [3]), if

$$\sum_{n=1}^{\infty} \frac{1}{nP_n^k} \left| \sum_{v=1}^{n} p_{n-v} v a_v \right|^k < \infty \ . \tag{1.10}$$

It is quite reasonable to give the following definition:

For $p \in M$, the series $\sum a_n$ is said to be $\left|\overline{N},p,q\right|_k$ –summable $k \geq 1$ if

$$\sum_{n=1}^{\infty} \frac{1}{nR_n^k} \left| \sum_{v=1}^{n} v p_{n-v} q_v a_v \right|^k < \infty \ . \tag{1.11}$$

where $R_n = p_n q_0 + p_{n-1} q_1 + \ldots + p_o q_n \to \infty$, as $n \to \infty$.

We also assume that (p_n), (q_n) are positive sequences of numbers such that

$$P_n = p_0 + p_1 + \ldots + p_n \to \infty, \text{ as } n \to \infty \ ,$$

$$Q_n = q_0 + q_1 + \ldots + q_n \to \infty, \text{ as } n \to \infty \ .$$

A positive sequence $\alpha = (\alpha_n)$ is said to be a quasi-f-power increasing sequence, $f = (f_n)$, if there exists a constant $K = K(\alpha, f)$ such that

$$K f_n \alpha_n \geq f_m \alpha_m \ ,$$

holds for $n \geq m \geq 1$(see [4]).

Das [1], in 1996, proved the following result

Theorem 1.1. *Let* $(p_n) \in M$, $q_n \geq 0$. *Then if* $\sum a_n$ *is* $|N,p,q|$ *–summable it is* $\left|\overline{N},q_n\right|$ *–summable.*

Recently Singh and Sharma [2] proved the following theorem

Theorem 1.2. *Let $(p_n) \in M$, $q_n > 0$ and let (q_n) be a monotonic non-decreasing sequence for $n \geq 0$. The necessary and sufficient condition that $\sum a_n \lambda_n$ is $\left|\overline{N}, q_n\right|$ −summable whenever*

$$\sum a_n = O(1)(N,p,q),$$

$$\sum_{n=0}^{\infty} \frac{q_n}{Q_n} \left|\lambda_n\right| < \infty,$$

$$\sum_{n=0}^{\infty} \left|\Delta \lambda_n\right| < \infty,$$

$$\sum_{n=0}^{\infty} \frac{Q_{n+1}}{q_{n+1}} \left|\Delta^2 \lambda_n\right| < \infty,$$

is that

$$\sum_{n=1}^{\infty} \frac{q_n}{Q_n} \left|s_n\right| \left|\lambda_n\right| < \infty.$$

2. Lemmas

Lemma 2.1. *Let (p_n) be non-increasing, $n = O(P_n)$, $P_n = O(R_n)$. Then for $r > 0$, $k \geq 1$,*

$$\sum_{n=v+1}^{\infty} \frac{p_{n-v}^k}{n^r R_n^k} = O(\frac{1}{v^{r+k-1}}). \tag{2.1}$$

In particular,

$$\sum_{n=v+1}^{\infty} \frac{p_{n-v}}{n R_n} = O(\frac{1}{v}).$$

Proof. Since p_n is non-increasing, then $np_n = O(P_n)$. Therefore

$$\begin{aligned}
\sum_{n=v+1}^{\infty} \frac{p_{n-v}^k}{n^r R_n^k} &= O(1) \sum_{n=v+1}^{\infty} \frac{p_{n-v}^k}{n^r P_n^k} = O(1) \sum_{n=v+1}^{2v} \frac{p_{n-v}^k}{n^r P_n^k} + O(1) \sum_{n=2v+1}^{\infty} \frac{p_{n-v}^k}{n^r P_n^k} \\
\sum_{n=v+1}^{2v} \frac{p_{n-v}^k}{n^r P_n^k} &= O(1) \frac{1}{v^r P_v^k} \sum_{n=v+1}^{2v} p_{n-v}^k = O(1) \frac{1}{v^r P_v^k} \sum_{m=1}^{v} p_m^k \\
&= O(1) \frac{1}{v^r P_v^k} \sum_{m=1}^{v} p_m = O(1) \frac{1}{v^r P_v^{k-1}} = O\left(\frac{1}{v^{r+k-1}}\right). \\
\sum_{n=2v+1}^{\infty} \frac{p_{n-v}^k}{n^r P_n^k} &= O(1) \sum_{m=v+1}^{\infty} \frac{p_m^k}{(m+v)^r P_{m+v}^k} = \sum_{m=v+1}^{\infty} \frac{p_m^k}{m^r P_m^k} \\
&= O(1) \sum_{m=v+1}^{\infty} \frac{1}{m^{r+k}} = O(1) \int_v^{\infty} x^{-r-k} dx = O\left(\frac{1}{v^{r+k-1}}\right).
\end{aligned}$$

Therefore

$$\sum_{m=v+1}^{\infty} \frac{p_{n-v}^k}{n^r P_n^k} = O\left(\frac{1}{v^{r+k-1}}\right).$$

□

Lemma 2.2. *For $p \in M$,*

$$\sum_{v=0}^{\infty} |\Delta_v p_{n-v}| < \infty.$$

Proof. Since $p \in M$, then (p_n) is non-increasing and hence

$$\sum_{v=0}^{m} |\Delta_v p_{n-v}| = \sum_{v=0}^{\infty} (p_{n-v-1} - p_{n-v}) = p_n - p_{m-v-1} = O(1).$$

□

Lemma 2.3. [4]. *If (X_n) is a quasi-f-increasing sequence , where $f = (f_n) = \left(n^{\beta} (\log n)^{\gamma}\right)$, $\gamma \geq 0$, $0 < \beta < 1$ then under the conditions*

$$X_m |\lambda_m| = O(1), \quad m \to \infty, \tag{2.2}$$

$$\sum_{n=1}^{m} nX_n \left|\Delta^2 \lambda_n\right| = O(1), \quad m \to \infty, \tag{2.3}$$

we have

$$nX_n |\Delta \lambda_n| = O(1), \tag{2.4}$$

$$\sum_{n=1}^{\infty} X_n |\Delta \lambda_n| < \infty.$$

3. Result

Our aim is to present the following new general result

Theorem 3.1. *Let $p \in M$, and let (X_n) be a quasi-f-increasing sequence , where $f = (f_n) = \left(n^{\beta} (\log n)^{\gamma}\right)$, $\gamma \geq 0$, $0 < \beta < 1$ and (2.2) and (2.3) and*

$$\sum_{v=1}^{n} \frac{q_v |s_v|^k}{v X_v^{k-1}} = O(X_n), \tag{3.1}$$

$$\Delta q_v = O(v^{-1} q_v), \tag{3.2}$$

$$q_{v+1} = O(q_v), \tag{3.3}$$

$$v = O(P_v), \tag{3.4}$$

$$P_n = O(R_n), \tag{3.5}$$

are all satisfied then the series $\sum a_n \lambda_n$ is summable $|N, p, q|_k$, $k \geq 1$.

Proof. Let (T_n) be the sequence of $(\overline{N}, p.q)$ transform of the series $\sum a_n\lambda_n$. Then, we have

$$\begin{aligned}
T_n &= \sum_{v=1}^{n} v p_{n-v} q_v a_v \lambda_v \\
&= \sum_{v=0}^{n-1}\left(\sum_{r=0}^{v} a_r\right)\Delta_v\left(v p_{n-v} q_v \lambda_v\right) + \left(\sum_{v=0}^{n} a_v\right) n p_0 q_n \lambda_n \\
&= \sum_{v=0}^{n-1} s_v\left(-p_{n-v} q_v \lambda_v + (v+1)\Delta q_v p_{n-v}\lambda_v + (v+1) q_{v+1}\Delta_v p_{n-v}\lambda_v\right. \\
&\quad \left. +(v+1) q_{v+1} p_{n-v-1}\Delta\lambda_v\right) + n p_0 q_n s_n \lambda_n \\
&= T_{n1} + T_{n2} + T_{n3} + T_{n4} + T_{n5}.
\end{aligned}$$

In order to prove the result, it is sufficient, by Minkowski's inequality, to show that

$$\sum_{n=1}^{\infty} \frac{1}{nP_n^k}\left|T_{nj}\right|^k < \infty, \quad j = 1,2,3,4,5.$$

Applying Hölder's inequality, we have

$$\begin{aligned}
\sum_{n=1}^{m} \frac{1}{nP_n^k}\left|T_{n1}\right|^k &= \sum_{n=1}^{m} \frac{1}{nP_n^k}\left|\sum_{v=0}^{n-1} p_{n-v} q_v s_v \lambda_v\right|^k \\
&\leq \sum_{n=1}^{m} \frac{1}{nP_n^k}\sum_{v=0}^{n-1} p_{n-v} q_v^k \left|s_v\right|^k \left|\lambda_v\right|^k \left(\sum_{v=0}^{n-1} p_{n-v}\right)^{k-1} \\
&= O(1)\sum_{n=1}^{m} \frac{P_n^{k-1}}{nP_n^k}\sum_{v=0}^{n-1} p_{n-v} q_v^k \left|s_v\right|^k \left|\lambda_v\right|^k \\
&= O(1)\sum_{v=0}^{m} q_v^k \left|s_v\right|^k \left|\lambda_v\right|^k \sum_{n=v+1}^{\infty} \frac{p_{n-v}}{nP_n} \\
&= O(1)\sum_{v=0}^{m} v^{-1} q_v^k \left|s_v\right|^k \left|\lambda_v\right|^k \\
&= O(1)\sum_{v=0}^{m} \frac{q_v^k \left|s_v\right|^k}{vX_v^{k-1}}\left|\lambda_v\right|\left|\lambda_v\right|^{k-1} X_v^{k-1} \\
&= O(1)\sum_{v=1}^{m} \frac{q_v^k \left|s_v\right|^k}{vX_v^{k-1}}\left|\lambda_v\right| \\
&= O(1)\sum_{v=1}^{m-1} \Delta\left|\lambda_v\right| \sum_{r=0}^{v} \frac{q_r^k \left|s_r\right|^k}{rX_r^{k-1}} + \left|\lambda_n\right| \sum_{v=0}^{m} \frac{q_v^k \left|s_v\right|^k}{vX_v^{k-1}} \\
&= O(1)\sum_{v=1}^{m-1} \left|\Delta\lambda_v\right| X_v + \left|\lambda_n\right| X_n = O(1).
\end{aligned}$$

$$\begin{aligned}
\sum_{n=1}^{m}\frac{1}{nP_n^k}\left|T_{n2}\right|^k &= \sum_{n=1}^{m}\frac{1}{nP_n^k}\left|\sum_{v=0}^{n-1}(v+1)\,p_{n-v}\Delta q_v s_v\lambda_v\right|^k \\
&\leq \sum_{n=1}^{m}\frac{1}{nP_n^k}\sum_{v=0}^{n-1}v^k p_{n-v}\left|\Delta q_v\right|^k\left|s_v\right|^k\left|\lambda_v\right|^k\left(\sum_{v=0}^{n-1}p_{n-v}\right)^{k-1} \\
&= O(1)\sum_{n=1}^{m}\frac{P_{n-1}^{k-1}}{nP_n^k}\sum_{v=0}^{n-1}v^k p_{n-v}\left|\Delta q_v\right|^k\left|s_v\right|^k\left|\lambda_v\right|^k \\
&= O(1)\sum_{v=0}^{m}v^k\left|\Delta q_v\right|^k\left|s_v\right|^k\left|\lambda_v\right|^k\sum_{n=v+1}^{\infty}\frac{p_{n-v}}{nP_n} \\
&= O(1)\sum_{v=0}^{m}v^{k-1}\left|\Delta q_v\right|^k\left|s_v\right|^k\left|\lambda_v\right|^k \\
&= O(1)\sum_{v=1}^{m}\frac{q_v^k\left|s_v\right|^k}{vX_v^{k-1}}\left|\lambda_v\right| \\
&= O(1),\ \text{as in the case of } T_{n1}.
\end{aligned}$$

$$\begin{aligned}
\sum_{n=1}^{\infty}\frac{1}{nP_n^k}\left|T_{n3}\right|^k &= \sum_{n=1}^{m}\frac{1}{nP_n^k}\left|\sum_{v=0}^{n-1}(v+1)\,\Delta_v p_{n-v}q_{v+1}s_v\lambda_v\right|^k \\
&\leq \sum_{n=1}^{m}\frac{1}{nP_n^k}\sum_{v=0}^{n-1}v^k\Delta_v p_{n-v}q_{v+1}^k\left|s_v\right|^k\left|\lambda_v\right|^k\left(\sum_{v=0}^{n-1}\left|\Delta p_{n-v}\right|\right)^{k-1} \\
&= O(1)\sum_{n=1}^{m}\frac{1}{nP_n^k}\sum_{v=0}^{n-1}v^k\Delta_v p_{n-v}q_{v+1}^k\left|s_v\right|^k\left|\lambda_v\right|^k \\
&= O(1)\sum_{v=0}^{m}v^k q_{v+1}^k\left|s_v\right|^k\left|\lambda_v\right|^k\sum_{n=v+1}^{\infty}\frac{\Delta_v p_{n-v}}{nP_n^k} \\
&= O(1)\sum_{v=0}^{m}v^{k-1}P_v^{-k}q_{v+1}^k\left|s_v\right|^k\left|\lambda_v\right|^k \\
&= O(1)\sum_{v=0}^{m}v^{-1}q_v^k\left|s_v\right|^k\left|\lambda_v\right|^k \\
&= O(1),\text{as in the case of } T_{n1}.
\end{aligned}$$

$$\begin{aligned}
\sum_{n=1}^{m}\frac{1}{nP_n^k}\left|T_{n4}\right|^k &= \sum_{n=1}^{m}\frac{1}{nP_n^k}\left|\sum_{v=0}^{n-1}(v+1)\,p_{n-v-1}q_{v+1}s_v\Delta\lambda_v\right|^k \\
&\le \sum_{n=1}^{m}\frac{1}{nP_n^k}\sum_{v=0}^{n-1}v^k p_{n-v-1}^k q_{v+1}^k\left|s_v\right|^k\left|\Delta\lambda_v\right|X_v^{1-k}\left(\sum_{v=0}^{n-1}X_v\left|\Delta\lambda_v\right|\right)^{k-1} \\
&= O(1)\sum_{n=1}^{m}\frac{1}{nP_n^k}\sum_{v=0}^{n-1}v^k p_{n-v-1}^k q_{v+1}^k\left|s_v\right|^k\left|\Delta\lambda_v\right| \\
&= O(1)\sum_{v=0}^{m}\frac{v^k q_{v+1}^k\left|s_v\right|^k}{X_v^{k-1}}\left|\Delta\lambda_v\right|\sum_{n=v+1}^{\infty}\frac{p_{n-v-1}}{nP_n^k} \\
&= O(1)\sum_{v=0}^{m}\frac{q_{v+1}^k\left|s_v\right|^k}{vX_v^{k-1}}v\left|\Delta\lambda_v\right| \\
&= O(1)\sum_{v=0}^{m-1}\Delta\left(v\left|\Delta\lambda_v\right|\right)\sum_{r=0}^{v}\frac{q_r^k\left|s_r\right|^k}{rX_r^{k-1}}+m\left|\Delta\lambda_m\right|\sum_{v=0}^{m}\frac{q_v^k\left|s_v\right|^k}{vX_v^{k-1}} \\
&= O(1)\sum_{v=0}^{m-1}\left|\Delta\lambda_v\right|X_v+O(1)\sum_{v=0}^{m-1}v\left|\Delta^2\lambda_v\right|X_v+O(1)m\left|\Delta\lambda_m\right|X_m=O(1).
\end{aligned}$$

$$\begin{aligned}
\sum_{n=1}^{\infty}\frac{1}{nP_n^k}\left|T_{n5}\right|^k &= \sum_{n=1}^{m}\frac{1}{nP_n^k}\left|np_0q_ns_n\lambda_n\right|^k \\
&= O(1)\sum_{n=1}^{m}n^{k-1}P_n^{-k}q_n^k\left|s_n\right|^k\left|\lambda_n\right|^k \\
&= O(1)\sum_{n=1}^{m}n^{k-1}q_n^k\left|s_n\right|^k\left|\lambda_n\right|^k \\
&= O(1), \text{as in the case of } T_{n1}.
\end{aligned}$$

This completes the proof of the theorem. □

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