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THE KOMATU INTEGRAL OPERATOR AND STRONGLY CLOSE-TO-CONVEX FUNCTIONS

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Abstract. In this paper we introduce some new subclasses of strongly close-to-convex functions defined by using the Komatu integral operator and study their inclusion relationships with the integral preserving properties.

Theorem[section] [theorem]Lemma [theorem]Proposition [theorem]Corollary Remark

1. Introduction

Let A_1 denote the class of functions of the form :

$$f(z)=z+\sum_{k=2}^{\infty}a_kz^k \tag{1.1}$$

which are analytic in the open unit disc $U=\{z:|z|<1\}$. If $f(z)$ and $g(z)$ are analytic in U, we say that $f(z)$ is subordinate to $g(z)$, written $f\prec g$ or $f(z)\prec g(z)$, if there exists a Schwarz function $w(z)$ in U such that $f(z)=g(w(z))$.

A function $f(z)\in A_1$ is said to be starlike of order η if it satisfies

$$\text{Re}\left\{\frac{zf^{'}(z)}{f(z)}\right\}>\eta \quad (z\in U) \tag{1.2}$$

for some $\eta(0\leq\eta<1)$. We denote by $S^*(\eta)$ the subclass of A_1 consisting of functions which are starlike of order η in U. Also a function $f(z)\in A_1$ is said to be convex of order η if it satisfies

$$\text{Re}\left\{1+\frac{zf^{''}(z)}{f^{'}(z)}\right\}>\eta \quad (z\in U) \tag{1.3}$$

for some $\eta(0\leq\eta<1)$. We denote by $C(\eta)$ the subclass of A_1 consisting of all functions which are convex of order η in U.

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It follows from (1.2) and (1.3) that

$$f(z) \in C(\eta) \Leftrightarrow zf^{'}(z) \in S^{*}(\eta)\,. \tag{1.4}$$

The classes $S^{*}(\eta)$ and $C(\eta)$ are introduced by Robertson [17] (see also Srivastava and Owa [21]).

Let $f(z) \in A_1$ and $g(z) \in S^{*}(\eta)$. Then $f(z) \in K(\gamma,\eta)$ if and only if

$$\mathrm{Re}\left\{\frac{zf^{'}(z)}{g(z)}\right\} > \gamma \quad (z \in U)\ , \tag{1.5}$$

where $0 \le \gamma < 1$ and $0 \le \eta < 1$. Such functions are called close-to-convex functions of order γ and type η. The class $K(\gamma,\eta)$ was introduced by Libera [8] (see also Noor and Alkhorasani [13] and Silverman [19]).

If $f(z) \in A_1$ satisfies

$$\left|\arg(\frac{zf^{'}(z)}{f(z)} - \eta)\right| < \frac{\pi}{2}\beta \quad (z \in U) \tag{1.6}$$

for some $\eta(0 \le \eta < 1)$ and $\beta(0 < \beta \le 1)$, then $f(z)$ is said to be strongly starlike of order β and type η in U. We denote this by $S^{*}(\beta,\eta)$.

If $f(z) \in A_1$ satisfies

$$\left|\arg(1 + \frac{zf^{''}(z)}{f^{'}(z)} - \eta)\right| < \frac{\pi}{2}\beta \quad (z \in U) \tag{1.7}$$

for some $\eta(0 \le \eta < 1)$ and $\beta(0 < \beta \le 1)$, then we say that $f(z)$ is strongly convex of order β and type η in U. We denote this class by $C(\beta,\eta)$ (see also Liu [10] and Nunokawa [14]). In particular, the classes $S^{*}(\beta,0)$ and $C(\beta,0)$ have been extensively studied by Mocanu [12] and Nunokawa [14].

It follows from (1.6) and (1.7) that

$$f(z) \in C(\beta,\eta) \Leftrightarrow zf^{'}(z) \in S^{*}(\beta,\eta)\ . \tag{1.8}$$

Also we note that $S^{*}(1,\eta) = S^{*}(\eta)$ and $C(1,\eta) = C(\eta)$.

Recently, Komatu [7] introduced a certain integral operator $I_a^{\lambda}(a > 0; \lambda \ge 0)$ defined by

$$I_a^{\lambda}f(z) = \frac{a^{\lambda}}{\Gamma(\lambda)}\int_0^1 t^{a-2}(\log\frac{1}{t})^{\lambda-1}f(zt)dt \tag{1.9}$$

$$(z \in U; a > 0; \lambda \ge 0; f \in A_1)\,.$$

Thus , if $f(z) \in A_1$ is of the form (1.1), it is easily seen from (1.9) that

$$I_a^{\lambda}f(z) = z + \sum_{k=2}^{\infty}\left(\frac{a}{a+k-1}\right)^{\lambda} a_k z^k \quad (a > 0; \lambda \ge 0)\,. \tag{1.10}$$

Using the above relation, it is easy verify that

$$z(I_a^{\lambda+1}f(z))^{'} = aI_a^{\lambda}f(z) - (a-1)I_a^{\lambda+1}f(z) \quad (a > 0; \lambda \ge 0)\,. \tag{1.11}$$

We note that :

(i) For $a = 1$ and $\lambda = n$ (n is any integer), the multiplier transformation $I_1^n f(z) = I^n f(z)$ was studied by Flett [5] and Salagean [18];

(ii) For $a = 1$ and $\lambda = -n\,(n \in N_0 \in \{0,1,2,...\})$, the differential operator $I_1^{-n}f(z) = D^n f(z)$ was studied by Salagean [18];

(iii) For $a = 2$ and $\lambda = n$ (n is any integer), the operator $I_2^n f(z) = L^n f(z)$ was studied by Uralegaddi and Somanatha [22];

(iv) For $a = 2$, the multiplier transformation $I_2^\lambda f(z) = I^\lambda f(z)$ was studied by Jung et al. [6].

For $a > 0$ and $\lambda \geq 0$, let $K_\lambda^a(\gamma, \delta, \eta, A, B)$ be the class of functions $f(z) \in A_1$ satisfying the condition

$$\left|\arg\left(\frac{z(I_a^\lambda f(z))'}{I_a^\lambda g(z)} - \gamma\right)\right| < \frac{\pi}{2}\delta \quad (0 \leq \gamma < 1; 0 < \delta \leq 1; z \in U), \tag{1.12}$$

for some $f(z) \in S_\lambda^a(\eta, A, B)$, where

$$S_\lambda^a(\eta, A, B) = \left\{g \in A_1 : \frac{1}{1-\eta}\left(\frac{z(I_a^\lambda g(z))'}{I_a^\lambda g(z)} - \eta\right) \prec \frac{1+Az}{1+Bz}\right\}$$
$$(0 \leq \eta < 1; -1 \leq B < A \leq 1; z \in U). \tag{1.13}$$

We note that $K_0^a(\gamma, 1, \eta, 1, -1) = K(\gamma, \eta)$. We also note that $K_0^a(0, \delta, 0, 1, -1)$ is the class of strongly close-to-convex functions of order δ in the sense of Pommerenke [16]. Also the class $S_0^a(\eta, A, B) = S(\eta, A, B)$ was studied by Aouf [1].

In the present paper, using the technique of Cho [3], we give some argument properties of analytic functions belonging to A_1 which contain the basic inclusion relationships among the classes $K_\lambda^a(\gamma, \delta, \eta, A, B)$. The integral preserving properties in connection with the operators I_a^λ defined by (1.10) are also considered. Furthermore, we obtain the previous results given by Bernardi [2] and Libera [9] as special cases.

2. Main Results

In proving our main results, we need the following lemmas.

Lemma 1. *[4]. Let $h(z)$ be convex univalent in U with $h(0) = 1$ and $\operatorname{Re}\{\nu h(z) + \mu\} > 0\, (\nu, \mu \in C)$. If $p(z)$ is analytic in U with $p(0) = 1$, then*

$$p(z) + \frac{zp'(z)}{\nu p(z) + \mu} \prec h(z) \quad (z \in U),$$

implies

$$p(z) \prec h(z) \quad (z \in U).$$

Lemma 2. *[11]. Let $h(z)$ be convex univalent in U and $w(z)$ be analytic in U with $\operatorname{Re} w(z) \geq 0$. If $p(z)$ is analytic in U and $p(0) = h(0)$, then*

$$p(z) + w(z)zp'(z) \prec h(z) \quad (z \in U),$$

implies

$$p(z) \prec h(z) \quad (z \in U).$$

Lemma 3. *[15]. Let $p(z)$ be analytic in U with $p(0) = 1$ and $p(z) \neq 0$ in U. If there exist two points $z_1, z_2 \in U$ such that*

$$-\frac{\pi}{2}\alpha_1 = \arg p(z_1) < \arg p(z) < \arg p(z_2) = \frac{\pi}{2}\alpha_2 \tag{2.1}$$

for some $\alpha_1, \alpha_2\, (\alpha_1, \alpha_2 > 0)$ and for all $z\, (|z| < |z_1| = |z_2|)$, then we have

$$\frac{z_1 p'(z_1)}{p(z_1)} = -i\frac{\alpha_1 + \alpha_2}{2}m \quad \text{and} \quad \frac{z_2 p'(z_2)}{p(z_2)} = i\frac{\alpha_1 + \alpha_2}{2}m, \tag{2.2}$$

where

$$m \geq \frac{1-|c|}{1+|c|} \quad \text{and} \quad c = i \tan\frac{\pi}{4}\,(\frac{\alpha_2-\alpha_1}{\alpha_1+\alpha_2}). \tag{2.3}$$

At first, with the help of Lemma 1, we obtain the following :

Proposition 4. *Let $a \geq 1$ and $h(z)$ be convex univalent in U with $h(0) = 1$ and $\operatorname{Re} h(z) > 0$. If a function $f(z) \in A_1$ satisfies the condition*

$$\frac{1}{1-\eta}\left(\frac{z(I_a^\lambda f(z))'}{I_a^\lambda f(z)} - \eta\right) \prec h(z) \quad (0 \leq \eta < 1; z \in U),$$

then

$$\frac{1}{1-\eta}\left(\frac{z(I_a^{\lambda+1} f(z))'}{I_a^{\lambda+1} f(z)} - \eta\right) \prec h(z) \quad (0 \leq \eta < 1; z \in U)\,.$$

Proof. Let

$$p(z) = \frac{1}{1-\eta}\left(\frac{z(I_a^{\lambda+1} f(z))'}{I_a^{\lambda+1} f(z)} - \eta\right) \ (z \in U)\ , \tag{2.4}$$

where $p(z)$ is analytic function in U with $p(0) = 1$. By using (1.11), we get

$$a - 1 + \eta + (1-\eta)p(z) = a\frac{I_a^\lambda f(z)}{I_a^{\lambda+1} f(z)}\ . \tag{2.5}$$

Differentiating (2.5) logarithmically with respect to z and multiplying by z, we obtain

$$p(z) + \frac{zp'(z)}{a-1+\eta+(1-\eta)p(z)} = \frac{1}{1-\eta}\left(\frac{z(I_a^\lambda f(z))'}{I_a^\lambda f(z)} - \eta\right) \quad (z \in U).$$

By using Lemma 1, it follows that $p(z) \prec h(z)$, that is,

$$\frac{1}{1-\eta}\left(\frac{z(I_a^{\lambda+1} f(z))'}{I_a^{\lambda+1} f(z)} - \eta\right) \prec h(z) \quad (z \in U)\ .$$

□

Taking $h(z) = \dfrac{1+Az}{1+Bz}$ $(-1 \leq B < A \leq 1)$, in Proposition 1, we have

Corollary 5. *The inclusion relation, $S_\lambda^a(\eta, A, B) \subset S_{\lambda+1}^a(\eta, A, B)$, holds for any $a > 0$ and $\lambda \geq 0$.*

Proposition 6. *Let $h(z)$ be convex univalent in U with $h(0) = 1$ and $\operatorname{Re} h(z) > 0$. If a function $f(z) \in A_1$ satisfies the condition*

$$\frac{1}{1-\eta}\left(\frac{z(I_a^\lambda f(z))'}{I_a^\lambda f(z)} - \eta\right) \prec h(z) \quad (0 \leq \eta < 1; z \in U),$$

then

$$\frac{1}{1-\eta}\left(\frac{z(I_a^\lambda L_\theta f(z))'}{I_a^\lambda L_\theta f(z)} - \eta\right) \prec h(z) \quad (0 \leq \eta < 1; z \in U)\,,$$

where $L_\theta(f)$ is the integral operator defined by

$$L_\theta(f) = L_\theta f(z) = \frac{\theta+1}{z^\theta}\int_0^z t^{\theta-1} f(t)dt \quad (\theta \geq 0)\ . \tag{2.6}$$

Proof. From (2.6), we have

$$z(I_a^\lambda L_\theta f(z))^{'} = (\theta+1)I_a^\lambda f(z) - \theta I_a^\lambda L_\theta(f)(z)\ . \tag{2.7}$$

Let

$$p(z) = \frac{1}{1-\eta}\left(\frac{z(I_a^\lambda L_\theta f(z))^{'}}{I_a^\lambda L_\theta f(z)} - \eta\right)\ (z\in U)\ ,$$

where $p(z)$ is analytic function in U with $p(0)=1$. Then, by using (2.7), we have

$$\theta+\eta+(1-\eta)p(z) = (\theta+1)\frac{I_a^\lambda f(z)}{I_a^\lambda L_\theta(f)(z)}\ . \tag{2.8}$$

Differentiating (2.8) logarithmically with respect to z and multiplying by z, we have

$$p(z) + \frac{zp^{'}(z)}{\theta+\eta+(1-\eta)p(z)} = \frac{1}{1-\eta}\left(\frac{z(I_a^\lambda f(z))^{'}}{I_a^\lambda f(z)} - \eta\right)\ (z\in U).$$

Therefore, by using Lemma 1, we obtain that

$$\frac{1}{1-\eta}\left(\frac{z(I_a^\lambda L_\theta f(z))^{'}}{I_a^\lambda L_\theta f(z)} - \eta\right) \prec h(z)\ (z\in U).$$

□

Taking $h(z) = \dfrac{1+Az}{1+Bz}$ $(-1\le B<A\le 1)$, in Proposition 2, we have immediately :

Corollary 7. *If $f(z)\in S_\lambda^a(\eta,A,B)$, then $L_\theta(f)\in S_\lambda^a(\eta,A,B)$, where $L_\theta(f)$ is the integral operator defined by (2.6).*

We now derive:

Theorem 8. *Let $f(z)\in A_1$ and $0<\delta_1,\delta_2\le 1, 0\le\gamma<1$. If*

$$-\frac{\pi}{2}\delta_1 < \arg\left(\frac{z(I_a^\lambda f(z))^{'}}{I_a^\lambda g(z)} - \gamma\right) < \frac{\pi}{2}\delta_2$$

for some $g(z)\in S_\lambda^a(\eta,A,B)$, then

$$-\frac{\pi}{2}\alpha_1 < \arg\left(\frac{z(I_a^{\lambda+1} f(z))^{'}}{I_a^{\lambda+1} g(z)} - \gamma\right) < \frac{\pi}{2}\alpha_2\,,$$

where α_1 and α_2 $(0<\alpha_1,\alpha_2\le 1)$ are the solutions of the equations

$$\delta_1 = \begin{cases} \alpha_1 + \frac{2}{\pi}\tan^{-1}\left\{\frac{(\alpha_1+\alpha_2)(1-|c|)\cos\frac{\pi}{2}t_1}{2\left(\frac{(1-\eta)(1+A)}{1+B}+\eta+a-1\right)(1+|c|)+(\alpha_1+\alpha_2)(1-|c|)\sin\frac{\pi}{2}t_1}\right\} & \textit{for}\ \ B\neq -1, \\ \alpha_1 & \textit{for}\ \ B=-1, \end{cases} \tag{2.9}$$

and

$$\delta_2 = \begin{cases} \alpha_2 + \frac{2}{\pi}\tan^{-1}\left\{\frac{(\alpha_1+\alpha_2)(1-|c|)\cos\frac{\pi}{2}t_1}{2\left(\frac{(1-\eta)(1+A)}{1+B}+\eta+a-1\right)(1+|c|)+(\alpha_1+\alpha_2)(1-|c|)\sin\frac{\pi}{2}t_1}\right\} & \textit{for}\ \ B\neq -1, \\ \alpha_2 & \textit{for}\ \ B=-1, \end{cases} \tag{2.10}$$

where c is given by (2.3) and

$$t_1 = \frac{2}{\pi}\sin^{-1}\left(\frac{(1-\eta)(1-B)}{(1-\eta)(1-AB)+(\eta+a-1)(1-B^2)}\right). \tag{2.11}$$

Proof. Let

$$p(z) = \frac{1}{1-\gamma}\left(\frac{z(I_a^{\lambda+1}f(z))'}{I_a^{\lambda+1}g(z)} - \gamma\right).$$

Using the identity (1.11) and simplifying, we have

$$[(1-\gamma)p(z)+\gamma]I_a^{\lambda+1}g(z) = aI_a^{\lambda}f(z) - (a-1)I_a^{\lambda+1}f(z). \tag{2.12}$$

Differentiating (2.12) with respect to z and multiplying by z, we obtain

$$(1-\gamma)zp'(z)I_a^{\lambda+1}g(z)+[(1-\gamma)p(z)+\gamma]z(I_a^{\lambda+1}g(z))' = az(I_a^{\lambda}f(z))'-(a-1)z(I_a^{\lambda+1}f(z))'. \tag{2.13}$$

Since $g(z) \in S_\lambda^a(\eta, A, B)$, from Corollary 1, we know that $g(z) \in S_{\lambda+1}^a(\eta, A, B)$. Let

$$q(z) = \frac{1}{1-\eta}\left(\frac{z(I_a^{\lambda+1}g(z))'}{I_a^{\lambda+1}g(z)} - \eta\right) \quad (z \in U).$$

Then, using the identity (1.11) once again, we have

$$(1-\eta)q(z)+\eta+a-1 = a\frac{I_a^{\lambda}g(z)}{I_a^{\lambda+1}g(z)}. \tag{2.14}$$

From (2.13) and (2.14), we obtain

$$\frac{1}{1-\gamma}\left(\frac{z(I_a^{\lambda}f(z))'}{I_a^{\lambda}g(z)} - \gamma\right) = p(z) + \frac{zp'(z)}{(1-\eta)q(z)+\eta+a-1},$$

while, by using the result of Silverman and Silvia [20], we have

$$\left|q(z) - \frac{1-AB}{1-B^2}\right| < \frac{(A-B)}{1-B^2} \quad (z \in U; B \neq -1), \tag{2.15}$$

and

$$\mathrm{Re}\{q(z)\} > \frac{1-A}{2} \quad (z \in U; B = -1). \tag{2.16}$$

Then, from (2.15) and (2.16), we obtain

$$(1-\eta)q(z)+\eta+a-1 = \rho e^{i\frac{\pi\varphi}{2}},$$

where

$$\begin{cases} \frac{(1-\eta)(1-A)}{1-B}+\eta+a-1 < \rho < \frac{(1-\eta)(1+A)}{1+B}+\eta+a-1, \\ -t_1 < \varphi < t_1 \quad \text{for } B \neq -1, \end{cases}$$

when t_1 is given by (2.11), and

$$\begin{cases} \frac{(1-\eta)(1-A)}{2}+\eta+a-1 < \rho < \infty, \\ -1 < \varphi < 1 \quad \text{for } B = -1. \end{cases}$$

Here, we note that $p(z)$ is analytic in U with $p(0) = 1$ and $\mathrm{Re}\, p(z) > 0$ in U by applying the assumption and Lemma 2 with $w(z) = \dfrac{1}{(1-\eta)q(z)+\eta+a-1}$. Hence $p(z) \neq 0$ in U. If there exist two points $z_1, z_2 \in U$ such that the condition

(2.1) is satisfied, then (by Lemma 3) we obtain (2.2) under the restriction (2.3). At first, for the case $B \neq -1$, we obtain :

$$\begin{aligned}
&\arg\left(p(z_1) + \frac{z_1 p^{'}(z_1)}{(1-\eta)q(z_1)+\eta+a-1}\right) \\
= \quad & -\frac{\pi}{2}\alpha_1 + \arg\left(1 - i\frac{\alpha_1+\alpha_2}{2}m(\rho e^{i\frac{\pi\pi}{2}})^{-1}\right) \\
\leq \quad & -\frac{\pi}{2}\alpha_1 - \tan^{-1}\left\{\frac{(\alpha_1+\alpha_2)m\sin\frac{\pi}{2}(1-\varphi)}{2\rho+(\alpha_1+\alpha_2)m\cos\frac{\pi}{2}(1-\varphi)}\right\} \\
\leq \quad & -\frac{\pi}{2}\alpha_1 - \tan^{-1}\left\{\frac{(\alpha_1+\alpha_2)(1-|c|)\cos\frac{\pi}{2}t_1}{2\left(\frac{(1-\eta)(1+A)}{1+B}+\eta+a-1\right)(1+|c|)+(\alpha_1+\alpha_2)(1-|c|)\sin\frac{\pi}{2}t_1}\right\} \\
= \quad & -\frac{\pi}{2}\delta_1 \ ,
\end{aligned}$$

and

$$\begin{aligned}
&\arg\left(p(z_2) + \frac{z_2 p^{'}(z_2)}{(1-\eta)q(z_2)+\eta+a-1}\right) \\
\geq \quad & \frac{\pi}{2}\alpha_2 + \tan^{-1}\left\{\frac{(\alpha_1+\alpha_2)(1-|c|)\cos\frac{\pi}{2}t_1}{2\left(\frac{(1-\eta)(1+A)}{1+B}+\eta+a-1\right)(1+|c|)+(\alpha_1+\alpha_2)(1-|c|)\sin\frac{\pi}{2}t_1}\right\} \\
= \quad & \frac{\pi}{2}\delta_2 \ ,
\end{aligned}$$

where we have used the inequality (2.3), and δ_1, δ_2 and t_1 are given by (2.9), (2.10) and (2.11), respectively. Similarly, for the case $B = -1$, we obtain

$$\arg\left(p(z_1) + \frac{z_1 p^{'}(z_1)}{(1-\eta)q(z_1)+\eta+a-1}\right) \leq \frac{-\pi}{2}\alpha_1$$

and

$$\arg\left(p(z_2) + \frac{z_2 p^{'}(z_2)}{(1-\eta)q(z_2)+\eta+a-1}\right) \geq \frac{\pi}{2}\alpha_2 \ .$$

These are contradiction to the assumption of Theorem 1. This completes the proof of Theorem 1. □

Taking $\delta_1 = \delta_2 = \delta$ in Theorem 1, then we obtain :

Corollary 9. *The inclusion relation,* $K^a_\lambda(\gamma,\delta,\eta,A,B) \subset K^a_{\lambda+1}(\gamma,\delta,\eta,A,B)$ *holds for any* $a > 0$ *and* $\lambda \geq 0$.

Taking $\lambda = 0$, $a = 1$ and $\delta_1 = \delta_2 = \delta$ in Theorem 1, we obtain :

Corollary 10. *Let* $f(z) \in A_1$. *If*

$$\left|\arg(\frac{zf^{'}(z)}{g(z)} - \gamma)\right| < \frac{\pi}{2}\delta \quad (0 \leq \gamma < 1, 0 < \delta \leq 1),$$

for some $g \in S(\eta, A, B)$, *then*

$$\left|\arg(\frac{f(z)}{I^1_1 g(z)} - \gamma)\right| < \frac{\pi}{2}\alpha \ ,$$

where $\alpha\,(0<\alpha\le 1)$ *is the solution of the equation :*

$$\delta=\begin{cases}\alpha+\frac{2}{\pi}\tan^{-1}\left(\frac{\alpha\cos\frac{\pi}{2}t_1}{(\frac{(1-\eta)(1+A)}{1+B}+\eta)+\alpha\sin\frac{\pi}{2}t_1}\right), & \text{for } B\neq -1,\\ \alpha & \text{for } B=-1,\end{cases}$$

where t_1 *is given by (2.11) with* $a=1$.

Putting $\lambda=\gamma=0$, $a=1$, $B\to A\,(A<1)$, and $g(z)=z$ in Theorem 1, we obtain

Corollary 11. *Let* $f(z)\in A_1$ *and* $0<\delta_1,\delta_2\le 1$. *If*

$$-\frac{\pi}{2}\delta_1<\arg f'(z)<\frac{\pi}{2}\delta_2\ ,$$

then

$$-\frac{\pi}{2}\alpha_1<\arg\frac{f(z)}{z}<\frac{\pi}{2}\alpha_2\ ,$$

where α_1 *and* $\alpha_2\,(0<\alpha_1,\alpha_2\le 1)$ *are the solutions of the equations :*

$$\delta_1=\alpha_1+\frac{2}{\pi}\tan^{-1}\frac{(\alpha_1+\alpha_2)(1-|c|)}{2(1+|c|)}$$

and

$$\delta_2=\alpha_2+\frac{2}{\pi}\tan^{-1}\frac{(\alpha_1+\alpha_2)(1-|c|)}{2(1+|c|)}.$$

Next, we prove

Theorem 12. *Let* $f(z)\in A_1$ *and* $0<\delta_1,\delta_2\le 1, 0\le\gamma<1$. *If*

$$\frac{-\pi}{2}\delta_1<\arg\left(\frac{z(I_a^\lambda f(z))'}{I_a^\lambda g(z)}-\gamma\right)<\frac{\pi}{2}\delta_2$$

for some $g(z)\in S_\lambda^\alpha(\eta,A,B)$, *then*

$$\frac{-\pi}{2}\alpha_1<\arg\left(\frac{z(I_a^\lambda L_\theta(f)(z))'}{I_a^\lambda L_\theta(g)(z)}-\gamma\right)<\frac{\pi}{2}\alpha_2\ ,$$

where $L_\theta(f)$ *is defined by (2.6), and* α_1 *and* $\alpha_2\,(0<\alpha_1,\alpha_2\le 1)$ *are the solutions of the equations :*

$$\delta_1=\begin{cases}\alpha_1+\frac{2}{\pi}\tan^{-1}\left\{\frac{(\alpha_1+\alpha_2)(1-|c|)\cos\frac{\pi}{2}t_2}{2\left(\frac{(1-\eta)(1+A)}{1+B}+\eta+\theta\right)(1+|c|)+(\alpha_1+\alpha_2)(1-|c|)\sin\frac{\pi}{2}t_2}\right\} & \text{for } B\neq -1,\\ \alpha_1 & \text{for } B=-1,\end{cases}$$

and

$$\delta_2=\begin{cases}\alpha_2+\frac{2}{\pi}\tan^{-1}\left\{\frac{(\alpha_1+\alpha_2)(1-|c|)\cos\frac{\pi}{2}t_2}{2\left(\frac{(1-\eta)(1+A)}{1+B}+\eta+\theta\right)(1+|c|)+(\alpha_1+\alpha_2)(1-|c|)\sin\frac{\pi}{2}t_2}\right\} & \text{for } B\neq -1,\\ \alpha_2 & \text{for } B=-1,\end{cases}$$

where c *is given by (2.3) and* t_2 *is given by*

$$t_2=\frac{2}{\pi}\sin^{-1}\left\{\frac{(1-\eta)(A-B)}{(1-\eta)(1-AB)+(\eta+\theta)(1-B^2)}\right\}\ . \tag{2.17}$$

Proof. Let

$$p(z) = \frac{1}{1-\gamma}\left(\frac{z(I_a^\lambda L_\theta(f)(z))^{'}}{I_a^\lambda L_\theta(g)(z)} - \gamma\right) \quad (z \in U)\ .$$

Since $g(z) \in S_\lambda^a(\eta, A, B)$, we have from Corollary 2 that $L_\theta(g) \in S_\lambda^a(\eta, A, B)$. Using (2.7) we obtain

$$[(1-\gamma)p(z) + \gamma] I_a^\lambda L_\theta(g)(z) = (\theta+1) I_a^\lambda f(z) - \theta I_a^\lambda L_\theta(f)(z).$$

Then, by a simple calculation, we get

$$(1-\gamma)zp^{'}(z) + [(1-\gamma)p(z)+\gamma][(1-\eta)q(z)+\eta+\theta] =$$

$$(\theta+1)\frac{z(I_a^\lambda f(z))^{'}}{I_a^\lambda L_\theta(g)(z)}\ ,$$

where

$$q(z) = \frac{1}{1-\eta}\left(\frac{z(I_a^\lambda L_\theta(g)(z))^{'}}{I_a^\lambda L_\theta(g)(z)} - \eta\right)\ .$$

Hence we have

$$\frac{1}{1-\eta}\left(\frac{z(I_a^\lambda f(z))^{'}}{I_a^\lambda g(z)} - \eta\right) = p(z) + \frac{zp^{'}(z)}{(1-\eta)q(z)+\eta+\theta}\ .$$

The remaining part of the proof of Theorem 2 is similar to that of Theorem 1 and so we omit it. □

Taking $\delta_1 = \delta_2 = \delta$ in Theorem 2, we have

Corollary 13. *Let $f(z) \in A_1$ and $0 \le \gamma < 1$, $0 < \delta \le 1$. If*

$$\left|\arg\left(\frac{z(I_a^\lambda f(z))^{'}}{I_a^\lambda g(z)} - \gamma\right)\right| < \frac{\pi}{2}\delta$$

for some $g(z) \in S_\lambda^a(\eta, A, B)$, then

$$\left|\arg\left(\frac{z(I_a^\lambda L_\theta(f)(z))^{'}}{I_a^\lambda L_\theta(g)(z)} - \gamma\right)\right| < \frac{\pi}{2}\alpha\ ,$$

where $L_\theta(f)$ is defined by (2.6), and $\alpha\,(0 < \alpha \le 1)$ is the solution of the equation:

$$\delta = \begin{cases} \alpha + \frac{2}{\pi}\tan^{-1}\left(\frac{\alpha\cos\frac{\pi}{2}t_2}{\left(\frac{(1-\eta)(1+A)}{1+B}+\eta+\theta\right)+\alpha\sin\frac{\pi}{2}t_2}\right), & \text{for } B \ne -1, \\ \alpha & \text{for } B = -1, \end{cases}$$

where t_2 is given by (2.17).

From Corollary 6, we see easily the following corollary.

Corollary 14. *$f(z) \in K_\lambda^a(\gamma, \delta, \eta, A, B) \implies L_\theta(f) \in K_\lambda^a(\gamma, \alpha, \eta, A, B)$, where $L_\theta(f)$ is the integral operator defined by (2.6) and α is the solution of equation in Corollary 6.*

Taking $\lambda = 0$, $\delta = 1$, $A = 1$ and $B = -1$ in Corollary 7, we obtain :

Corollary 15. *Let $f(z) \in A_1$. If*

$$\operatorname{Re}\left\{\frac{zf'(z)}{g(z)}\right\} > \gamma \quad (0 \le \gamma < 1),$$

then

$$\operatorname{Re}\left\{\frac{z(L_\theta(f)(z))'}{L_\theta(g)(z)}\right\} > \gamma \quad (0 \le \gamma < 1)\ ,$$

where $L_\theta(f)$ is the integral operator defined by (2.6) and $g(z) \in S^*(\eta)\,(0 \le \eta < 1)$.

Remark 1. *Taking $\lambda = \gamma = \eta = 0$, $A = \delta = 1$ and $B = -1$ in Corollary 7, we obtain the classical result obtained by Bernardi [2], which implies the result studied by Libera [8].*

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