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DIFFERENTIAL SANDWICH THEOREMS FOR MULTIVALENT ANALYTIC FUNCTIONS DEFINED BY THE SRIVASTAVA–ATTIYA OPERATOR

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Abstract. In this paper, we obtain some applications of the theory of differential subordination and superordination results involving the operator $J_{s,b}^{\lambda,p}$ and other linear operators for certain normalized p-valent analytic functions associated with that operator.

1. Introduction

Let $H(U)$ be the class of analytic functions in the open unit disc $U = \{z : z \in \mathbb{C},\ |z| < 1\}$ and let $H[a, p]$ be the subclass of $H(U)$ consisting of functions of the form:

$$f(z) = a + a_p z^p + a_{p+1} z^{p+1} + \ldots \quad (a \in \mathbb{C}). \tag{1.1}$$

Also, let $A(p)$ denote the class of functions of the form:

$$f(z) = z^p + \sum_{k=1}^{\infty} a_{k+p} z^{k+p} \quad (p \in \mathbb{N} = \{1, 2, ...\}), \tag{1.2}$$

and let $A_1 = A(1)$.

If f , $g \in A(p)$, we say that f is subordinate to g, written $f \prec g$ if there exists a Schwarz function w, which (by definition) is analytic in U with $w(0) = 0$ and $|w(z)| < 1$ for all $z \in U$, such that $f(z) = g(w(z))$, $z \in U$. Furthermore, if the function g is univalent in U, then we have the following equivalence (cf., e.g., [7] ,[12] and [13]):

$$f(z) \prec g(z) \Leftrightarrow f(0) = g(0) \text{ and } f(U) \subset g(U).$$

Let $k, h \in H(U)$ and let $\varphi(r, s, t; z) : \mathbb{C}^3 \times U \to \mathbb{C}$. If k and $\varphi(k(z), zk'(z), z^2 k''(z); z)$ are univalent functions in U and if k satisfies the second-order superordination

$$h(z) \prec \varphi(k(z), zk'(z), z^2 k''(z); z), \tag{1.3}$$

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then p is a solution of the differential superordination (1.3). Note that if f subordinate to g, then g is superordinate to f. An analytic function q is called a subordinant of (1.3), if $q(z) \prec k(z)$ for all functions p satisfying (1.3). An univalent subordinant $\widetilde{q}$ that satisfies $q(z) \prec \widetilde{q}(z)$ for all subordinants of (1.3) is called the best subordinant. Recently, Miller and Mocanu [14] obtained sufficient conditions on the functions h, q and φ for which the following implication holds:

$$h(z) \prec \varphi\left(k(z), zk'(z), z^2k''(z); z\right) \Rightarrow q(z) \prec k(z). \tag{1.4}$$

Using the results of Miller and Mocanu [14], Bulboaca [6] considered certain classes of first order differential superordinations as well as superordination-preserving integral operators [5]. Ali et al. [1], have used the results of Bulboaca [6] to obtain sufficient conditions for normalized analytic functions to satisfy:

$$q_1(z) \prec \frac{zf'(z)}{f(z)} \prec q_2(z),$$

where q_1 and q_2 are given univalent functions in U with $q_1(0) = q_2(0) = 1$. Also, Tuneski [28] obtained a sufficient condition for starlikeness of f in terms of the quantity $\frac{f''(z)f(z)}{(f'(z))^2}$. Recently, Shanmugam et al. [22] obtained sufficient conditions for the normalized analytic functions f to satisfy

$$q_1(z) \prec \frac{f(z)}{zf'(z)} \prec q_2(z)$$

and

$$q_1(z) \prec \frac{z^2f'(z)}{\{f(z)\}^2} \prec q_2(z).$$

They [22] also obtained results for functions defined by using Carlson-Shaffer operator .

For functions f given by (1.1) and $g \in A(p)$ given by $g(z) = z^p + \sum_{k=1}^{\infty} b_{k+p} z^{k+p}$, the Hadamard product (or convolution) of f and g is defined by

$$(f * g)(z) = z^p + \sum_{k=1}^{\infty} a_{k+p} b_{k+p} z^{k+p} = (g * f)(z).$$

We begin our investigation by recalling that a general Hurwitz-Lerch Zeta function $\Phi(z, s, a)$ defined by (see [26])

$$\Phi(z, s, a) = \sum_{k=0}^{\infty} \frac{z^k}{(k+a)^s}\ , \tag{1.5}$$

$$a \in \mathbb{C} \backslash \mathbb{Z}_0^- = \{0, -1, -2, ...\}; \mathbb{Z}_0^- = \mathbb{Z} \backslash \mathbb{N}, \mathbb{Z} = \left\{0, \pm 1, \pm 2, ...\right\}; s \in \mathbb{C}$$

$$when\ \ |z| < 1; \Re\{s\} > 1\ when\ \ |z| = 1.$$

Recently, the Srivastava and Attiya [25] (see also [11], [17] and [18]) introduced and investigated the linear operator $J_{s,b}(f) : A_1 \to A_1$, defined in terms of the Hadamard product by

$$J_{s,b}f(z) = G_{s,b}(z) * f(z)\ (z \in U; b \in \mathbb{C} \backslash \mathbb{Z}_0^-; s \in \mathbb{C}),$$

where for convenience,

$$G_{s,b} = (1+b)^s [\Phi(z,s,b) - b^{-s}] \ (z \in U).$$

In [29], Wang et al. defined the operator $J_{s,b}^{\lambda,p} : A(p) \to A(p)$ by

$$J_{s,b}^{\lambda,p} f(z) = f_{s,b}^{\lambda,p}(z) * f(z) \tag{1.6}$$

$$(z \in U; b \in \mathbb{C}\backslash\mathbb{Z}_0^-; s \in C; \lambda > -p; p \in \mathbb{N}; f \in A(p)),$$

where

$$f_{s,b}^{p}(z) * f_{s,b}^{\lambda,p}(z) = \frac{z^p}{(1-z)^{\lambda+p}} \tag{1.7}$$

and

$$f_{s,b}^{p}(z) = z^p + \sum_{k=1}^{\infty} \left(\frac{p+k+b}{p+b}\right) z^{k+p} \qquad (z \in U; p \in \mathbb{N}). \tag{1.8}$$

It is easy to obtain from (1.6), (1.7) and (1.8) that

$$J_{s,b}^{\lambda,p} f(z) = z^p + \sum_{k=1}^{\infty} \frac{(\lambda+p)_k}{k!} \left(\frac{p+b}{k+p+b}\right)^s a_{k+p} z^{k+p}, \tag{1.9}$$

where $(\gamma)_k$, is the Pochhammer symbol defined in terms of the Gamma function Γ, by

$$(\gamma)_k = \frac{\Gamma(\gamma+n)}{\Gamma(\gamma)} = \begin{cases} 1 & (k=0) \\ \gamma(\gamma+1)....(\gamma+k-1) & (k \in \mathbb{N}). \end{cases}$$

We note that $J_{0,b}^{1-p,p} f(z) = f(z) \ (f \in A(p))$.

Using (1.9), it is easy to verify that (see [29])

$$z\left(J_{s+1,b}^{\lambda,p} f\right)'(z) = (p+b) J_{s,b}^{\lambda,p}(f)(z) - b J_{s+1,b}^{\lambda,p}(f)(z) \tag{1.10}$$

and

$$z\left(J_{s,b}^{\lambda,p} f\right)'(z) = (p+\lambda) J_{s,b}^{\lambda+1,p}(f)(z) - \lambda J_{s,b}^{\lambda,p}(f)(z). \tag{1.11}$$

It should be remarked that the linear operator $J_{s,b}^{\lambda,p} f(z)$ is generalization of many other linear operators considered earlier. We have:

(1) $J_{0,b}^{\lambda,p} f(z) = D^{\lambda+p-1} f(z) \ (\lambda > -p, p \in \mathbb{N})$, where $D^{\lambda+p-1}$ is the $(\lambda+p-1)$-th order Ruscheweyh derivative of a function $f(z) \in A(p)$ (see [10]);

(2) $J_{1,v}^{1-p,p} f(z) = J_{v,p} f(z) \ (v > -p)$, where the generalized Bernardi-Libera-Livingston operator $J_{v,p}$ was studied by Choi et al. [8];

(3) $J_{m,0}^{1-p,p} f(z) = I_p^m f(z) = z^p + \sum_{k=1}^{\infty} \left(\frac{p}{k+p}\right)^m a_{k+p} z^{k+p} \ (m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\})$, where for $p = 1$ the integral operator $I_1^m = I^m$ was introduced and studied by Salagean [20];

(4) $J_{\sigma,1}^{1-p,p} f(z) = I_p^{\sigma} f(z) \ (\sigma > 0)$, where the integral operator I_p^{σ} was studied by Shams et al. [21] and Aouf et al. [4];

(5) $J_{\gamma,\tau}^{0,1} f(z) = P_{\tau}^{\gamma} f(z) \ (\gamma \geq 0, \tau > 1)$, where the integral operator P_{τ}^{γ} was introduced and studied by Patel and Sahoo [16].

2. Definitions and preliminaries

In order to prove our results, we shall need the following definition and lemmas.

Definition 1 [14]. *Let Q be the set of all functions f that are analytic and injective on $\overline{U} \setminus E(f)$, where $E(f) = \{\zeta \in \partial U : \lim_{z\to\zeta} f(z) = \infty\}$, and are such that $f'(\zeta) \neq 0$ for $\zeta \in \partial U \setminus E(f)$.*

Lemma 1 [12]. *Let q be univalent in the unit disc U, and let θ and φ be analytic in a domain D containing $q(U)$, with $\varphi(w) \neq 0$ when $w \in q(U)$. Set $\mathbb{C}$*

$$Q(z) = zq'(z)\varphi(q(z)) \text{ and } h(z) = \theta(q(z)) + Q(z) \tag{2.1}$$

suppose that

(i) Q is a starlike function in U,

(ii) $\operatorname{Re}\left\{\frac{zh'(z)}{Q(z)}\right\} > 0,\ z \in U$.

If p is analytic in U with $k(0) = q(0)$, $p(U) \subseteq D$ and

$$\theta(k(z)) + zk'(z)\varphi(k(z)) \prec \theta(q(z)) + zq'(z)\varphi(q(z)), \tag{2.2}$$

then $k(z) \prec q(z)$, and q is the best dominant of (2.2).

Lemma 2 [24]**.** *Let $\xi, \beta \in \mathbb{C}$* with $\beta \neq 0$ and *let q be a convex function in U with*

$$\operatorname{Re}\left\{1 + \frac{zq''(z)}{q'(z)}\right\} > \max\{0; -\operatorname{Re}\frac{\xi}{\beta}\}.$$

If p is analytic in U and

$$\xi k(z) + \beta zk'(z) \prec \xi q(z) + \beta zq'(z), \tag{2.3}$$

then $k \prec q$ and q is the best dominant of (2.3).

Lemma 3 [6]. *Let q be a univalent function in U and let θ and φ be analytic in a domain D containing $q(U)$. Suppose that*

(i) $\operatorname{Re}\left\{\frac{\theta'(q(z))}{\varphi(q(z))}\right\} > 0$ for $z \in U$,

(ii) $Q(z) = zq'(z)\varphi(q(z))$ is starlike univalent in U.

If $k \in H[q(0), 1] \cap Q$, with $k(\mathrm{U}) \subseteq D$, $\theta(k(z)) + zk'(z)\varphi(k(z))$ is univalent in U, *and*

$$\theta(q(z)) + zq'(z)\varphi(q(z)) \prec \theta(k(z)) + zk'(z)\varphi(k(z)), \tag{2.4}$$

then $q(z) \prec k(z)$ and q is the best subordinant of (2.4).

Lemma 4 [14]**.** *Let q be convex univalent in U and let $\beta \in \mathbb{C}$*, with $\operatorname{Re}\{\beta\} > 0$. *If $k \in H[q(0), 1] \cap Q$, $k(z) + \beta zk'(z)$ is univalent in U and*

$$q(z) + \beta zq'(z) \prec k(z) + \beta zk'(z), \tag{2.5}$$

then $q \prec k$ and q is the best subordinant of (2.5).

Lemma 5 [19]. *The function $q(z) = (1 - z)^{-2ab}$ $(a, b \in \mathbb{C}^* = \mathbb{C}\setminus\{0\})$ is univalent in U if and only if $|2ab - 1| \leq 1$ or $|2ab + 1| \leq 1$.*

3. Subordinant results for analytic functions

Unless otherwise mentioned, we shall assume in the reminder of this paper that $b \in \mathbb{C}\backslash\mathbb{Z}_0^-, s \in \mathbb{C}, p \in \mathbb{N}, \lambda > -p,\ \gamma \in \mathbb{C}^*, z \in U$ and the powers are understood as principle values.

Theorem 1. *Let $q(z)$ be univalent in U, with $q(0) = 1$ and suppose that $\frac{zq'(z)}{q(z)}$ is starlike univalent in U. Let*

$$\operatorname{Re}\left\{1 + \frac{zq''(z)}{q'(z)}\right\} > \max\left\{0; -p\operatorname{Re}\left(\frac{b+p}{\gamma}\right)\right\}. \tag{3.1}$$

If $f(z) \in A(p)$ satisfies the subordination

$$\frac{\gamma}{p}\left(\frac{J_{s-1,b}^{\lambda,p}f(z)}{z^p}\right) + \frac{p-\gamma}{p}\left(\frac{J_{s,b}^{\lambda,p}f(z)}{z^p}\right) \prec q(z) + \frac{\gamma zq'(z)}{p(b+p)}. \tag{3.2}$$

Then

$$\frac{J_{s,b}^{\lambda,p}f(z)}{z^p} \prec q(z)$$

and q is the best dominant of (3.2).

Proof. Define a function $k(z)$ by

$$k(z) = \frac{J_{s,b}^{\lambda,p}f(z)}{z^p} \quad (z \in U), \tag{3.3}$$

by differentiating (3.3) logarithmically with respect to z, we obtain that

$$\frac{zk'(z)}{k(z)} = \frac{z\left(J_{s,b}^{\lambda,p}f(z)\right)'}{J_{s,b}^{\lambda,p}f(z)} - p. \tag{3.4}$$

From (3.4) and (1.10), a simple computation shows that

$$\frac{\gamma}{p}\left(\frac{J_{s-1,b}^{\lambda,p}f(z)}{z^p}\right) + \frac{p-\gamma}{p}\left(\frac{J_{s,b}^{\lambda,p}f(z)}{z^p}\right) = k(z) + \frac{\gamma zk'(z)}{p(b+p)}$$

hence *the subordination* (3.2) is equivalent to

$$k(z) + \frac{\gamma zk'(z)}{p(b+p)} \prec q(z) + \frac{\gamma zq'(z)}{p(b+p)}.$$

Combining this last relation together with Lemma 2 for the special case $\beta = \frac{\gamma}{p(b+p)}$ and $\xi = 1$ we obtain our result. ∎

Taking $q(z) = \frac{1+Az}{1+Bz}$ $(-1 \le B < A \le 1)$ in Theorem 1, the condition (3.1) reduces to

$$\operatorname{Re}\left\{\frac{1-Bz}{1+Bz}\right\} > \max\left\{0; -p\operatorname{Re}\left(\frac{b+p}{\gamma}\right)\right\}. \tag{3.5}$$

It is easy to check that the function $\psi(\zeta) = \frac{1-\zeta}{1+\zeta}$, $|\zeta| < |B|$, is convex in U and since $\psi(\bar{\zeta}) = \overline{\psi(\zeta)}$ for all $|\zeta| < |B|$, it follows that the image $\psi(U)$ is convex domain symmetric with respect to the real axis, hence

$$\inf\left\{\operatorname{Re}\frac{1-Bz}{1+Bz}\right\} = \frac{1-|B|}{1+|B|} > 0. \tag{3.6}$$

Then the inequality (3.5) is equivalent to $\frac{|B|-1}{|B|+1} \le p\operatorname{Re}\left(\frac{b+p}{\gamma}\right)$, hence, we obtain the following corollary.

Corollary 2. *Let $f(z) \in A(p)$, $-1 \le B < A \le 1$ and $\max\left\{0; -p\operatorname{Re}\left(\frac{b+p}{\gamma}\right)\right\} \le \frac{1-|B|}{1+|B|}$, then*

$$\frac{\gamma}{p}\left(\frac{J_{s-1,b}^{\lambda,p}f(z)}{z^p}\right) + \frac{p-\gamma}{p}\left(\frac{J_{s,b}^{\lambda,p}f(z)}{z^p}\right) \prec \frac{1+Az}{1+Bz} + \frac{\gamma}{p(b+p)}\frac{(A-B)z}{(1+Bz)^2}, \tag{3.7}$$

implies

$$\frac{J_{s,b}^{\lambda,p}f(z)}{z^p} \prec \frac{1+Az}{1+Bz}$$

and $\frac{1+Az}{1+Bz}$ *is the best dominant of (3.7).*

Taking $q(z)=\frac{1+z}{1-z}$ in Theorem 1 (or putting $A=1$ and $B=-1$ in Corollary 1), the condition (3.1) reduces to

$$p\,\mathrm{Re}\left(\frac{b+p}{\gamma}\right)\geq 0, \tag{3.8}$$

hence, we obtain the following corollary.

Corollary 3. *Let $f(z)\in A(p)$, assume that (3.8) holds true and*

$$\frac{\gamma}{p}\left(\frac{J_{s-1,b}^{\lambda,p}f(z)}{z^p}\right)+\frac{p-\gamma}{p}\left(\frac{J_{s,b}^{\lambda,p}f(z)}{z^p}\right)\prec\frac{1+z}{1-z}+\frac{2\gamma z}{p(p+b)(1-z)^2}, \tag{3.9}$$

then

$$\frac{J_{s,b}^{\lambda,p}f(z)}{z^p}\prec\frac{1+z}{1-z}$$

and $\dfrac{1+z}{1-z}$ *is the best dominant of (3.9).*

Now, by appealing to Lemma 1 it can be easily prove the following theorem.

Theorem 4. *Let $q(z)$ be univalent in U, with $q(0)=1$ and $q(z)\neq 0$ for all $z\in U$. Let $\mu,\delta\in\mathbb{C}^*$ and $\alpha,\tau\in\mathbb{C}$, with $\alpha+\tau\neq 0$. Let $f(z)\in A(p)$ and suppose that f and q satisfy the next conditions:*

$$\frac{\alpha J_{s-1,b}^{\lambda,p}f(z)+\tau J_{s,b}^{\lambda,p}f(z)}{(\alpha+\tau)z^p}\neq 0\quad(z\in U) \tag{3.10}$$

and

$$\mathrm{Re}\left\{1+\frac{zq^{''}(z)}{q^{'}(z)}-\frac{zq^{'}(z)}{q(z)}\right\}>0\quad(z\in U). \tag{3.11}$$

If

$$1+\delta\mu\left\{\frac{\alpha z\left(J_{s-1,b}^{\lambda,p}f(z)\right)^{'}+\tau z\left(J_{s,b}^{\lambda,p}f(z)\right)^{'}}{\alpha J_{s-1,b}^{\lambda,p}f(z)+\tau J_{s,b}^{\lambda,p}f(z)}-p\right\}\prec 1+\delta\frac{zq^{'}(z)}{q(z)}, \tag{3.12}$$

then

$$\left(\frac{\alpha J_{s-1,b}^{\lambda,p}f(z)+\tau J_{s,b}^{\lambda,p}f(z)}{(\alpha+\tau)z^p}\right)^{\mu}\prec q(z)$$

and q is the best dominant of (3.12).

Taking $q(z)=\frac{1+Az}{1+Bz}$ $(-1\leq B<A\leq 1)$, $\alpha=0$ and $\tau=\delta=1$ in Theorem 2, the condition (3.11) reduces to

$$\left\{1-\frac{2Bz}{1+Bz}-\frac{(A-B)z}{(1+Az)(1+Bz)}\right\}>0. \tag{3.13}$$

hence, we obtain the following corollary.

Corollary 5. *Let $f(z)\in A(p)$, assume that (3.13) holds true, $-1\leq B<A\leq 1$, $\mu\in\mathbb{C}^*$ and suppose that $\frac{J_{s,b}^{\lambda,p}f(z)}{z^p}\neq 0$ $(z\in U)$. If*

$$1+\mu\left\{\frac{z\left(J_{s,b}^{\lambda,p}f(z)\right)^{'}}{J_{s,b}^{\lambda,p}f(z)}-p\right\}\prec 1+\frac{(A-B)z}{(1+Az)(1+Bz)}, \tag{3.14}$$

then

$$\left(\frac{J_{s,b}^{\lambda,p}f(z)}{z^p}\right)^{\mu} \prec \frac{1+Az}{1+Bz}, \tag{3.15}$$

and $\dfrac{1+Az}{1+Bz}$ *is the best dominant of (3.14).*

Putting $\alpha = 0, \tau = 1,\ \delta = \frac{1}{ab}\,(a, b \in \mathbb{C}^*)\,, \mu = a, s = 0, \lambda = 1-p\,(p \in \mathbb{N})$ and $q(z) = (1-z)^{-2ab}$ in Theorem 2, hence combining this together with Lemma 5, we obtain the following corollary.

Corollary 6. *Let $f(z) \in A(p)$, assume that (3.11) holds true and $a, b \in \mathbb{C}^*$ such that $|2ab-1| \le 1$ or $|2ab+1| \le 1$. If*

$$1 + \frac{1}{b}\left(\frac{zf'(z)}{f(z)} - p\right) \prec \frac{1+z}{1-z}, \tag{3.16}$$

then

$$\left(\frac{f(z)}{z^p}\right)^{a} \prec (1-z)^{-2ab}$$

and $(1-z)^{-2ab}$ is the best dominant of (3.16).

Remark 1. *(i) For $p = 1$, Corollary 4 reduces to the result obtained by Obradović et al. [15, Theorem 1], the recent result of Aouf and Bulboacă et [3, Corollary 3.3] and the recent result of El-Ashwah and Aouf [9, Corollary 4];*

(ii) For $p = a = 1$, Corollary 4 reduces to the recent result of Srivastava and Lashin [27, Theorem 3] and the recent result of Shanmugam et al. [23, Corollary 3.6].

Remark 2. *Putting $p = 1,\ s = 0,\ \delta = \frac{e^{i\lambda}}{ab\cos\lambda}\ \left(a, b \in \mathbb{C}^*;\ |\lambda| < \dfrac{\pi}{2}\right)$, $\mu = a$ and $q(z) = (1-z)^{-2ab\cos\lambda e^{-i\lambda}}$ in Theorem 2, we obtain the result obtained by Aouf et al. [2, Theorem 1], the recent result of Aouf and Bulboacă et [3, Corollary 3.5] and the recent result of El-Ashwah and Aouf [9, Corollary 6].*

Putting $\alpha = 0, \tau = \delta = 1, s = 0, \lambda = 1-p$ and $q(z) = (1+Bz)^{\frac{\mu(A-B)}{B}}$ $(\mu \in \mathbb{C}^*,\ -1 \le B < A \le 1,\ B \ne 0)$ in Theorem 2, it is easy to check that the assumption (3.11) holds, hence we get the next corollary:

Corollary 7. *Let $f \in A(p)$, $\mu \in \mathbb{C}^*$, $-1 \le B < A \le 1$, with $B \ne 0$ and suppose that $\left|\frac{\mu(A-B)}{B} - 1\right| \le 1$ or $\left|\frac{\mu(A-B)}{B} + 1\right| \le 1$. If*

$$1 + \mu\left(\frac{zf'(z)}{f(z)} - p\right) \prec \frac{1+[B+\mu(A-B)z]}{1+Bz}, \tag{3.17}$$

then

$$\left(\frac{f(z)}{z^p}\right)^{\mu} \prec (1+Bz)^{\frac{\mu(A-B)}{B}}$$

and $(1+Bz)^{\frac{\mu(A-B)}{B}}$ is the best dominant of (3.17).

Remark 3. *For $p = 1$, Corollary 5 reduces to the result obtained by Aouf and Bulboaca [3, Corollary 3.4] and the recent result of El-Ashwah and Aouf [9, Corollary 5].*

By using Lemma 1, we obtain the following result.

Theorem 8. *Let $q(z)$ be univalent in U, with $q(0) = 1$, let $\mu, \delta \in \mathbb{C}^*$ and let $\alpha, \tau, \sigma, \eta \in \mathbb{C}$, with $\alpha + \tau \neq 0$. Let $f(z) \in A(p)$ and suppose that f and q satisfy the next two conditions:*

$$\frac{\alpha J_{s-1,b}^{\lambda,p} f(z) + \tau J_{s,b}^{\lambda,p} f(z)}{(\alpha+\tau) z^p} \neq 0 \quad (z \in U) \tag{3.18}$$

and

$$\operatorname{Re}\left\{1 + \frac{zq''(z)}{q'(z)}\right\} > \max\{0; -\operatorname{Re}\frac{\sigma}{\delta}\} \quad (z \in U). \tag{3.19}$$

If

$$\mathfrak{F}(z) = \left(\frac{\alpha J_{s-1,b}^{\lambda,p} f(z) + \tau J_{s,b}^{\lambda,p} f(z)}{(\alpha+\tau) z^p}\right)^{\mu} \cdot \left[\sigma + \delta\mu\left(\frac{\alpha z\left(J_{s-1,b}^{\lambda,p} f(z)\right)' + \tau z\left(J_{s,b}^{\lambda,p} f(z)\right)'}{\alpha J_{s-1,b}^{\lambda,p} f(z) + \tau J_{s,b}^{\lambda,p} f(z)} - p\right)\right] + \eta \tag{3.20}$$

and

$$\mathfrak{F}(z) \prec \sigma q(z) + \delta z q^{'}(z) + \eta \tag{3.21}$$

then

$$\left(\frac{\alpha J_{s-1,b}^{\lambda,p} f(z) + \tau J_{s,b}^{\lambda,p} f(z)}{(\alpha+\tau) z^p}\right)^{\mu} \prec q(z) \tag{3.22}$$

and q is the best dominant of (3.22).

Taking $q(z) = \frac{1+Az}{1+Bz}$ $(-1 \leq B < A \leq 1)$ and using (3.6), the condition (3.22) reduces to

$$\max\left\{0; -\operatorname{Re}\frac{\sigma}{\delta}\right\} \leq \frac{1-|B|}{1+|B|}, \tag{3.23}$$

hence, putting $\delta = \alpha = 1$ and $\tau = 0$ in Theorem 3, we obtain the following corollary.

Corollary 9. *Let $f(z) \in A(p)$, $-1 \leq B < A \leq 1$ and let $\sigma \in \mathbb{C}$ with $\max\{0; -\operatorname{Re}\sigma\} \leq \frac{1-|B|}{1+|B|}$, suppose that $\frac{J_{s-1,b}^{\lambda,p} f(z)}{z^p} \neq 0$ $(z \in U)$ and let $\mu \in \mathbb{C}^*$. If*

$$\left(\frac{J_{s-1,b}^{\lambda,p} f(z)}{z^p}\right)^{\mu} \cdot \left[\sigma + \mu\left(\frac{z\left(J_{s-1,b}^{\lambda,p} f(z)\right)'}{J_{s-1,b}^{\lambda,p} f(z)} - p\right)\right] + \eta \prec \sigma\frac{1+Az}{1+Bz} + \frac{(A-B)z}{(1+Bz)^2} + \eta, \tag{3.24}$$

then

$$\left(\frac{J_{s-1,b}^{\lambda,p} f(z)}{z^p}\right)^{\mu} \prec \frac{1+Az}{1+Bz}$$

and $\frac{1+Az}{1+Bz}$ is the best dominant of (3.24).

Putting $s = 0, \lambda = 1 - p\,(p \in \mathbb{N}), \delta = \tau = 1$, $\alpha = 0$ and $q(z) = \dfrac{1+z}{1-z}$ in Theorem 3, we obtain the following corollary.

Corollary 10. *Let $f(z) \in A(p)$ such that $\frac{f(z)}{z^p} \neq 0$ for all $z \in U$ and let $\mu \in \mathbb{C}^*$. If*

$$\left(\frac{f(z)}{z^p}\right)^{\mu} \cdot \left[\sigma + \mu\left(\frac{zf^{'}(z)}{f(z)} - p\right)\right] + \eta \prec \sigma\frac{1+z}{1-z} + \frac{2z}{(1-z)^2} + \eta, \tag{3.25}$$

then

$$\left(\frac{f(z)}{z^p}\right)^{\mu} \prec \frac{1+z}{1-z}$$

and $\dfrac{1+z}{1-z}$ is the best dominant of (3.25).

Remark 4. *For $p=1$, Corollary 7 reduces to the result obtained by Aouf and Bulboaca [3, Corollary 3.7] and the recent result of El-Ashwah and Aouf [9, Corollary 8].*

4. Superordination and sandwich results

Theorem 11. *Let $q(z)$ be convex in U, with $q(0)=1$ and*

$$p^{-1}\operatorname{Re}\left(\frac{\gamma}{b+p}\right)>0. \tag{4.1}$$

Let $f(z)\in A(p)$ and suppose that $\frac{J_{s,b}^{\lambda,p}f(z)}{z^p}\in H[q(0),1]\cap Q$. If the function

$$\frac{\gamma}{p}\left(\frac{J_{s-1,b}^{\lambda,p}f(z)}{z^p}\right)+\frac{p-\gamma}{p}\left(\frac{J_{s,b}^{\lambda,p}f(z)}{z^p}\right)$$

is univalent in U, and

$$q(z)+\frac{\gamma z q'(z)}{p(b+p)}\prec\frac{\gamma}{p}\left(\frac{J_{s-1,b}^{\lambda,p}f(z)}{z^p}\right)+\frac{p-\gamma}{p}\left(\frac{J_{s,b}^{\lambda,p}f(z)}{z^p}\right). \tag{4.2}$$

Then

$$q(z)\prec\frac{J_{s,b}^{\lambda,p}f(z)}{z^p}$$

and q is the best subordinant of (4.2).

Proof. Define a function $g(z)$ by

$$g(z)=\frac{J_{s,b}^{\lambda,p}f(z)}{z^p}\quad(z\in U).$$

From the assumption of the theorem, the function g is analytic in U and differentiating logarithmically with respect to z the above definition, we obtain

$$\frac{zg'(z)}{g(z)}=\frac{z\left(J_{s,b}^{\lambda,p}f(z)\right)'}{J_{s,b}^{\lambda,p}f(z)}-p. \tag{4.3}$$

After some computations and using the identity (1.10), from (4.3), we obtain

$$\frac{\gamma}{p}\left(\frac{J_{s-1,b}^{\lambda,p}f(z)}{z^p}\right)+\frac{p-\gamma}{p}\left(\frac{J_{s,b}^{\lambda,p}f(z)}{z^p}\right)=g(z)+\frac{\gamma z g'(z)}{p(b+p)}$$

and now, by using Lemma 4 we get the desired result. ∎

Taking $q(z)=\frac{1+Az}{1+Bz}$ $(-1\le B<A\le 1)$ in Theorem 4, we obtain the following corollary.

Corollary 12. *Let $q(z)$ be convex in U, with $q(0)=1$ and $\left[p^{-1}\operatorname{Re}\left(\frac{\gamma}{b+p}\right)\right]>0$. Let $f(z)\in A(p)$ and suppose that $\frac{J_{s,b}^{\lambda,p}f(z)}{z^p}\in H[q(0),1]\cap Q$. If the function*

$$\frac{\gamma}{p}\left(\frac{J_{s-1,b}^{\lambda,p}f(z)}{z^p}\right)+\frac{p-\gamma}{p}\left(\frac{J_{s,b}^{\lambda,p}f(z)}{z^p}\right)$$

is univalent in U, and

$$\frac{1+Az}{1+Bz}+\frac{\gamma}{p(b+p)}\frac{(A-B)z}{(1+Bz)^2}\prec\frac{\gamma}{p}\left(\frac{J_{s-1,b}^{\lambda,p}f(z)}{z^p}\right)+\frac{p-\gamma}{p}\left(\frac{J_{s,b}^{\lambda,p}f(z)}{z^p}\right). \tag{4.4}$$

Then

$$\frac{1+Az}{1+Bz}\prec\frac{J_{s,b}^{\lambda,p}f(z)}{z^p}$$

and $\frac{1+Az}{1+Bz}$ $(-1 \leq B < A \leq 1)$ *is the best subordinant of (4.4).*

By applying Lemma 3, we obtain the following result.

Theorem 13. *Let $q(z)$ be convex in U, with $q(0) = 1$, let $\mu, \delta \in \mathbb{C}^*$ and let $\alpha, \tau, \sigma, \eta \in \mathbb{C}$, with $\alpha + \tau \neq 0$ and $\operatorname{Re} \frac{\sigma}{\delta} > 0$. Let $f(z) \in A(p)$ and suppose that f satisfies the next conditions:*

$$\frac{\alpha J_{s-1,b}^{\lambda,p} f(z) + \tau J_{s,b}^{\lambda,p} f(z)}{(\alpha+\tau) z^p} \neq 0 \ \ (z \in U)$$

and

$$\left(\frac{\alpha J_{s-1,b}^{\lambda,p} f(z) + \tau J_{s,b}^{\lambda,p} f(z)}{(\alpha+\tau) z^p} \right)^{\mu} \in H[q(0), 1] \cap Q.$$

If the function $\mathfrak{F}$ given by (3.20) is univalent in U and

$$\sigma q(z) + \delta z q^{'}(z) + \eta \prec \mathfrak{F}(z) \tag{4.5}$$

then

$$q(z) \prec \left(\frac{\alpha J_{s-1,b}^{\lambda,p} f(z) + \tau J_{s,b}^{\lambda,p} f(z)}{(\alpha+\tau) z^p} \right)^{\mu}$$

and q is the best subordinant of (4.5).

Combining Theorem 1 and Theorem 4, we get the following sandwich theorem.

Theorem 14. *Let q_1 and q_2 be two convex functions in U, with $q_1(0) = q_2(0) = 1$ and $\left[p^{-1} \operatorname{Re} \left(\frac{\gamma}{b+p} \right) \right] > 0$. Let $f(z) \in A(p)$ and suppose that $\frac{J_{s,b}^{\lambda,p} f(z)}{z^p} \in H[q(0), 1] \cap Q$. If the function $\frac{\gamma}{p} \left(\frac{J_{s-1,b}^{\lambda,p} f(z)}{z^p} \right) + \frac{p-\gamma}{p} \left(\frac{J_{s,b}^{\lambda,p} f(z)}{z^p} \right)$ is univalent in U, and*

$$q_1(z) + \frac{\gamma z q_1^{'}(z)}{p(b+p)} \prec \frac{\gamma}{p} \left(\frac{J_{s-1,b}^{\lambda,p} f(z)}{z^p} \right) + \frac{p-\gamma}{p} \left(\frac{J_{s,b}^{\lambda,p} f(z)}{z^p} \right) \prec q_2(z) + \frac{\gamma z q_2^{'}(z)}{p(b+p)}. \tag{4.6}$$

Then

$$q_1(z) \prec \frac{J_{s,b}^{\lambda,p} f(z)}{z^p} \prec q_2(z)$$

and q_1 and q_2 are, respectively, the best subordinant and dominant of (4.6).

Combining Theorem 3 and Theorem 5, we get the following sandwich theorem.

Theorem 15. *Let q_1 and q_2 be two convex functions in U, with $q_1(0) = q_2(0) = 1$, let $\mu, \delta \in \mathbb{C}^*$ and let $\alpha, \tau, \sigma, \eta \in \mathbb{C}$, with $\alpha+\tau \neq 0$ and $\operatorname{Re} \frac{\sigma}{\delta} > 0$. Let $f(z) \in A(p)$ satisfies $\frac{\alpha J_{s-1,b}^{\lambda,p} f(z) + \tau J_{s,b}^{\lambda,p} f(z)}{(\alpha+\tau) z^p} \neq 0$ $(z \in U)$ and $\left(\frac{\alpha J_{s-1,b}^{\lambda,p} f(z) + \tau J_{s,b}^{\lambda,p} f(z)}{(\alpha+\tau) z^p} \right)^{\mu} \in H[q(0), 1] \cap Q$. If the function $\mathfrak{F}$ given by (3.20) is univalent in U and*

$$\sigma q_1(z) + \delta z q_1^{'}(z) + \eta \prec \mathfrak{F}(z) \prec \sigma q_2(z) + \delta z q_2^{'}(z) + \eta \tag{4.7}$$

then

$$q_1(z) \prec \left(\frac{\alpha J_{s-1,b}^{\lambda,p} f(z) + \tau J_{s,b}^{\lambda,p} f(z)}{(\alpha+\tau) z^p} \right)^{\mu} \prec q_2(z)$$

and q_1 and q_2 are, respectively, the best subordinant and dominant of(4.7).

Remark 5. *Specializing s, λ and b, in the above results, we obtain the corresponding results for the corresponding operators (1-5) defined in the introduction.*

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