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STABILITY OF A QUARTIC AND ORTHOGONALLY QUARTIC FUNCTIONAL EQUATION

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Abstract. In this paper, the authors investigate the generalized Hyers-Ulam-Aoki-Rassias stability of a quartic functional equation

$$\begin{aligned} &g(2x+y+z)+g(2x+y-z)+g(2x-y+z)+g(-2x+y+z)+16g(y)+16g(z)\\ &=8[g(x+y)+g(x-y)+g(x+z)+g(x-z)]+2[g(y+z)+g(y-z)]+32g(x). \end{aligned} \tag{1}$$

The above equation (1) is modified and its Hyers-Ulam-Aoki-Rassias stability for the following quartic functional equation

$$\begin{aligned} &f(2x+y+z)+f(2x+y-z)+f(2x-y+z)+f(-2x+y+z)+f(2y)+f(2z)\\ &=8[f(x+y)+f(x-y)+f(x+z)+f(x-z)]+2[f(y+z)+f(y-z)]+32f(x) \end{aligned} \tag{2}$$

for all $x, y, z \in X$ with $x \perp y, y \perp z$ and $z \perp x$ is discussed in orthogonality space in the sense of Rätz.

1. INTRODUCTION AND PRELIMINARIES

In 1940, S.M. Ulam [30] posed the stability problem. In 1941, D.H. Hyers [12] gave a partial answer to the question of Ulam. In 1950, Aoki [5] generalized the Hyers theorem for additive mappings. In 1978, Th.M. Rassias [22] proved a further generalization of Hyers theorem for linear mappings by considering an unbounded Cauchy difference for sum of powers of norms $\epsilon\left(||x||^p+||y||^p\right)$.

The result of Th.M. Rassias influenced the development stability theory and now it is called the Hyers-Ulam-Rassias stability for functional equations. Following the spirit of the innovative approach of Th.M. Rassias, a similar stability theorem was proved by J.M. Rassias [19, 20] in which he replaced the term $||x||^p+||y||^p$ by $||x||^p||y||^p$ (product of norms). Later this stability result is called the Ulam-Gavruta-Rassias stability of functional equations (see [27]).

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All the above stability results are further generalized by P. Gavruta [9] in 1994 considering the control function as function of variables $\phi(x, y)$ and he proved the following theorem.

Theorem 1.1. [9] *Let $(G, +)$ be an Abelian group, $(X, ||\cdot||)$ be a Banach space and $\phi : G \times G \to [0, \infty)$ be a mapping such that*

$$\Phi(x, y) = \sum_{k=0}^{\infty} 2^{-k}\phi(2^k x, 2^k y) < \infty. \tag{1.1}$$

If a function $f : G \to E$ satisfies the inequality

$$\|f(x + y) - f(x) - f(y)\| \leq \phi(x, y) \tag{1.2}$$

for any $x, y \in G$, then there exists a unique additive function $T : G \to E$ such that

$$\|f(x) - T(x)\| \leq \frac{1}{2}\Phi(x, x) \tag{1.3}$$

for all x in G. If moreover $f(tx)$ is continuous in t for each fixed $x \in G$, then T is linear.

This result is called Generalized Hyers - Ulam - Aoki - Rassias stability of functional equation $f(x+y) = f(x)+f(y)$. Very recently J.M. Rassias [27] introduced a new concept on stability called JMRassias Mixed type product-sum of powers of norms stability. This stability result is called JMRassias stability for functional equations.

The functional equation

$$f(x + y) + f(x - y) = 2f(x) + 2f(y) \tag{1.4}$$

is said to be quadratic functional equation because the quadratic function $f(x) = ax^2$ is a solution of the functional equation (1.4). It is well known that a function f is a solution of (1.4) if and only if there exists a unique symmetric biadditive function B such that $f(x) = B(x, x)$ for all x (see [16]). The biadditive function B is given by

$$B(x, y) = \frac{1}{4}\left[f(x + y) + f(x - y)\right].$$

In Section 2, authors investigate the general solution and the generalized Hyers-Ulam-Aoki-Rassias stability of a quartic functional equation (1). Also in Section 3, the Hyers-Ulam-Aoki-Rassias stability of the modified orthogonally quartic functional equation (2) for all $x, y, z \in X$ with $x \perp y, y \perp z$ and $z \perp x$ in the sense of Rätz is discussed.

2. GENERAL SOLUTION AND STABILITY OF THE FUNCTIONAL EQUATION (2.3)

The quartic functional equation

$$f(x + 2y) + f(x - 2y) + 6f(x) = 4[f(x + y) + f(x - y) + 6f(y)] \tag{2.1}$$

was first introduced by J.M. Rassias [21], who solved its Ulam stability problem. Later P.K. Sahoo and J.K. Chung [28], S.H. Lee et al., [17] remodified J.M. Rassias's equation and obtained its general solution.

The general solution of the generalized quartic functional equation

$$f(ax+by)+f(ax-by) = (ab)^2[f(x+y)+f(x-y)]+2(b^2-a^2)[b^2f(y)-a^2f(x)] \quad (2.2)$$

for all $x, y \in \mathbb{R}, a \neq b$, and $a, b \neq 0, \pm 1$ using Fréchet functional equation was discussed in [29]. Very recently the generalized Hyers-Ulam-Aoki-Rassias stability for a 3 dimensional quartic functional equation

$$\begin{aligned} &g(2x + y + z) + g(2x + y - z) + g(2x - y + z) + g(-2x + y + z) + 16g(y) + 16g(z) \\ &= 8[g(x + y) + g(x - y) + g(x + z) + g(x - z)] + 2[g(y + z) + g(y - z)] + 32g(x) \end{aligned} \quad (2.3)$$

in fuzzy normed space was discussed by M. Arunkumar [7].

2.1. **GENERAL SOLUTION OF THE FUNCTIONAL EQUATION (2.3).**

In this section, the authors discussed the general solution of the functional equation (2.3) by considering X and Y are real vector spaces.

Theorem 2.1. *If $g : X \to Y$ satisfies the functional equation (2.3) then there exists a a unique symmetric multi - additive function $Q : X \times X \times X \times X \to Y$ such that $g(x) = Q(x, x, x, x)$ for all $x \in X$. .*

Proof. Letting $x = y = z = 0$ in (2.3), we get $f(0) = 0$. Replacing (x, y) by $(0, 0)$, we obtain

$$g(-z) = g(z) \quad (2.4)$$

for all $z \in X$. Again replacing (y, z) by $(0, 0)$ and $(x, 0)$ respectively, we get

$$g(2x) = 16g(x) \qquad \textit{and} \qquad g(3x) = 81g(x) \quad (2.5)$$

for all $x \in X$. In general for any positive integer n, we obtain

$$g(nx) = n^4\ g(x)$$

for all $x \in X$. Letting $z = 0$ in (2.3) and using (2.4), (2.5), we arrive

$$g(2x + y) + g(2x - y) = 4[g(x + y) + g(x - y)] + 24g(x) - 6g(y) \quad (2.6)$$

for all $x, y, z \in X$. By Theorem 2.1 of [17], there exist a unique symmetric multi - additive function $Q : X \times X \times X \times X \to Y$ such that $g(x) = Q(x, x, x, x)$ for all $x \in X$. □

2.2. **GENERALIZED HYERS - ULAM - AOKI - RASSIAS STABILITY OF THE FUNCTIONL EQUATION (2.3).**

In this section, let X be a normed space and Y be a Banach space. Define a difference operator $Dg : X \times X \times X \to Y$ by

$$\begin{aligned} Dg(x, y, z) = &\ g(2x + y + z) + g(2x + y - z) + g(2x - y + z) + g(-2x + y + z) \\ &+ 16g(y) + 16g(z) - 8[g(x + y) + g(x - y) + g(x + z) + g(x - z)] \\ &- 2[g(y + z) + g(y - z)] - 32g(x) \end{aligned}$$

for all $x, y, z \in X$ and investigate its generalized Hyers - Ulam - Aoki - Rassias stability of the functional equation (2.3).

Theorem 2.2. *Let $j \in \{-1, 1\}$. Let $\alpha : X^3 \to [0, \infty)$ be a function such that*

$$\lim_{n\to\infty} \frac{\alpha\left(4^{nj}x, 4^{nj}y, 4^{nj}z\right)}{4^{4nj}} = 0 \quad and \quad \sum_{n=0}^{\infty} \frac{\alpha\left(4^{nj}x, 4^{nj}y, 4^{nj}z\right)}{4^{4nj}} \quad converges \tag{2.7}$$

for all $x, y, z \in X$ and let $g : X \to Y$ be a function satisfies the inequality

$$\|Dg\,(x, y, z)\| \leq \alpha\,(x, y, z) \tag{2.8}$$

for all $x, y, z \in X$. Then there exists a unique quartic function $Q : X \to Y$ such that

$$\|g(x) - Q(x)\| \leq \frac{1}{4^4} \sum_{k=0}^{\infty} \frac{\beta(4^{kj}x)}{4^{4kj}} \tag{2.9}$$

where

$$\beta(4^{kj}x) = \alpha(4^{kj}x, 4^{kj}y, 4^{kj}z) + 4\alpha(4^{kj}x, 0, 0)$$

for all $x \in X$. The mapping $Q(x)$ is defined by

$$Q(x) = \lim_{n\to\infty} \frac{g(4^{nj}x)}{4^{4nj}} \tag{2.10}$$

for all $x \in X$.

Proof. Assume $j = 1$. Replacing (x, y, z) by (x, x, x) in (2.8), we get

$$\|g(4x) - 16g(2x)\| \leq \alpha\,(x, x, x) \tag{2.11}$$

for all $x \in X$. Again replacing (x, y, z) by $(x, 0, 0)$ in (2.8), we obtain

$$\|4g(2x) - 64g(x)\| \leq \alpha\,(x, 0, 0) \tag{2.12}$$

for all $x \in X$. From (2.11) and (2.12), we arrive

$$\begin{aligned} \|g(4x) - 256g(x)\| &= \|g(4x) - 16g(2x) + 16g(2x) - 256g(x)\| \\ &\leq \|g(4x) - 16g(2x)\| + 4\,\|4g(2x) - 64g(x)\| \\ &\leq \alpha\,(x, x, x) + 4\alpha\,(x, 0, 0) \end{aligned} \tag{2.13}$$

for all $x \in X$. Hence from (2.13), we get

$$\left\|\frac{g(4x)}{4^4} - g(x)\right\| \leq \frac{1}{4^4}\left[\alpha\,(x, x, x) + 4\alpha\,(x, 0, 0)\right] \tag{2.14}$$

for all $x \in X$. Hence it follows from (2.14), we obtain

$$\left\|\frac{g(4x)}{4^4} - g(x)\right\| \leq \frac{\beta(x)}{4^4} \tag{2.15}$$

where

$$\beta(x) = \alpha\,(x, x, x) + 4\alpha\,(x, 0, 0)$$

for all $x \in X$. Now replacing x by $4x$ and dividing by 4^4 in (2.15), we get

$$\left\|\frac{g(4^2x)}{4^8} - \frac{g(4x)}{4^4}\right\| \leq \frac{\beta(4x)}{4^8} \tag{2.16}$$

for all $x \in X$. From (2.15) and (2.16), we arrive

$$\begin{aligned}\left\|\frac{g(4^2x)}{4^8} - g(x)\right\| &\le \left\|\frac{g(4^2x)}{4^8} - \frac{g(4x)}{4^4}\right\| + \left\|\frac{g(4x)}{4^4} - g(x)\right\| \\ &\le \frac{1}{4^4}\left[\beta(x) + \frac{\beta(4x)}{4^4}\right] \qquad (2.17)\end{aligned}$$

for all $x \in X$. In general for any positive integer n , we get

$$\begin{aligned}\left\|\frac{g(4^nx)}{4^{4n}} - g(x)\right\| &\le \frac{1}{4^4}\sum_{k=0}^{n-1}\frac{\beta(4^kx)}{4^{4k}} \qquad (2.18)\\ &\le \frac{1}{4^4}\sum_{k=0}^{\infty}\frac{\beta(4^kx)}{4^{4k}}\end{aligned}$$

for all $x \in X$. In order to prove the convergence of the sequence $\left\{\frac{g(4^nx)}{4^{4n}}\right\}$, replace x by 4^mx and divided by 4^{4m} in (2.18), for any $m, n > 0$, we arrive

$$\begin{aligned}\left\|\frac{g(4^{n+m}x)}{4^{4(n+m)}} - \frac{g(4^mx)}{4^{4m}}\right\| &= \frac{1}{4^{4m}}\left\|\frac{g(4^n4^mx)}{4^{4n}} - g(4^mx)\right\| \\ &\le \frac{1}{4^4}\sum_{k=0}^{\infty}\frac{\beta(4^{k+m}x)}{4^{4(k+m)}} \\ &\to 0 \quad as \ \ m \to \infty \qquad (2.19)\end{aligned}$$

for all $x \in X$. Hence the sequence $\left\{\frac{g(4^nx)}{4^{4n}}\right\}$ is Cauchy sequence. Since Y is complete, there exists a mapping $Q : X \to Y$ such that

$$Q(x) = \lim_{n\to\infty}\frac{g(4^nx)}{4^{4n}} \ \ \forall \ x \in X.$$

Letting $n \to \infty$ in (2.18) we see that (2.9) holds for all $x \in X$. To prove Q satisfies (2.3), replacing (x, y, z) by $(4^nx, 4^ny, 4^nz)$ and divided by 4^{4n} in (2.8), we arrive

$$\begin{aligned}&\frac{1}{4^{4n}}\Big\|g(4^n(2x+y+z)) + g(4^n(2x+y-z)) + g(4^n(2x-y+z)) + g(4^n(-2x+y+z)) \\ &+ 16g(4^ny) + 16g(4^nz) - 8[g(4^n(x+y)) + g(4^n(x-y)) + g(4^n(x+z)) + g(4^n(x-z))] \\ &\quad - 2[g(4^n(y+z)) + g(4^n(y-z))] - 32g(4^nx)\Big\| \le \frac{1}{4^{4n}}\alpha(4^nx, 4^ny, 4^nz)\end{aligned}$$

for all $x, y, z \in X$. Letting $n \to \infty$ and in the above inequality we see that

$$\begin{aligned}&Q(2x+y+z) + Q(2x+y-z) + Q(2x-y+z) + Q(-2x+y+z) + 16Q(y) + 16Q(z) \\ &\quad = 8[Q(x+y) + Q(x-y) + Q(x+z) + Q(x-z)] + 2[Q(y+z) + Q(y-z)] + 32Q(x)\end{aligned}$$

Hence Q satisfies (2.3) for all $x, y, z \in X$. To prove Q is unique, let $R(x)$ be another quartic mapping satisfying (2.3) and (2.9), we arrive

$$\begin{aligned}
\|Q(x) - R(x)\| &= \frac{1}{4^{4n}} \|Q(4^n x) - R(4^n x)\| \\
&\leq \frac{1}{4^{4n}} \{\|Q(4^n x) - g(4^n x)\| + \|g(4^n x) - R(4^n x)\|\} \\
&\leq \frac{2}{4^4} \sum_{k=0}^{\infty} \frac{\beta(4^{k+n}x)}{4^{4(k+n)}} \\
&\to 0 \quad as \;\; n \to \infty
\end{aligned}$$

for all $x \in X$. Hence Q is unique.

For $j = -1$, we can prove the similar stability result. This completes the proof of the theorem. □

The following corollary is a immediate consequence of Theorems 3.1 concerning the stability of (2.3).

Corollary 2.3. *Let λ and s be nonnegative real numbers. If a function $g : X \to Y$ satisfies the inequality*

$$\|Dg(x, y, z)\| \leq \begin{cases} \lambda, & \\ \lambda\left\{||x||^s + ||y||^s + ||z||^s\right\}, & s < 4 \text{ or } s > 4; \\ \lambda \; ||x||^s||y||^s||z||^s, & s < \frac{4}{3} \text{ or } s > \frac{4}{3}; \\ \lambda\left\{||x||^s||y||^s||z||^s + \left\{||x||^{3s} + ||y||^{3s} + ||z||^{3s}\right\}\right\}, & s < \frac{4}{3} \text{ or } s > \frac{4}{3}; \end{cases} \tag{2.20}$$

for all $x, y, z \in X$. Then there exists a unique quartic function $Q : X \to Y$ such that

$$\|f(x) - Q(x)\| \leq \begin{cases} \dfrac{\lambda}{255}, \\ \dfrac{7\lambda||x||^s}{|4^4 - 4^s|}, \\ \dfrac{\lambda||x||^{3s}}{|4^4 - 4^{3s}|} \\ \dfrac{8\lambda||x||^{3s}}{|4^4 - 4^{3s}|} \end{cases} \tag{2.21}$$

for all $x \in X$.

3. STABILITY OF ORTHOGONALLY QUARTIC FUNCTIONAL EQUATION (2)

Now, we introduce some basic concepts of orthogonality and orthogonality normed spaces.

Definition 3.1. [11] A vector space X is called an *orthogonality vector space* if there is a relation $x \perp y$ on X such that

(i) $x \perp 0$, $0 \perp x$ for all $x \in X$;

(ii) if $x \perp y$ and $x, y \neq 0$, then x, y are linearly independent;

(iii) $x \perp y$, $ax \perp by$ for all $a, b \in \mathbb{R}$;
(iv) if P is a two-dimensional subspace of X ; then
(a) for every $x \in P$ there exists $0 \neq y \in P$ such that $x \perp y$;
(b) there exists vectors $x, y \neq 0$ such that $x \perp y$ and $x + y \perp x - y$.

Any vector space can be made into an orthogonality vector space if we define $x \perp 0, 0 \perp x$ for all x and for non zero vector $x,\ y$ define $x \perp y$ iff $x,\ y$ are linearly independent. The relation $\perp$ is called symmetric if $x \perp y$ implies that $y \perp x$ for all $x,\ y \in X$. The pair $(x, \perp)$ is called an *orthogonality space*. It becomes *orthogonality normed space* when the orthogonality space is equipped with a norm.

The orthogonal Cauchy functional equation

$$f(x+y) = f(x) + f(y), x \perp y \tag{3.1}$$

in which $\perp$ is an abstract orthogonally was first investigated by S. Gudder and D. Strawther [11]. R. Ger and J. Sikorska discussed the orthogonal stability of the equation (3.1) in [10].

Definition 3.2. Let X be an orthogonality space and Y be a real Banach space. A mapping $f : X \to Y$ is called *orthogonally quadratic* if it satisfies the so called orthogonally Euler-Lagrange quadratic functional equation

$$f(x+y) + f(x-y) = 2f(x) + 2f(y) \tag{3.2}$$

for all $x, y \in X$ with $x \perp y$. The orthogonality Hilbert space for the orthogonally quadratic functional equation (3.2) was first investigated by F. Vajzovic [31].

C.G. Park [18] proved the orthogonality quartic functional equation (2.1) where $\perp$ is the orthogonality in the sense of Rätz [26].

Here after, let $(A, \perp)$ denote an orthogonality normed space with norm $\|\cdot\|_A$ and $(B, \|\cdot\|_B)$ is a Banach space. We define

$$\begin{aligned} Df(x,y,z) = {} & f(2x+y+z) + f(2x+y-z) + f(2x-y+z) + f(-2x+y+z) \\ & + f(2y) + f(2z) - 8[f(x+y) + f(x-y) + f(x+z) + f(x-z)] \\ & - 2[f(y+z) + f(y-z)] - 32f(x) \end{aligned}$$

for all $x, y, z \in X$ with $x \perp y, y \perp z$ and $z \perp x$ in the sense of Ratz.

3.1. HYERS - ULAM - AOKI - RASSIAS STABILITY OF MODIFIED ORTHOGONALLY QUARTIC FUNCTIONAL EQUATION (2).

In this section, we present the Hyers - Ulam - Aoki - Rassias stability of the functional equation (2) involving sum of powers of norms.

Theorem 3.3. *Let μ and $s(s < 4)$ be nonnegative real numbers. Let $f : A \to B$ be a mapping fulfilling*

$$\|D\ f\,(x,y,z)\|_B \leq \mu\,\{\|x\|_A^s + \|y\|_A^s + \|z\|_A^s\} \tag{3.3}$$

for all $x, y, z \in A$, with $x \perp y, y \perp z$ and $z \perp x$. Then there exists a unique orthogonally quartic mapping $Q : A \to B$ such that

$$\|f(y) - Q(y)\|_B \leq \frac{\mu}{16 - 2^s}\,\|y\|_A^s \tag{3.4}$$

for all $y \in A$. The function $Q(y)$ is defined by

$$Q(y) = \lim_{n\to\infty} \frac{f(2^n y)}{16^n} \tag{3.5}$$

for all $y \in A$.

Proof. Replacing (x, y, z) by $(0, 0, 0)$ in (3.3), we get $f(0) = 0$. Setting (x, y, z) by $(0, y, 0)$ in (3.3), we obtain

$$\|f(2y) - 16f(y)\|_B \le \mu \|y\|_A^s \tag{3.6}$$

for all $y \in A$. Since $y \perp 0$, we have

$$\left\| \frac{f(2y)}{16} - f(y) \right\|_B \le \frac{\mu}{16} \|y\|_A^s \tag{3.7}$$

for all $y \in A$. Now replacing y by $2y$ and dividing by 16 in (3.7) and summing resulting inequality with (3.7), we arrive

$$\left\| \frac{f(2^2 y)}{16^2} - f(y) \right\|_B \le \frac{\mu}{16} \left\{ 1 + \frac{2^s}{16} \right\} \|y\|_A^s \tag{3.8}$$

for all $y \in A$. In general, using induction on a positive integer n, we obtain that

$$\begin{aligned} \left\| \frac{f(2^n y)}{16^n} - f(y) \right\|_B &\le \frac{\mu}{16} \sum_{k=0}^{n-1} \left(\frac{2^s}{16} \right)^k \|y\|_A^s \qquad (3.9) \\ &\le \frac{\mu}{16} \sum_{k=0}^{\infty} \left(\frac{2^s}{16} \right)^k \|y\|_A^s \end{aligned}$$

for all $y \in A$. In order to prove the convergence of the sequence $\{f(2^n y)/16^n\}$, replace y by $2^m y$ and divide by 16^m in (3.9), for any $n, m > 0$, we obtain

$$\begin{aligned} \left\| \frac{f(2^n 2^m y)}{16^{(n+m)}} - \frac{f(2^m y)}{16^m} \right\|_B &= \frac{1}{16^m} \left\| \frac{f(2^n 2^m y)}{16^n} - f(2^m y) \right\|_B \\ &\le \frac{1}{16^m} \frac{\mu}{16} \sum_{k=0}^{n-1} \left(\frac{2^s}{16} \right)^k \|2^m y\|_A^s \\ &\le \frac{\mu}{16} \sum_{k=0}^{\infty} \left(\frac{2^s}{16} \right)^{k+m} \|y\|_A^s . \qquad (3.10) \end{aligned}$$

As $s < 4$, the right hand side of (3.10) tends to 0 as $m \to \infty$ for all $y \in A$. Thus $\{f(2^n y)/16^n\}$ is a Cauchy sequence. Since B is complete, there exists a mapping $Q : A \to B$ such that

$$Q(y) = \lim_{n\to\infty} \frac{f(2^n y)}{16^n} \quad \forall\, y \in A.$$

In order to prove Q is unique and it satisfies the (2), the proof is similar to that of Theorem 3.1. This completes the proof of the theorem. □

Theorem 3.4. *Let μ and $s(s > 4)$ be nonnegative real numbers. Let $f : A \to B$ be a mapping satisfying (3.3) for all $x, y, z \in A$, with $x \perp y, y \perp z$ and $z \perp x$. Then there exists a unique orthogonally quartic mapping $Q : A \to B$ such that*

$$\|f(y) - Q(y)\|_B \leq \frac{\mu}{2^s - 16} \|y\|_A^s \tag{3.11}$$

for all $y \in A$. The function $Q(y)$ is defined by

$$Q(y) = \lim_{n\to\infty} 16^n f\left(\frac{y}{2^n}\right) \tag{3.12}$$

for all $y \in A$.

Proof. Replacing y by $\frac{y}{2}$ in (3.6), the rest of the proof is similar to that of Theorem 4.1. □

3.2. **JMRASSIAS STABILITY OF (2).**

In this section, we investigate the JMRassias mixed type product - sum of powers of norms stability of the functional equation (2).

Theorem 3.5. *Let $f : A \to B$ be a mapping satisfying the inequality*

$$\|D\ f(x, y, z)\|_B \leq \mu \left(\|x\|_A^s \|y\|_A^s \|z\|_A^s + \left\{\|x\|_A^{3s} + \|y\|_A^{3s} + \|z\|_A^{3s}\right\}\right) \tag{3.13}$$

for all $x, y, z \in A$, with $x \perp y, y \perp z$ and $z \perp x$ where μ and s are constants with, $\mu, s > 0$ and $s < \frac{4}{3}$. Then the limit

$$Q(y) = \lim_{n\to\infty} \frac{f(2^n y)}{16^n} \tag{3.14}$$

exists for all $y \in A$ and $Q : A \to B$ is the unique orthogonally quartic mapping such that

$$\|f(y) - Q(y)\|_B \leq \frac{\mu}{16 - 2^{3s}} \|y\|_A^{3s} \tag{3.15}$$

for all $y \in A$.

Proof. Letting (x, y, z) by $(0, 0, 0)$ in (3.13), we get $f(0) = 0$. Again substituting (x, y, z) by $(0, y, 0)$ in (3.13), we obtain

$$\left\|\frac{f(2y)}{16} - f(y)\right\|_B \leq \frac{\mu}{16} \|y\|_A^{3s} \tag{3.16}$$

for all $y \in A$. Now replacing y by $2y$ and dividing by 16 in (3.16) and summing resulting inequality with (3.16), we arrive

$$\left\|\frac{f(2^2 y)}{16^2} - f(y)\right\|_B \leq \frac{\mu}{16}\left\{1 + \frac{2^{3s}}{16}\right\} \|y\|_A^{3s} \tag{3.17}$$

for all $y \in A$. Using induction on a positive integer n, we obtain that

$$\begin{aligned}\left\|\frac{f(2^n y)}{16^n} - f(y)\right\|_B &\leq \frac{\mu}{16}\sum_{k=0}^{n-1}\left(\frac{2^{3s}}{16}\right)^k \|y\|_A^{3s} \\ &\leq \frac{\mu}{16}\sum_{k=0}^{\infty}\left(\frac{2^{3s}}{16}\right)^k \|y\|_A^{3s}\end{aligned} \tag{3.18}$$

for all $y \in A$. In order to prove the convergence of the sequence $\{f(2^n y)/16^n\}$, replace y by $2^m y$ and divide by 16^m in (3.18), for any $n, m > 0$, we obtain

$$\begin{aligned} \left\| \frac{f\left(2^n 2^m y\right)}{16^{(n+m)}} - \frac{f\left(2^m y\right)}{16^m} \right\|_B &= \frac{1}{16^m} \left\| \frac{f\left(2^n 2^m y\right)}{16^n} - f(2^m y) \right\|_B \\ &\leq \frac{\mu}{16} \sum_{k=0}^{n-1} \left(\frac{2^{3s}}{16} \right)^{k+m} \|y\|_A^{3s} \\ &\leq \frac{\mu}{16} \sum_{k=0}^{\infty} \frac{1}{2^{(4-3s)(k+m)}} \|y\|_A^{3s} \end{aligned} \tag{3.19}$$

As $s < \frac{4}{3}$, the right hand side of (3.19) tends to 0 as $m \to \infty$ for all $y \in A$. Thus $\{f(2^n y)/16^n\}$ is a Cauchy sequence. Since B is complete, there exists a mapping $Q : A \to B$ such that

$$Q(y) = \lim_{n\to\infty} \frac{f(2^n y)}{16^n} \quad \forall\, y \in E.$$

Letting $n \to \infty$ in (3.18), we arrive the formula (3.15) for all $y \in A$. To show that Q is unique and it satisfies (2), the rest of the proof is similar to that of Theorem 3.1 □

Theorem 3.6. *Let $f : A \to B$ be a mapping satisfying the inequality (3.13) for all $x, y, z \in A$, with $x \perp y, y \perp z$ and $z \perp x$, where μ and s are constants with $\mu, s > 0$ and $s > \frac{4}{3}$. Then the limit*

$$Q(y) = \lim_{n\to\infty} 16^n\ f\left(\frac{y}{2^n}\right) \tag{3.20}$$

exists for all $y \in A$ and $Q : A \to B$ is the unique quartic mapping such that

$$\|f(y) - Q(y)\|_B \leq \frac{\mu}{2^{3s} - 16} \|y\|_A^{3s} \tag{3.21}$$

for all $y \in A$.

Proof. Replacing y by $\frac{y}{2}$ in (3.16), the proof is similar to that of Theorem 5.1. □

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