AN EXTENSION OF A RESULT ABOUT THE ORDER OF CONVERGENCE

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ABSTRACT. In this note, we analize the context in which the calculation of limits of sequences is performed using the integration of product of Riemann integrable functions. We will also study the degree of convergence of such sequences.

1. INTRODUCTION

The integration is one of the most useful concept in introductory calculus. Its development was started with Newton which considered this process like a reverse of differentiation. Later, Leibniz and Riemann have seen the integration as "summation of many quantities". This approach led to the current definition of Riemann integral (see e.g. [5], pag. 216).

To evaluate Riemann integral without derivative is not easy. Therefore, in this article, we will analize the Riemann integral as a limit. First, we recall a classic result(see e.g. [8], pag. 49).

Lemma 1.1. *Let $f : [a, \ b] \to \mathbb{R}$ be Riemann integrable. Then*

$$\lim_{n\to\infty} \frac{b-a}{n} \sum_{k=1}^{n} f\left(a + \frac{k}{n}(b-a)\right) = \int_{a}^{b} f(x)dx.$$

In [8], p. 49, Polya and Szego have formulated the problem of "the degree of approximation" (see problem 10) and gave an answer (p. 232) which is presented in the next lemma.

Lemma 1.2. *Let $f : [a, b] \to \mathbb{R}$ be continuously differentiable. Then*

$$\lim_{n \to \infty} n \left(\frac{b - a}{n} \sum_{k=1}^{n} f \left(a + \frac{k}{n}(b - a) \right) - \int_a^b f(x)\, dx \right) = (b - a) \frac{f(b) - f(a)}{2}.$$

In this paper we want to extend these results for product of two functions. In order to do this, we recall the next results about the product of two integrable functions (see e.g. [9], theorem 7.9).

Theorem 1.3. *If $f, g : [a, b] \to \mathbb{R}$ are two Riemann integrable functions, then fg is Riemann integrable.*

Since the transformations $x \to \frac{x - a}{b - a}$ change the interval $[a, b]$ into $[0, 1]$, we will present all our results for the functions defined on the interval $[0, 1]$. In this context, by using theorem 1.3, lemmas 1.1 and 1.2 admit the following extensions:

Lemma 1.4. *If $f, g : [0, 1] \to \mathbb{R}$ are two Riemann integrable functions, then*

$$\lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} f \left(\frac{k}{n} \right) g \left(\frac{k}{n} \right) = \int_0^1 f(x)\, g(x) dx.$$

Lemma 1.5. *If $f, g : [0, 1] \to \mathbb{R}$ are two continuously differentiable functions, then*

$$\lim_{n \to \infty} n \left(\frac{1}{n} \sum_{k=1}^{n} f \left(\frac{k}{n} \right) g \left(\frac{k}{n} \right) - \int_0^1 f(x)\, g(x)\, dx \right) = \frac{f(1)\, g(1) - f(0)\, g(0)}{2}.$$

We will generalize these lemmas in the next section. The main result is the proposition 3.1, which will be presented in section 3. This result will give more information about the convergence's order of a sequence, more general then the one from lemma 1.5.

2. SEQUENCES WHOSE LIMITS ARE CALCULATED USING RIEMANN INTEGRALS

First, we formulate the next lemma to establish a general framework for the main results of this note.

Lemma 2.1. *Let $f, g : [0, 1] \to \mathbb{R}$ be two Riemann integrable functions. For any $n \in \mathbb{N}^*$ and $k \in \{1, 2, ..., n\}$, we denote $I_k = \left[\frac{k-1}{n}, \frac{k}{n} \right]$. We define $A_k = \sup_{x \in I_k} f(x)$, $a_k = \inf_{x \in I_k} f(x)$, $B_k = \sup_{x \in I_k} g(x)$ and $b_k = \inf_{x \in I_k} g(x)$. We consider the sequences*

$$x_n = \frac{1}{n} \sum_{k=1}^{n} a_k b_k$$

and

$$y_n = \frac{1}{n} \sum_{k=1}^{n} A_k B_k.$$

Then:

a) $\lim_{n \to \infty} (y_n - x_n) = 0;$

b) $\lim_{n \to \infty} x_n = \lim_{n \to \infty} y_n = \int_0^1 f(x) g(x) dx.$

Proof. As the functions are Riemann integrable, we obtain from Darboux's criterion, that $\lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} (A_k - a_k) = 0$ and $\lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} (B_k - b_k) = 0.$

a) First, we assume that the functions f and g are nonnegative. Then

$$0 \leqslant y_n - x_n = \frac{1}{n} \sum_{k=1}^{n} (A_k B_k - a_k b_k)$$

$$= \frac{1}{n} \sum_{k=1}^{n} (A_k - a_k) B_k + \frac{1}{n} \sum_{k=1}^{n} (B_k - b_k) a_k$$

$$\leqslant \sup_{x \in [0,1]} g(x) \cdot \frac{1}{n} \sum_{k=1}^{n} (A_k - a_k) + \sup_{x \in [0,1]} f(x) \cdot \frac{1}{n} \sum_{k=1}^{n} (B_k - b_k).$$

But f, g are integrable, so they are bounded. Then, we obtain

$$\lim_{n \to \infty} \left(\sup_{x \in [0,1]} g(x) \cdot \frac{1}{n} \sum_{k=1}^{n} (A_k - a_k) + \sup_{x \in [0,1]} f(x) \cdot \frac{1}{n} \sum_{k=1}^{n} (B_k - b_k) \right) = 0$$

and $\lim_{n \to \infty} (y_n - x_n) = 0.$

In the absence of the nonnegative assumption, we note $a = \inf_{x \in [0,1]} f(x)$ and, respectively $b = \inf_{x \in [0,1]} g(x)$. Consider the functions $F : [0, 1] \to \mathbb{R}, F(x) = f(x) - a$ and, $G : [0, 1] \to \mathbb{R}, G(x) = g(x) - b$, which are nonnegative functions. Then :

$$\lim_{n \to \infty} (y_n - x_n) = \lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} (A_k B_k - a_k b_k)$$

$$= \lim_{n \to \infty} \left(\frac{1}{n} \sum_{k=1}^{n} ((A_k - a)(B_k - b) - (a_k - a)(b_k - b)) + \frac{1}{n} \sum_{k=1}^{n} a(B_k - b_k) + \frac{1}{n} \sum_{k=1}^{n} b(A_k - a_k) \right) = 0,$$

because

$$\lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} ((A_k - a)(B_k - b) - (a_k - a)(b_k - b)) = 0,$$

in accordance with previous calculations for the functions F, respectively G.

b) If f, g are nonnegative, then for any interval $I \subset [0, 1]$, the following inequalities are satisfied:

$$\sup_{x \in I} (f(x) g(x)) \leqslant \sup_{x \in I} f(x) \cdot \sup_{x \in I} g(x)$$

and

$$\inf_{x \in I} f(x) g(x) \geqslant \inf_{x \in I} f(x) \cdot \inf_{x \in I} g(x).$$

This implies that

$$x_n \leqslant \int\limits_0^1 f(x) g(x) \, dx \leqslant y_n,$$

for all $n \in \mathbb{N}^*$, as any Riemann sum attached to the function fg lies between x_n and y_n. So, we obtain

$$0 \leqslant y_n - \int\limits_0^1 f(x) g(x) \, dx \leqslant y_n - x_n$$

which means that

$$\lim_{n \to \infty} y_n = \int\limits_0^1 f(x) g(x) dx.$$

Similarly

$$0 \leqslant \int\limits_0^1 f(x) g(x) \, dx - x_n \leqslant y_n - x_n$$

and we obtain

$$\lim_{n \to \infty} x_n = \int\limits_0^1 f(x) g(x) dx.$$

If f, g are not necessarily nonnegative, we obtain the same result by applying the same reasoning for functions F and G, defined at the previous point.

$\square$

The result from 2.1 help us to prove the next lemma. We should mention that this result was presented by Bényi and Niţu in [3] but without a solution.

Lemma 2.2. *Let $f, g : [0, 1] \to \mathbb{R}$ be two Riemann integrable functions. For any $k \in \{1, 2, 3, ..., n\}$ and for any $\alpha_k, \beta_k \in \left[\frac{k-1}{n}, \frac{k}{n} \right]$, the sequence*

$$z_n = \frac{1}{n} \sum_{k=1}^n f(\alpha_k) g(\beta_k)$$

converges to $\int\limits_0^1 f(x) g(x) \, dx$.

Proof. We have two cases. First, we assume that the functions f and g are nonnegative. We choose the sequences x_n and y_n from lemma 2.1. Obviously we have

$$x_n \leqslant z_n \leqslant y_n$$

and the conclusion follows.

In the absence of the nonnegative assumption, we note $a = \inf\limits_{x \in [0, 1]} f(x)$ and, respectively $b = \inf\limits_{x \in [0, 1]} g(x)$. Similar with the proof of lemma 2.1., we consider the

functions $F : [0,\ 1] \to \mathbb{R}, F(x) = f(x) - a$ and $G : [0,\ 1] \to \mathbb{R}, G(x) = g(x) - b$, which are nonnegative functions. Then

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} f(\alpha_k) g(\beta_k) = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} (F(\alpha_k) + a)(G(\beta_k) + b)$$

$$= \lim_{n\to\infty} \left(\frac{1}{n} \sum_{k=1}^{n} F(\alpha_k) G(\beta_k) + \frac{b}{n} \sum_{k=1}^{n} F(\alpha_k) + \frac{a}{n} \sum_{k=1}^{n} G(\beta_k) + ab \right)$$

$$= \int_0^1 F(x) G(x)\, dx + b \int_0^1 F(x)\, dx + a \int_0^1 G(x)\, dx + ab$$

$$= \int_0^1 (F(x) + a)(G(x) + b)\, dx = \int_0^1 f(x) g(x)\, dx$$

which conclude the proof. $\square$

A similar result could be found in [10], where Spivak has posed the same problem (see problem 1, p.263) but for continuous functions . Evidently, our result is more general.

A consequence of lemma 2.2 is the following result from [3].

Corollary 2.3.(Bényi, Niţu) *Let $f : [0,\ 1] \to \mathbb{R}$ be a Riemann integrable functions.*Then

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} f\left(\frac{k-1}{n}\right) f\left(\frac{k}{n}\right) = \int_0^1 f^2(x)\, dx.$$

Proof. We apply 2.2 for $g = f$ and the sequences $\alpha_n = \frac{k-1}{n}$ and $\beta_n = \frac{k}{n}$. $\square$

3. THE ORDER OF CONVERGENCE

Lemma 2.2 gives us an answer about the limits of sequences of the type

$$\frac{1}{n} \sum_{k=1}^{n} f(a_k) g(b_k),$$

but we want to know more details about these limits. We would like to study their order of convergence and we can deliver a good estimation in the case of the sequence

$$\frac{1}{n} \sum_{k=1}^{n} f\left(\frac{k-\alpha}{n}\right) g\left(\frac{k-\beta}{n}\right),$$

with $\alpha, \beta \in [0,\ 1]$. The results are included in next proposition.

Proposition 3.1. *Let $f, g : [0,\ 1] \to \mathbb{R}$ be two continuously differentiable functions. Let define $\alpha, \beta \in [0,\ 1]$ and the sequence*

$$a_n = \int_0^1 f(x) g(x)\, dx - \frac{1}{n} \sum_{k=1}^n f\left(\frac{k-\alpha}{n}\right) g\left(\frac{k-\beta}{n}\right),$$

for all $n \in \mathbb{N}^*$. Then:

$$\lim_{n \to \infty} n a_n = \frac{1}{2}\left(f(0)g(0) - f(1)g(1)\right) + \alpha \int_0^1 f'(x)g(x)\, dx + \beta \int_0^1 f(x)g'(x)\, dx.$$

Proof. After some algebra we have

$$n a_n = n \int_0^1 f(x) g(x)\, dx - \sum_{k=1}^n f\left(\frac{k-\alpha}{n}\right) g\left(\frac{k-\beta}{n}\right)$$

$$= n \left(\int_0^1 f(x) g(x)\, dx - \frac{1}{n} \sum_{k=1}^n f\left(\frac{k}{n}\right) g\left(\frac{k}{n}\right) \right) + \sum_{k=1}^n f\left(\frac{k}{n}\right) \left(g\left(\frac{k}{n}\right) - g\left(\frac{k-\beta}{n}\right) \right)$$

$$+ \sum_{k=1}^n g\left(\frac{k-\beta}{n}\right) \left(f\left(\frac{k}{n}\right) - f\left(\frac{k-\alpha}{n}\right) \right).$$

But

$$\lim_{n \to \infty} n \left(\int_0^1 f(x) g(x)\, dx - \frac{1}{n} \sum_{k=1}^n f\left(\frac{k}{n}\right) g\left(\frac{k}{n}\right) \right) = \frac{f(0)g(0) - f(1)g(1)}{2}.$$

However, for any $k \in \{1, 2, 3.., n\}$ and any interval $\left[\frac{k-\beta}{n}, \frac{k}{n}\right]$, we apply mean value theorem for the function g and we find $c_k \in \left(\frac{k-1}{n}, \frac{k}{n}\right)$ which verifies the relation

$$g\left(\frac{k}{n}\right) - g\left(\frac{k-\beta}{n}\right) = g'(c_k) \frac{\beta}{n}.$$

So

$$\sum_{k=1}^n f\left(\frac{k}{n}\right) \left(g\left(\frac{k}{n}\right) - g\left(\frac{k-\beta}{n}\right) \right) = \frac{\beta}{n} \sum_{k=1}^n f\left(\frac{k}{n}\right) g'(c_k).$$

From 2.2, we obtain

$$\lim_{n \to \infty} \left(\frac{\beta}{n} \sum_{k=1}^n f\left(\frac{k}{n}\right) g'(c_k) \right) = \beta \int_0^1 f(x) g'(x)\, dx.$$

Similarly, for the function f on the interval $\left[\frac{k-\alpha}{n}, \frac{k}{n}\right]$, applying mean value theorem, we find $b_k \in \left(\frac{k-1}{n}, \frac{k}{n}\right)$ so that

$$f\left(\frac{k}{n}\right) - f\left(\frac{k-\alpha}{n}\right) = f'(b_k) \frac{\alpha}{n}.$$

Then

$$\sum_{k=1}^n g\left(\frac{k-\beta}{n}\right) \left(f\left(\frac{k}{n}\right) - f\left(\frac{k-\alpha}{n}\right) \right) = \frac{\alpha}{n} \sum_{k=1}^n f'(b_k) g\left(\frac{k-\beta}{n}\right).$$

Applying Lemma 2.2. we find that

$$\lim_{n\to\infty} \left(\frac{\alpha}{n} \sum_{k=1}^{n} f'(b_k) \, g\left(\frac{k-\beta}{n}\right) \right) = \alpha \int_{0}^{1} f'(x) \, g(x) \, dx.$$

Finally, we obtain

$$\lim_{n\to\infty} na_n = \frac{1}{2} \left(f(0) g(0) - f(1) g(1) \right) + \alpha \int_{0}^{1} f'(x) g(x) \, dx + \beta \int_{0}^{1} f(x) g'(x) \, dx.$$

$\square$

If we choose the particular values for the numbers α and β from proposition 3.1, we obtain some results as:

For $\alpha = 0$ and $\beta = 1$, we obtain

$$\lim_{n\to\infty} n \left(\int_{0}^{1} f(x) g(x) \, dx - \frac{1}{n} \sum_{k=1}^{n} f\left(\frac{k}{n}\right) g\left(\frac{k-1}{n}\right) \right)$$

$$= \frac{1}{2} \left(f(0) g(0) - f(1) g(1) \right) + \int_{0}^{1} f(x) g'(x) \, dx;$$

For $\alpha = \frac{1}{2}$ and $\beta = 1$, we obtain

$$\lim_{n\to\infty} n \left(\int_{0}^{1} f(x) g(x) \, dx - \frac{1}{n} \sum_{k=1}^{n} f\left(\frac{2k-1}{2n}\right) g\left(\frac{k-1}{n}\right) \right)$$

$$= \frac{1}{2} \left(f(0) g(0) - f(1) g(1) \right) + \frac{1}{2} \int_{0}^{1} f'(x) g(x) \, dx + \int_{0}^{1} f(x) g'(x) \, dx.$$

Using the formula of integration by parts, we obtain

$$\lim_{n\to\infty} n \left(\int_{0}^{1} f(x) g(x) \, dx - \frac{1}{n} \sum_{k=1}^{n} f\left(\frac{2k-1}{2n}\right) g\left(\frac{k-1}{n}\right) \right) = \frac{1}{2} \int_{0}^{1} f(x) g'(x) \, dx.$$

If we choose $g(x) = 1$ we obtain the next result, which represent the theorem 1 from [4].

Corollary 3.2. (Chiţescu) *Let* $f : [0, 1] \to \mathbb{R}$ *be a continuously differentiable functions. Let define* $\alpha \in [0, 1]$ *and the sequence*

$$a_n = \int_{0}^{1} f(x) \, dx - \frac{1}{n} \sum_{k=1}^{n} f\left(\frac{k-\alpha}{n}\right),$$

for all $n \in \mathbb{N}^*$. *Then:*

$$\lim_{n\to\infty} na_n = \left(\alpha - \frac{1}{2} \right) [f(1) - f(0)].$$

Proof. We apply 3.1 for $g(x) = 1$. $\square$

4. APPLICATIONS

At the end of this note, we present three applications whose solutions are obtained using the results from the previous paragraph. The first problem reads

Problem 1. *Let the sequence*

$$a_n = \sum_{k=1}^{n} \frac{k^2 - k}{n^3 + kn^2},$$

for any $n \in \mathbb{N}^*$. Show that $\lim\limits_{n \to \infty} a_n = \ln 2 - \frac{1}{2}$ and evaluate $\lim\limits_{n \to \infty} n \left(\ln 2 - \frac{1}{2} - a_n \right)$.

Solutions: By using Lema 2.2 for the functions $f, g : [0, 1] \to \mathbb{R}$, $f(x) = \frac{x}{1+x}$, $g(x) = x$, we have

$$\frac{k^2 - k}{n^3 + kn^2} = \frac{1}{n} \cdot \frac{k(k-1)}{n(n+k)} = \frac{1}{n} \cdot \frac{\frac{k}{n}}{1 + \frac{k}{n}} \cdot \frac{k-1}{n},$$

so

$$\lim_{n \to \infty} a_n = \lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} \frac{\frac{k}{n}}{1 + \frac{k}{n}} \cdot \frac{k-1}{n} = \int_0^1 x \cdot \frac{x}{1+x} dx,$$

But

$$\int_0^1 \frac{x^2}{1+x} dx = \int_0^1 \left(x - 1 + \frac{1}{1+x} \right) dx = \frac{x^2}{2} \Big|_0^1 - x \Big|_0^1 + \ln(1+x) \Big|_0^1 = \ln 2 - \frac{1}{2}.$$

From Prop. 3.1, we obtain

$$\lim_{n \to \infty} n \left(\ln 2 - \frac{1}{2} - a_n \right)$$

$$= \lim_{n \to \infty} n \left(\int_0^1 f(x)g(x) \, dx - \frac{1}{n} \sum_{k=1}^{n} f\left(\frac{k}{n} \right) g\left(\frac{k-1}{n} \right) \right)$$

$$= \frac{1}{2} \left(f(0) \, g(0) - f(1) \, g(1) \right) + \int_0^1 f(x) \, g'(x) \, dx$$

$$= -\frac{1}{4} + \int_0^1 \frac{x}{1+x} dx = -\frac{1}{4} + \int_0^1 \left(1 - \frac{1}{1+x} \right) dx = -\frac{1}{4} + x \Big|_0^1 - \ln(x+1) \Big|_0^1 = \frac{3}{4} - \ln 2.$$

Problem 2. *Let $n \in \mathbb{N}$, $n \geqslant 2$ and the function $f : [0, n] \to \mathbb{R}$, $f(x) = \sqrt{x} e^{-\frac{[x]}{n}}$. We note V_n, the volume of the solid obtained by rotating the graph of f about x axis. Find $\lim\limits_{n \to \infty} \frac{V_n}{n^2}$.*

Solution: As $V_n = \pi \int\limits_0^n f^2(x)\,dx = \pi \int\limits_0^n xe^{-\frac{2[x]}{n}}\,dx$, we evaluate first $\int\limits_0^n xe^{-\frac{2[x]}{n}}\,dx$.
We have

$$\int\limits_0^n xe^{-\frac{2[x]}{n}}\,dx = \sum_{k=1}^n \int\limits_{k-1}^k xe^{-\frac{2[x]}{n}}\,dx = \sum_{k=1}^n e^{-\frac{2(k-1)}{n}} \int\limits_{k-1}^k x\,dx = \sum_{k=1}^n \frac{2k-1}{2} e^{-\frac{2(k-1)}{n}}.$$

Then

$$\lim_{n\to\infty} \frac{V_n}{n^2} = \pi \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^n \frac{2k-1}{2n} e^{-\frac{2(k-1)}{n}} = \int\limits_0^1 xe^{-2x}\,dx.$$

But

$$\int\limits_0^1 xe^{-2x}\,dx = -\frac{xe^{-2x}}{2} \bigg|_0^1 + \frac{1}{2} \int\limits_0^1 e^{-2x}\,dx = -\frac{xe^{-2x}}{2} \bigg|_0^1 - \frac{e^{-2x}}{4} \bigg|_0^1 = \frac{1 - 3e^{-2}}{4},$$

so

$$\lim_{n\to\infty} \frac{V_n}{n^2} = \pi \frac{1 - 3e^{-2}}{4}.$$

Problem 3. (Pr. 11535 from AMM 9/2010) *Let f be a continuously differentiable function on [0, 1]. Let $A = f(1)$ and let $B = \int\limits_0^1 x^{-1/2} f(x)\,dx$. Evaluate*

$$\lim_{n\to\infty} n \left(\int\limits_0^1 f(x)\,dx - \sum_{k=1}^n \left(\frac{k^2}{n^2} - \frac{(k-1)^2}{n^2} \right) f\left(\frac{(k-1)^2}{n^2} \right) \right)$$

in terms of A and B.

Solution: We perform the change of variables $x = y^2$, so we have $\int\limits_0^1 f(x)\,dx = \int\limits_0^1 2yf(y^2)\,dt$ and $\int\limits_0^1 x^{-1/2} f(x)\,dx = \int\limits_0^1 2f(y^2)\,dt$. Using the notation $g(y) = 2f(y^2)$, the hypothesis becomes $g(1) = 2A$ and $\int\limits_0^1 g(y)\,dy = B$.

Then, we compute

$$\lim_{n\to\infty} n \left(\int\limits_0^1 f(x)\,dx - \sum_{k=1}^n \left(\frac{k^2}{n^2} - \frac{(k-1)^2}{n^2} \right) f\left(\frac{(k-1)^2}{n^2} \right) \right)$$

$$= \lim_{n\to\infty} n \left(\int\limits_0^1 yg(y)\,dy - \frac{1}{n} \sum_{k=1}^n \frac{2k-1}{2n} g\left(\frac{k-1}{n} \right) \right)$$

$$= \frac{1}{2} \int\limits_0^1 yg'(y)\,dy$$

from proposition 3.1. But

$$\int\limits_0^1 yg'\left(y\right)dy = yg\left(y\right)\Big|_0^1 - \int\limits_0^1 g\left(y\right)dy = g\left(1\right) - \int\limits_0^1 g\left(y\right)dy = 2A - B.$$

Finally we obtain that the limit value is $A - \frac{B}{2}$.

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