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VARIATIONAL ITERATION METHOD FOR EXACT SOLUTION OF GAS DYNAMIC EQUATION USING HE'S POLYNOMIALS

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Abstract. In this paper, we apply the variational iteration method using He's polynomials for finding the analytical solution of gas dynamic equation. The proposed method is an elegant combination of He's variational iteration and the homotopy perturbation methods. The suggested algorithm is quite efficient and is practically well suited for use in such problems. The proposed iterative scheme finds the solution without any discretization, linearization or restrictive assumptions. A clear advantage of this technique over the decomposition method is that no calculation of Adomian's polynomials is needed.

1. Introduction

Analytical methods that commonly use to solve nonlinear equations are very restricted and numerical techniques are involving discretization of the variables and give rise to rounding off errors. The basic motivation of this paper is to apply the variational iteration method coupled with He's polynomials (VIMHP) [1, 2, 13, 14, 15] for finding the solution of gas dynamic equation . In this algorithm, the correct functional is developed [3, 4, 8, 9, 10, 11] and the Lagrange multipliers are calculated optimally via variational theory. The use of Lagrange multipliers reduces the successive application of the integral operator and the cumbersome of huge computational work while still maintaining a very high level of accuracy. Finally, the He's polynomials are introduced in the correct functional and the comparison of like powers of p gives solutions of various orders. The developed algorithm takes full advantage of He's variational iteration and the homotopy perturbation methods. It is worth mentioning that the VIMHP is applied without any discretization, restrictive assumption or transformation and is free from round off errors. Unlike the method of separation of variables that require initial and boundary conditions, the VIMHP provides an analytical solution by using the initial conditions only. The proposed method work efficiently and the results so far are very encouraging and reliable. The fact that VIMHP solves nonlinear

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problems without using Adomian's polynomials can be considered as a clear advantage of this technique over the decomposition method. The proposed VIMHP solves effectively, easily and accurately a large class of linear, nonlinear, partial, deterministic or stochastic differential equations with approximate solutions which converge very rapidly to accurate solutions. The main solution procedure is that of the variational iteration method, while the homotopy perturbation method is applied to deal with the nonlinear terms where He's polynomials are used. In the present paper, VIMHP is employed to solve the following equation[5]

$$\frac{\partial u}{\partial t} + 1/2\frac{\partial (u^2)}{\partial x} = u(1-u) + g(x,t); \qquad 0 \le x \le 1, t > 0 \tag{1.1}$$

2. Variational iteration method

For the purpose of illustration of the methodology to the proposed method, using variational iteration method, we begin by considering a differential equation in the formal form,

$$L[u(x,t)] + N[u(x,t)] = g(x,t), \tag{2.1}$$

where L is a linear operator, N a nonlinear operator and $g(x,t)$ is the source inhomogeneous term. According to the variational iteration method, we can construct a correction functional for (2.1) as follows;

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t \lambda \left\{ Lu_n(x,\tau) + N\tilde{u_n}(x,\tau) - g(x,\tau) \right\} d\tau, \qquad n \ge 0,$$

where λ is a general Lagrangian multiplier [12], which can be identified optimally via the variational theory, the subscript n denotes the nth order approximation, and $\tilde{u_n}$ is considered as a restricted variation [10, 12] i.e., $\delta \tilde{u_n} = 0$. Therefore, we first determine the Lagrange multiplier λ that will be identified optimally via integration by parts. The successive approximations $u_n(x,t), n \ge 0$ of the solution $u(x,t)$ will be readily obtained upon using the obtained Lagrange multiplier and by using any selective function u_0. Consequently, the exact solution may be obtained by using

$$u(x,t) = \lim_{n\to\infty} u_n(x,t).$$

3. Homotopy perturbation method

In this section to illustrate the basic ideas of this method, we consider the following equation :

$$L[u(x,t)] + N[u(x,t)] = g(x,t), \ r \in \Omega, \tag{3.1}$$

with the boundary condition of:

$$B(u, \frac{\partial u}{\partial n}) = 0, \ r \in \Gamma, \tag{3.2}$$

where L is a linear operator, N a nonlinear operator and $g(x,t)$ is the source inhomogeneous term, B is a boundary operator and Γ is the boundary of the domain Ω. Homotopy perturbation structure is shown as follows:

$$H(v,p) = (1-p) * [L(v) - L(u_0)] + p\ [L(u) + N(u) - g(x,t)] = 0, \tag{3.3}$$

In Eq.(3.3), $p \in [0,1]$ is an embedding parameter and is the first approximation that satisfies the boundary conditions. We can assume that the solution of Eq. (3.3) can be written as a power series in p, as following:

$$v = v_0 + p\ v_1 + p^2 v_2 + \ \dots, \qquad (3.4)$$

The comparisons of like powers of p give solutions of various orders and the best approximation is:

$$u = \lim_{p \longrightarrow 1}\ v = v_0 + \ v_1 + \ v_2 + \ \dots. \qquad (3.5)$$

The convergence of series (3.5) is discussed in [7]. The method considers the nonlinear term $N[u]$ as

$$N[u] = \sum_{i=0}^{+\infty} p^i H_i = H_0 + pH_1 + p^2 H_2 + \cdots$$

where H_n's are the so-called He's polynomials [1], which can be calculated by using the formula

$$H_n(u_0, u_1, \cdots, u_n) = \frac{1}{n!}\frac{\partial^n}{\partial p^n}\left(N\left(\sum_{i=0}^{n} p^i u_i\right)\right)_{p=0}, \ n = 0, 1, 2, \cdots$$

4. Variational Iteration Method Using He's Polynomials (VIMHP)

To illustrate the basic idea of the variational homotopy perturbation method, we consider the following general differential equation:

$$L[u(x,t)] + N[u(x,t)] = g(x,t), \qquad (4.1)$$

where L is a linear operator, N a nonlinear operator and $g(x,t)$ is the source inhomogeneous term. According to section (2) we can construct a correct functional as follows:

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t \lambda \left\{Lu_n(x,\tau) + N\tilde{u_n}(x,\tau) - g(x,\tau)\right\} d\tau, \qquad n \geq 0, \ (4.2)$$

Now, we apply the homotopy perturbation method,

$$\sum_{n=0}^{\infty} p^n v_n(x,t) = u_0(x,t) + p\int_0^t \lambda \left[\sum_{n=0}^{\infty} p^n (L(v_n(x,\tau)) + N(v_n(x,\tau)) - g(x,\tau)\right] d\tau.$$

which is the variational homotopy perturbation method.

5. Applications

In order to assess the advantages and the accuracy of VIMHP for solving nonlinear equations,we will consider the following two examples. For the sake of comparison, we take the same examples as used in [5].

5.1. **Homogeneous gas Dynamic equation.** To apply the VIMHP, first we rewrite Eq. (1.1) with $g(x,t)=0$ in the following form

$$L[u(x,t)]+N[u(x,t)]=0, \tag{5.1}$$

where the notations $Lu=\frac{\partial u}{\partial t}$, $Nu=1/2\frac{\partial(u^2)}{\partial x}-u+u^2$, symbolize the linear and nonlinear terms, respectively. The correction functional for Eq. (5.1) reads

$$u_{n+1}(x,t)=u_n(x,t)+\int_0^t \lambda(\tau)\left\{\frac{\partial}{\partial\tau}(u_n(x,\tau))+N(\tilde{u}_n(x,\tau))\right\}d\tau, \quad n\geq 0, \tag{5.2}$$

Taking variation with respect to the independent variable u_n, noticing that $\delta N(\widetilde{u}_n)=0$,

$$\begin{aligned}\delta u_{n+1}(x,t) &= \delta u_n(x,t)+\delta\int_0^t \lambda(\tau)\left\{\frac{\partial}{\partial\tau}(u_n(x,\tau))+N(\tilde{u}_n(x,\tau))\right\}d\tau \\ &= \delta u_n(x,t)+[\lambda(\tau)\delta u_n(x,\tau)]_{\tau=t}-\int_0^t \lambda'(\tau)\delta u_n(x,\tau)\ d\tau=0,\end{aligned}$$

This yields the stationary conditions

$$1+\lambda(\tau)=0, \tag{5.3}$$

$$[\lambda'(\tau)]_{\tau=t}=0. \tag{5.4}$$

Eq. (5.3) is called Lagrange-Euler equation, and Eq. (5.4) natural boundary condition. The Lagrange multiplier can be identified as $\lambda=-1$, and the following variational homotopy perturbation formula can be obtained:

$$\begin{aligned}\sum_{n=0}^{\infty}p^n v_n &= u_0(x,t)-p\int_0^t\left[\frac{\partial}{\partial t}(\sum_{n=0}^{\infty}p^n v_n)+(\sum_{n=0}^{\infty}p^n v_n)\frac{\partial}{\partial x}(\sum_{n=0}^{\infty}p^n v_n)\right. \\ &- \left.(\sum_{n=0}^{\infty}p^n v_n)+(\sum_{n=0}^{\infty}p^n v_n)^2\right]\ d\tau, \quad n\geq 0,\end{aligned} \tag{5.5}$$

We start with an initial approximation $u_0(x,t)=e^{-x}$ and by comparing the coefficient of like powers of p, we can obtain directly the other components as

$$\begin{aligned}p^0: v_0(x,t) &= e^{-x}, \\ p^1: v_1(x,t) &= te^{-x}, \\ p^2: v_2(x,t) &= \frac{t^2}{2}e^{-x}, \\ p^3: v_3(x,t) &= \frac{t^3}{3!}e^{-x}, \\ &\vdots\end{aligned} \tag{5.6}$$

This gives the exact solution of (5.1) by

$$u(x,t)=v_0+v_1+v_2+\cdots=e^{-x}\left(1+t+\frac{t^2}{2}+\frac{t^3}{3!}+\cdots\right)=e^{t-x}, \tag{5.7}$$

obtained upon using the Taylor expansion for e^t. Which is exactly the same as obtained by Adomain decomposition method [5] and homotopy perturbation method [6].

5.2. **Inhomogeneous gas Dynamic equation.** Now we rewrite Eq. (1.1) in the following form

$$L[u(x,t)] + N[u(x,t)] = -e^{t-x}, \tag{5.8}$$

where the notations $Lu = \frac{\partial u}{\partial t}$, $Nu = 1/2\frac{\partial(u^2)}{\partial x} - u + u^2$, symbolize the linear and nonlinear terms, respectively. The correction functional for Eq. (5.8) reads

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t \lambda(\tau)\left\{\frac{\partial}{\partial\tau}(u_n(x,\tau)) + N(\tilde{u}_n(x,\tau) + e^{\tau-x})\right\}d\tau, \quad n \geq 0, \tag{5.9}$$

Taking variation with respect to the independent variable u_n, noticing that $\delta N(\widetilde{u}_n) = 0$,

$$\begin{aligned}\delta u_{n+1}(x,t) \quad &= \delta u_n(x,t) + \delta\int_0^t \lambda(\tau)\left\{\frac{\partial}{\partial\tau}(u_n(x,\tau)) + N(\tilde{u}_n(x,\tau) + e^{\tau-x})\right\}d\tau \\ &= \delta u_n(x,t) + [\lambda(\tau)\delta u_n(x,\tau)]_{\tau=t} - \int_0^t \lambda'(\tau)\delta u_n(x,\tau)\ d\tau = 0,\end{aligned}$$

This yields the stationary conditions

$$1 + \lambda(\tau) = 0, \tag{5.10}$$

$$[\lambda'(\tau)]_{\tau=t} = 0. \tag{5.11}$$

Eq. (5.10) is called Lagrange-Euler equation, and Eq. (5.11) natural boundary condition. The Lagrange multiplier can be identified as $\lambda = -1$, and the following variational homotopy perturbation formula can be obtained:

$$\begin{aligned}\sum_{n=0}^{\infty} p^n v_v \quad &= \quad u_0(x,t) - p\int_0^t \left[\frac{\partial}{\partial t}(\sum_{n=0}^{\infty} p^n v_n) + (\sum_{n=0}^{\infty} p^n v_n)\frac{\partial}{\partial x}(\sum_{n=0}^{\infty} p^n v_n)\right. \\ &- \quad \left.(\sum_{n=0}^{\infty} p^n v_n) + (\sum_{n=0}^{\infty} p^n v_n)^2 + e^{\tau-x}\right]\ d\tau, \quad n \geq 0, \end{aligned} \tag{5.12}$$

We start with an initial approximation $u_0(x,t) = 1 - e^{-x}$ and by comparing the coefficient of like powers of p, we can obtain directly the other components as

$$\begin{aligned} p^0 : v_0(x,t) \quad &= \quad 1 - e^{-x}, \\ p^1 : v_1(x,t) \quad &= \quad -e^{t-x} + e^{-x}, \\ p^n : v_n(x,t) \quad &= \quad 0, \qquad n \geq 2 \end{aligned} \tag{5.13}$$

This gives the exact solution of (5.8) by

$$u(x,t) = v_0 + v_1 + v_2 + \cdots = 1 - e^{-x} - e^{t-x} + e^{-x} + 0 + 0 + \cdots = 1 - e^{t-x}, \tag{5.14}$$

Which is exactly the same as obtained by Adomain decomposition method [5] and homotopy perturbation method [6].

6. Conclusion

In this paper, we applied the He's variational iteration method coupled with He's polynomials (VIMHP) by combining the traditional variational iteration and the homotopy perturbation methods for finding the analytical solution of gas dynamic equation. The proposed method is employed without using linearization, discretization or restrictive assumptions. It may be concluded that the variational iteration method using He's polynomials is very powerful and efficient in finding the analytical solutions for a wide class of boundary value problems. The method gives more realistic series solutions that converge very rapidly in physical problems. The fact that the VIMHP solves nonlinear problems without using the Adomian's polynomials is a clear advantage of this technique over the decomposition method.

References

[1] A. Ghorbani, Beyond Adomian polynomials: He polynomials, Chaos Solitons Fractals, 39 (2009) 1486-1492.

[2] A. Ghorbani, J. Saberi-Nadjafi, He's homotopy perturbation method for calculating adomian polynomials, int. J. OF Non. Sci, and Num. Simul. 8 (2007) 229-232.

[3] A.M. Wazwaz, The variational iteration method: A reliable analytic tool for solving linear and nonlinear wave equations, Comput. and Math. with Appl. 54 (2007) 926-932.

[4] A.M. Wazwaz, The variational iteration method for rational solutions for KdV, K(2,2), Burgers, and cubic Boussinesq equations, J. Comput. Appl. Math. 207 (2007) 18-23.

[5] D.J. Evans and H. Bulut, A new approach to the gas dynamics equation:An application of the decomposition method, Appl. Comput. Math. 79 (2002) 817-822.

[6] H. Jafari, M. Zabihi, M. saidy, Application of Homotopy perturbation method for solving Gas Dynamic Equation, Appl. Math. scie. 2 (2008) 2393-2396.

[7] J.H. He, New Interpretation of homotopy-perturbation method, Int. J. of Modern Physics B, 20 (2006) 2561-2568.

[8] J.H. He, Variational iteration method -Some recent results and new interpretations, J. Comput. Appl. Math. 207 (2007) 3-7. with variable coefficients, Chaos, Solitons and Fractals, 19 (2004) 847-851.

[9] J.H. He and X.-H.Wu, Variational iteration method: new development and applications, Comput. and Math. with Appl. 54 (2007) 881-894.

[10] J.H. He, Variational iteration method for autonomous ordinary differential systems, Appl. Math. Comput. 114 (2000) 115-123.

[11] M.A. Abdou and A.A. Soliman, Variational iteration method for solving Burger's and coupled Burger's equations, J. Comput. Appl. Math. 181 (2005) 245-251.

[12] M. Inokuti, et al., General use of the Lagrange multiplier in non-linear mathematical physics, in: S. Nemat-Nasser (Ed.), Variational Method in the Mechanics of Solids, Pergamon Press, Oxford, (1978) 156-162.

[13] M. A. Noor, S.T. Mohyud-Din. Variational iteration method for unsteady flow of gas through a porous medium using He's polynomials and Pade approximants, Comput. Math. with Appl. 58 (2009) 2182-2189.

[14] S. T. Mohyud-Din, A. Yildirim, Variational Iteration Method for the Hirota-Satsuma Model Using He's Polynomials , Zeitschrift Fur Naturforschunge Section A, 65 (2010) 525-528.

[15] S. T. Mohyud-Din, M. A. Noor, KI. Noor. Variational Iteration Method for Burgers' and Coupled Burgers' Equations Using He's Polynomials, Zeitschrift Fur Naturforschunge Section A-A J. OF Phy. Sci. 65 (2010) 263-267.

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