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# GENERALIZED WEYL'S THEOREM FOR AN ELEMENTARY OPERATOR

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Abstract. Let $d_{A,B} \in L(L(H))$ denote either the generalized derivation $\delta_{A,B} = L_A - R_B$ or the elementary operator $\Delta_{A,B} = L_A.R_B - I$, where $L_A$ and $R_B$ are the left and right multiplication operators defined on $L(H)$ by $L_A(X) = AX$ and $R_B(X) = XB$ respectively. $A$ and $B$ are bounded linear operators on an infinite complex Hilbert space. This paper is concerned with the transmission of polaroid and generalized Weyl's theorem from bounded linear maps on Hilbert spaces to the elementary operator. We show that polaroid property is preserved from $A$ and $B$ to $d_{A,B}$, we also prove that $d_{A,B}$ do not inherit generalized Weyl's theorem from generalized Weyl's theorem for $A$ and $B$. Moreover we give necessary and sufficient conditions for $d_{A,B}$ to satisfy generalized Weyl's theorem. Some applications for paranormal operators are given.

## 1. Introduction

Let $T \in L(X)$ be a bounded linear operator on an infinite dimensional complex Banach space $X$ and denote by $\alpha(T)$ the dimension of the kernel $\ker T$, and by $\beta(T)$ the codimension of the range $R(T)$. $T \in L(X)$ is said to be an upper semi-Fredholm operators if $\alpha(T) < \infty$ and $R(T)$ is closed, while $T \in L(X)$ is said to be lower semi-Fredholm if $\beta(T) < \infty$. If $T \in L(X)$ is either an upper or a lower semi-Fredholm operator, then $T$ is called a semi-Fredholm, and the index of $T$ is defined by $ind(T) = \alpha(T) - \beta(T)$. If both $\alpha(T)$ and $\beta(T)$ are finite, then $T$ is called a Fredholm operator. An operator $T \in L(X)$ is said to be Weyl operator if it is Fredholm operator of index zero. The Weyl spectrum of $T$ is defined by

$$\sigma_W(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not Weyl operator}\}.$$

For $T \in L(X)$ and a nonnegative integer $n$ define $T_n$ to be the restriction of $T$ to $R(T^n)$ viewed as a map from $R(T^n)$ into $R(T^n)$. If for some integer $n$ the range space $R(T^n)$ is closed and $T_n$ is an upper (resp. a lower) semi -Fredholm operator, then $T$ is called an upper (resp. a lower) semi-B-Fredholm operator. In this case

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the index of $T$ is defined as the index of the semi-Fredholm operator $T_n$, see [5]. Moreover, if $T_n$ is a Fredholm operator, then $T$ is called a B-Fredholm operator. A semi-B-Fredholm operator is an upper or a lower semi-B-Fredholm operator. An operator $T \in L(X)$ is said to be B-Weyl operator if it is B-Fredholm operator of index zero. The B-Weyl spectrum of $T$ is defined by

$$\sigma_{BW}(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not B-Weyl operator}\}.$$

We say that generalized Weyl's theorem holds for $T$ if $\sigma(T) \setminus \sigma_{BW}(T) = E(T)$, where $E(T)$ is the set of isolated eigenvalues of $T$.
M. Berkani [5, Theorem 4.5] has shown that every normal operator $T$ acting on a Hilbert space satisfies generalized Weyl's theorem. This gives a generalization of the classical Weyl's theorem. Recall that the classical Weyl's theorem asserts that for every normal operator $T$ acting on a Hilbert space, $\sigma(T) \setminus \sigma_W(T) = E_0(T)$, where $E_0(T)$ is the set of isolated eigenvalues of finite multiplicity of $T$ [21].
Recall that the ascent $p(T)$ of an operator $T$, is defined by $p(T) = \inf\{n \in \mathbb{N} : \ker T^n = \ker T^{n+1}\}$ and the descent $q(T) = \inf\{n \in \mathbb{N} : R(T^n) = R(T^{n+1})\}$, with $\inf \emptyset = \infty$. It is well known that if $p(T)$ and $q(T)$ are both finite then $p(T) = q(T)$. We denote by $\Pi(T) = \{\lambda \in \mathbb{C} : p(T - \lambda I) = q(T - \lambda I) < \infty\}$ the set of poles of the resolvent. An operator $T \in L(X)$ is called Drazin invertible if and only if it has finite ascent and descent. The Drazin spectrum of an operator $T$ is defined by

$$\sigma_D(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not Drazin invertible}\}.$$

Clearly,

$$\sigma_{BW}(T) \subset \sigma_D(T) \text{ for all } T \in L(X).$$

Let $H$ be an infinite complex Hilbert space and consider two bounded linear operators $A, B \in L(H)$. Let $L_A \in L(L(H))$ and $R_B \in L(L(H))$ be the left and the right multiplication operators, respectively, and denote by $d_{A,B} \in L(L(H))$ either the elementary operator $\Delta_{A,B}(X) = AXB - X$ or the generalized derivation $\delta_{A,B}(X) = AX - XB$. The main objective of the present paper is the transmission of polaroid and generalized Weyl's theorem from $A$ and $B$ to $d_{A,B}$. In the second section of this paper, we show that polaroid property is preserved from $A$ and $B$ to $d_{A,B}$, and we give examples proving that $d_{A,B}$ do not inherit generalized Weyl's theorem from generalized Weyl's theorem for $A$ and $B$, moreover we give necessary and sufficient conditions for $d_{A,B}$ to satisfy generalized Weyl's theorem.
In the third section we give an application to paranormal operators. Our results generalize the following ones [11, Theorem 3.3] and [15, Corollary 2.6] and [10, Theorem 3.4].

## 2. Necessary and sufficient conditions for $d_{A,B}$ to satisfy generalized Weyl's theorem

In the sequel we shall denote by $accD$ and $isoD$, the set of accumulation points and the set of isolated points of $D \subset \mathbb{C}$, respectively.

**Definition 2.1.** *An operator $T \in L(X)$ is said to be polaroid if*

$$iso\sigma(T) \subseteq \Pi(T).$$

It is easily seen that, if $T \in L(X)$ is polaroid, then $\Pi(T) = E(T)$.
An important subspace in local spectral theory is the the quasi-nilpotent part of $T$

is defined by

$$H_0(T) = \{x \in X : \lim_{n\to\infty} \|T^n(X)\|^{\frac{1}{n}} = 0\}.$$

It is easily seen that $\ker T^n \subset H_0(T)$ for every $n \in \mathbb{N}$, see [1] for information on $H_0(T)$.

**Lemma 2.2.** *Suppose that $A, B \in L(H)$ are polaroid operators, then $d_{A,B}$ is polaroid.*

*Proof.* Recall from [16] that $\sigma(\delta_{A,B}) = \sigma(A) - \sigma(B)$ and $\sigma(\Delta_{A,B}) = \sigma(A)\sigma(B) - \{1\}$. If $\lambda \in iso\sigma(d_{A,B})$, then we have one of the following cases:

1. If $d_{A,B} = \delta_{A,B}$, then there exist finite sequences $\{\mu_i\}_{i=1}^n$ and $\{\nu_i\}_{i=1}^n$, of isolated points in $\sigma(A)$ and $\sigma(B)$, respectively such that $\lambda = \mu_i - \nu_i$, for all $1 \leq i \leq n$.
2. $d_{A,B} = \Delta_{A,B}$ and $\lambda = -1$, then either $0 \in iso\sigma(A)$ and $0 \in iso\sigma(B)$, or $0 \in iso\sigma(A)$ and $0 \notin \sigma(B)$, or $0 \in iso\sigma(B)$ and $0 \notin \sigma(A)$.
3. $d_{A,B} = \Delta_{A,B}$ and $\lambda \neq -1$, then there exist finite sequences $\{\mu_i\}_{i=1}^n$ and $\{\nu_i\}_{i=1}^n$, of isolated points in $\sigma(A)$ and $\sigma(B)$, respectively such that $\mu_i\nu_i = 1 + \lambda$, for all $1 \leq i \leq n$.

We stat by considering **Case 1**. If $\lambda \in iso\sigma(\delta_{A,B})$, then there exist finite sequences $\{\mu_i\}_{i=1}^n$ and $\{\nu_i\}_{i=1}^n$ such that $\mu_i \in iso\sigma(A)$ and $\nu_i \in iso\sigma(B)$. Since $A$ and $B$ are polaroid, then from [1, Theorem 3.74] $H_0(A - \mu_i I) = \ker(A - \mu_i I)^{p_i}$ and $H_0(B - \nu_i I) = \ker(B - \nu_i I)^{q_i}$,$1 \leq i \leq n$, for some integers $p_i, q_i \geq 1$, $\lambda = \mu_i - \nu_i$. The sets $E_1 = \{\mu_1, \mu_2, ....., \mu_n\}$ and $E_2 = \{\nu_1, \nu_2, ....., \nu_n\}$ are spectral sets of $\sigma(A)$ and $\sigma(B)$, respectively. Hence by the Riesz decomposition theorem there exist invariant subspaces $M_k$ and $N_k$, $k = 1, 2$ of $A$ and $B$ respectively such that $H = M_1 \oplus M_2 = N_1 \oplus N_2$, $\sigma(A_1) = \sigma(A|_{M_1}) = E_1$, $\sigma(B_1) = \sigma(B|_{N_1}) = E_2$, $\sigma(A_2) = \sigma(A|_{M_2}) = \sigma(A) \backslash E_1$ and $\sigma(B_2) = \sigma(B|_{N_2}) = \sigma(B) \backslash E_2$. Observe that $\mu_i$ is a pole of $A_1$ of order $p_i$ and $\nu_i$ is a pole of $B_1$ of order $q_i$, for all $1 \leq i \leq n$. Thus $A_1$ and $B_1$ are algebraic operators [1, Theorem 3.83]. Hence $M_1 = \oplus_{i=1}^n \ker(A_1 - \mu_i)^{p_i}$, and $N_1 = \oplus_{i=1}^n \ker(B_1 - \nu_i)^{p_i}$, let $M_{1i} = \ker(A_1 - \mu_i)^{p_i}$ and $N_{1i} = \ker(B_1 - \nu_i)^{p_i}$, for all $1 \leq i \leq n$, let $p = \max\{p_1, p_2, ....., p_n\}$ and $q = \max\{q_1, q_2, ....., q_n\}$, and set $p + q = r$.

Let $Y \in R((\delta_{A,B} - \lambda I)^r)$, then there exist $X \in L(N_1 \oplus N_2, M_1 \oplus M_2)$ have the representation $X = [X_{kl}]_{k,l=1}^2$ such that $Y = (\delta_{A,B} - \lambda I)^r(X)$. It will be proved that $q(\delta_{A,B} - \lambda I) \leq r$.

$$Y = (\delta_{A,B} - \lambda I)^r(X) = \begin{pmatrix} (\delta_{A_1,B_1} - \lambda I)^r(X_{11}) & (\delta_{A_1,B_2} - \lambda I)^r(X_{12}) \\ (\delta_{A_2,B_1} - \lambda I)^r(X_{21}) & (\delta_{A_2,B_2} - \lambda I)^r(X_{22}) \end{pmatrix}.$$

Observe that $\delta_{A_i,B_j} - \lambda I$ is invertible for all $1 \leq i, j \leq 2$ such that $i, j \neq 1$. Hence there exist operators $Z_{ij}$ such that

$$X_{ij} = (\delta_{A_i,B_j} - \lambda I)Z_{ij}, \tag{2.1}$$

for all $1 \leq i, j \leq 2$ such that $i, j \neq 1$. Let $X_{11} = [Y_{ij}]_{1\leq i,j\leq n} \in L(\oplus_{i=1}^n N_{1i}, \oplus_{i=1}^n M_{1i})$. Then for $1 \leq i, j \leq n$, We have

$$\begin{aligned}(\delta_{A_1,B_1} - \lambda I)^r(X_{11}) &= ((L_{A_1-\mu_i} - R_{B_1-\nu_j}) + (\mu_i - \nu_j - \lambda))^r[Y_{ij}]_{1\leq i,j\leq n} \\ &= \left(\sum_{k=0}^{r} \begin{pmatrix} r \\ k \end{pmatrix} (L_{A_1-\mu_i} - R_{B_1-\nu_j})^k(\mu_i - \nu_j - \lambda)^{r-k}\right)[Y_{ij}]_{1\leq i,j\leq n}\end{aligned}$$

since $(A_1 - \mu_i)^{p_i}|_{M_{1i}} = 0 = (B_1 - \nu_i)^{q_i}|_{N_{1i}}$ for all $1 \leq i \leq n$, observe that $\sigma(A_1|_{M_{1i}}) = \{\mu_i\}$, $\sigma(B_1|_{N_{1i}}) = \{\nu_i\}$, $\sigma(\delta_{A_1,B_1} - \lambda I) = \{\mu_i - \nu_j - \lambda, \mu_i \in \sigma(A), \nu_j \in \sigma(B), 1 \leq i,j \leq n\}$. Hence the operator $\delta_{A_1,B_1} - \lambda I|_{L(N_{1j},M_{1i})}$ is invertible for all $i \neq j$, $1 \leq i,j \leq n$ and the operator $\delta_{A_1,B_1} - \lambda I|_{L(N_{1i},M_{1i})}$ is nilpotent of order $r$ for all $1 \leq i \leq n$. Hence there exit $C_{ij} \in L(N_{1j}, M_{1i})$, for all $i \neq j$, $1 \leq i,j \leq n$ and choose $C_{ii} \in L(N_{1i}, M_{1i})$, arbitrarily for all $1 \leq i \leq n$, set $Z_{11} = [C_{ij}]_{1 \leq i,j \leq n}$. Thus

$$X_{11} = (\delta_{A_1,B_1} - \lambda I)Z_{11}. \tag{2.2}$$

Therefore from (2.1) and (2.2) it follows that there exist $Z = [Z_{ij}]_{1 \leq i,j \leq 2} \in L(N_1 \oplus N_2, M_1 \oplus M_2)$, such that $Y = (\delta_{A,B} - \lambda I)^{r+1}(Z)$, thus $Y \in R((\delta_{A,B} - \lambda I)^{r+1})$. Since the reverse inclusion holds for all operators. Then $R((\delta_{A,B} - \lambda I)^{r+1}) = R((\delta_{A,B} - \lambda I)^{r})$, i.e $q(\delta_{A,B} - \lambda I) \leq r$.
With the same decompositions we can easily prove that $\ker(\delta_{A,B} - \lambda I)^{r+1} \subset \ker(\delta_{A,B} - \lambda I)^{r}$. Since the reverse inclusion holds for all operators. Then $p(\delta_{A,B} - \lambda I) = q(\delta_{A,B} - \lambda I) \leq r$.
**Case 2**. Is proved by E. Boisso, B.P. Duggal and I. H. Jeon [9, Lemma 4.7].
**Case 3**. Since $L_A R_B$ is polaroid it follows from [14, Lemma 3.8] that $\Delta_{A,B}$ is polaroid. □

**Remark.** *The class of polaroid operators is large, it contains:*

(1) *The class of all operators $A \in L(H)$ such that for every complex number $\lambda$ there exists an integer $p_\lambda \geq 1$ for which the following condition holds*

$$H_0(A - \lambda I) = \ker(A - \lambda I)^{p_\lambda}.$$

(2) *$\mathcal{HN}$ the class of hereditarily normaloid, $A \in \mathcal{HN}$ if every part of $A$ is normaloid, a part of $A$ is its restriction to an invariant subspace*
(3) *$\mathcal{THN}$ the class of totally hereditarily normaloid. We say that $A \in \mathcal{HN}$ is totally hereditarily normaloid if also every invertible part of $A$ is normaloid*
(4) *$\mathcal{CHN}$ the class of completely totally hereditarily normaloid, $A \in \mathcal{CHN}$ if either $A$ is totally hereditarily normaloid or $A - \lambda I \in \mathcal{HN}$ for every $\lambda \in \mathbb{C}$*
(5) *$(p,k)-Q$ the class of $(p,k)$ quasi-hyponormal, $A \in (p,k)-Q$ if $A^{*k}(|A|^{2p} - |A^*|^{2p})A^k \geq 0$ for some positive integer $k$ and $0 < p \leq 1$.*

*From* [1, Theorem 3.74] *we can easily prove that $T \in L(X)$ is polaroid if and only if there exists $p = p(\lambda) \in \mathbb{N}$ such that $H_0(T - \lambda I) = \ker(T - \lambda I)^p$ for all $\lambda \in iso\sigma(T)$. This result allows us to deduce that Lemma 2.2 generalize the following results* [12, Theorem 3.2], [12, Theorem 3.3], [12, Theorem 3.4], [12, Theorem 3.5], [15, Theorem 2.3] *and* [10, Lemma 3.2].

$L_A$ inherit generalized Weyl's theorem from generalized Weyl's theorem for $A \in L(X)$ and $R_B$ inherit generalized Weyl's theorem from generalized Weyl's theorem for $B^* \in L(X)$ ($B^*$ is the dual of $B$) see [9, Theorem 3.5] and [9, Theorem 3.6]. But $d_{A,B}$ do not inherit generalized Weyl's theorem from generalized Weyl's theorem for $A$ and $B$ because the following examples shows that generalized Weyl's theorem is not preserved under products and sums of commuting operators.

**Example 2.3.** *Let $I_1$ and $I_2$ be the identities on $\mathbb{C}$ and $l^2(\mathbb{N})$, respectively. Let $S_1$ and $S_2$ defined on $l^2(\mathbb{N})$ by*

$$S_1(x_1, x_2, .....) = (0, \frac{1}{3}x_1, \frac{1}{3}x_2, .....), \quad S_2(x_1, x_2, .....) = (0, \frac{1}{2}x_1, \frac{1}{3}x_2, .....).$$

*Let* $T_1 = I_1 \oplus S_1$, $T_2 = S_2 - I_2$ *and* $T = T_1 \oplus T_2$, *from* [22, Example 1] *we have generalized Weyl's theorem holds for* $T$ *but it does not hold for* $T.T = T^2$.

**Example 2.4.** *Let* $S : l^2(\mathbb{N}) \to l^2(\mathbb{N})$ *be an injective quasinilpotent operator, and let* $U : l^2(\mathbb{N}) \to l^2(\mathbb{N})$ *be defined by* $U(x_1, x_2, x_3, .....) = (-x_1, 0, 0, .....)$. *define on* $l^2(\mathbb{N}) \oplus l^2(\mathbb{N})$ *the operators* $T$ *and* $F$ *by* $T = I \oplus S$ *and* $F = U \oplus 0$.
*Clearly,* $F$ *is a finite rank operator and* $FT = TF$, *it is easy to check that* $\sigma(T) = \{0, 1\}$, $E(T) = \{1\}$ *and it follows from* [7, Example 2] *that* $\sigma_{BW}(T) = \{0\}$. *Hence* $T$ *satisfies generalized Weyl's theorem. Since* $F$ *is a finite rank operator, then* $\sigma(F) = E(F) = \{0, -1\}$ *and* $\sigma_{BW}(F) = \emptyset$, *then generalized Weyl's theorem holds for* $F$, *and* $T + F$ *does not satisfy generalized Weyl's theorem.*

In the following results we give necessary and sufficient condition for $d_{A,B}$ to satisfy generalized Weyl's theorem.

**Theorem 2.5.** *Suppose that* $A, B \in L(H)$ *are polaroid operators which satisfy generalized Weyl's theorem, then a necessary and sufficient condition for* $\delta_{A,B}$ *to satisfy generalized Weyl's theorem is*

$$\sigma_{BW}(\delta_{A,B}) = (\sigma_{BW}(A) - \sigma(B)) \cup (\sigma(A) - \sigma_{BW}(B)).$$

*Proof.* Assume that $\sigma_{BW}(\delta_{A,B}) = (\sigma_{BW}(A) - \sigma(B)) \cup (\sigma(A) - \sigma_{BW}(B))$.
Let $\lambda \in \sigma(\delta_{A,B}) \backslash \sigma_{BW}(\delta_{A,B})$, then for $\lambda = \mu - \nu$ such that $\mu \in \sigma(A)$ and $\nu \in \sigma(B)$, $\mu \notin \sigma_{BW}(A)$ and $\nu \notin \sigma_{BW}(B)$. Consequently $\mu \in E(A) = \Pi(A)$ and $\nu \in E(B) = \Pi(B)$. Let $p(A - \mu I) = q(A - \mu I) = p_1$ and $p(B - \nu I) = q(B - \nu I) = q_1$, it follows that there exist decompositions $H = \ker(A - \mu I)^{p_1} \oplus R((A - \mu I)^{p_1}) = M_1 \oplus M_2$ and $H = \ker(B - \nu I)^{q_1} \oplus R((B - \nu I)^{q_1}) = N_1 \oplus N_2$ such that $\sigma(A_1) = \sigma(A|_{M_1}) = \{\mu\}$, $\sigma(B_1) = \sigma(B|_{N_1}) = \{\nu\}$, $\sigma(A_2) = \sigma(A|_{M_2}) = \sigma(A) \backslash \{\mu\}$ and $\sigma(B_2) = \sigma(B|_{N_2}) = \sigma(B) \backslash \{\nu\}$. Observe that $A_1 - \mu$ is nilpotent of order $p_1$ and $B_1 - \nu$ is nilpotent of order $q_1$. It will be proved that $p(\delta_{A,B} - \lambda I) = q(\delta_{A,B} - \lambda I) \leq r$, such that $r = p_1 + q_1$. Let $Y \in R((\delta_{A,B} - \lambda I)^r)$, then there exist $X \in L(N_1 \oplus N_2, M_1 \oplus M_2)$ have the representation $X = [X_{kl}]_{k,l=1}^2$ such that

$$Y = (\delta_{A,B} - \lambda I)^r(X) = \begin{pmatrix} (\delta_{A_1,B_1} - \lambda I)^r(X_{11}) & (\delta_{A_1,B_2} - \lambda I)^r(X_{12}) \\ (\delta_{A_2,B_1} - \lambda I)^r(X_{21}) & (\delta_{A_2,B_2} - \lambda I)^r(X_{22}) \end{pmatrix}.$$

The operator $\delta_{A_i,B_j} - \lambda I$ is invertible for all $1 \leq i, j \leq 2$ such that $i, j \neq 1$. We argue as in the proof of Lemma 2.2, we get that $Y \in R((\delta_{A,B} - \lambda I)^{r+1})$, i.e $q(\delta_{A,B} - \lambda I) \leq r$. With the same decompositions we can easily prove that $\ker(\delta_{A,B} - \lambda I)^{r+1} \subset \ker(\delta_{A,B} - \lambda I)^r$, i.e $p(\delta_{A,B} - \lambda I) \leq r$, consequently $\lambda \in \Pi(\delta_{A,B})$. Hence $\sigma(\delta_{A,B}) \backslash \sigma_{BW}(\delta_{A,B}) \subset \Pi(\delta_{A,B}) \subset E(\delta_{A,B})$. On the other hand the inclusion $\Pi(\delta_{A,B}) \subset \sigma(\delta_{A,B}) \backslash \sigma_{BW}(\delta_{A,B})$ holds for every operator and since $\delta_{A,B}$ is polaroid from Lemma 2.2, we have $E(\delta_{A,B}) = \Pi(\delta_{A,B})$, so $E(\delta_{A,B}) = \sigma(\delta_{A,B}) \backslash \sigma_{BW}(\delta_{A,B})$, it then follows that $\delta_{A,B}$ satisfies generalized Weyl's theorem.
Conversely suppose that $\delta_{A,B}$ satisfies generalized Weyl's theorem, then $E(\delta_{A,B}) = \sigma(\delta_{A,B}) \backslash \sigma_{BW}(\delta_{A,B})$. If $\lambda \notin (\sigma_{BW}(A) - \sigma(B)) \cup (\sigma(A) - \sigma_{BW}(B))$, then for $\lambda = \mu - \nu$ such that $\mu \in \sigma(A)$ and $\nu \in \sigma(B)$, we have $\mu \in E(A) = \Pi(A)$ and $\nu \in E(B) = \Pi(B)$, we argue as above we get, $\lambda \in \Pi(\delta_{A,B}) = E(\delta_{A,B}) = \sigma(\delta_{A,B}) \backslash \sigma_{BW}(\delta_{A,B})$. Hence $\sigma_{BW}(\delta_{A,B}) \subset (\sigma_{BW}(A) - \sigma(B)) \cup (\sigma(A) - \sigma_{BW}(B))$. For the reverse inclusion, let $\lambda \in \sigma(\delta_{A,B}) \backslash \sigma_{BW}(\delta_{A,B})$, then $\lambda \in E(\delta_{A,B})$, which implies that there exist finite sequences $\{\mu_i\}_{i=1}^n$ and $\{\nu_i\}_{i=1}^n$ of values $\mu_i \in iso\sigma(A) \subset E(A)$ and $\nu_i \in iso\sigma(B) \subset E(B)$ such that $\lambda = \mu_i - \nu_i$ for all $1 \leq i \leq n$, then $\mu_i \notin \sigma_{BW}(A)$ and $\nu_i \notin \sigma_{BW}(B)$, for all $1 \leq i \leq n$, consequently $\lambda \notin (\sigma_{BW}(A) - \sigma(B)) \cup (\sigma(A) - \sigma_{BW}(B))$. □

**Theorem 2.6.** *Suppose that $A, B \in L(H)$ are polaroid operators which satisfy generalized Weyl's theorem, then a necessary and sufficient condition for $\Delta_{A,B}$ to satisfy generalized Weyl's theorem, is*

$$\sigma_{BW}(\Delta_{A,B}) = \sigma(A)\sigma_{BW}(B) \cup \sigma_{BW}(A)\sigma(B) - \{1\}.$$

*Proof.* Assume that $\sigma_{BW}(\Delta_{A,B}) = \sigma(A)\sigma_{BW}(B) \cup \sigma_{BW}(A)\sigma(B) - \{1\}$.
Let $\lambda \in \sigma(\Delta_{A,B})\backslash\sigma_{BW}(\Delta_{A,B})$ such that $\lambda \neq -1$, then for $\lambda = \mu\nu - 1$ such that $\mu \in \sigma(A)$ and $\nu \in \sigma(B)$, it follows that $\mu \notin \sigma_{BW}(A)$ and $\nu \notin \sigma_{BW}(B)$, hence $\mu \in E(A) = \Pi(A)$ and $\nu \in E(B) = \Pi(B)$, we argue as in the proof of Theorem 2.5, we get

$$E(\Delta_{A,B}) = \sigma(\Delta_{A,B})\backslash\sigma_{BW}(\Delta_{A,B}),$$

it then follows that $\Delta_{A,B}$ satisfies generalized Weyl's theorem.
Conversely suppose that $\Delta_{A,B}$ satisfies generalized Weyl's theorem, then $E(\Delta_{A,B}) = \sigma(\Delta_{A,B})\backslash\sigma_{BW}(\Delta_{A,B})$. If $\lambda \notin \sigma(A)\sigma_{BW}(B)\cup\sigma_{BW}(A)\sigma(B)-\{1\}$, then for $\lambda = \mu\nu-1$ such that $\mu \in \sigma(A)$ and $\nu \in \sigma(B)$, hence $\mu \in E(A) = \Pi(A)$ and $\nu \in E(B) = \Pi(B)$, we argue as in proof of Theorem 2.5, we get $\lambda \in \Pi(\Delta_{A,B}) = E(\Delta_{A,B}) = \sigma(\Delta_{A,B})\backslash\sigma_{BW}(\Delta_{A,B})$. Hence $\sigma_{BW}(\Delta_{A,B}) \subset \sigma(A)\sigma_{BW}(B) \cup \sigma_{BW}(A)\sigma(B) - \{1\}$.
For the reverse inclusion and the case $\lambda = -1$, will be proved similarly. □

## 3. Application

A bounded linear operator $T$ on a complex Hilbert space $H$, is said to be p-hyponormal if $(T^*T)^p) \geq (TT^*)^p)$. Especially, a p-hyponormal operator $T$ is said to be hyponormal and semi-hyponormal if $p = 1$ and $p = \frac{1}{2}$, respectively. For positive numbers $s$ and $t$, an operator $T$ belongs to class $A(s,t)$ if $(|T^*|^t|T|^{2s}|T^*|^t)^{\frac{t}{s+t}} \geq |T^*|^{2t}$. Especially, we denote class $A(1,1)$ by class A. $A(\frac{1}{2},\frac{1}{2})$ is the class of w-hyponormal operators it was introduced by Aluthge and Wang [3], the class of w-hyponormal operators contains the class of p-hyponormal $(0 < p \leq 1)$ and log-hyponormal operators. An operator $T \in L(H)$ is said to be log-hyponormal if $T$ is invertible and satisfies $\log(T^*T) \geq \log(TT^*)$. Recall that $T \in L(H)$ is said to be paranormal if $\|Tx\|^2 \leq \|T^2x\|\|x\|$, for all $x \in H$. Inclusion relations among these classes are known as follows:

$$\begin{aligned} \{hyponorml\} &\subset \{p - hyponorml, 0 < p < 1\} \\ &\subset \{classA(s,t), s,t \in ]0,1]\} \\ &\subset \{classA\} \\ &\subset \{paranormal\}. \end{aligned}$$

It is proved in [11] that if $A, B^* \in L(H)$ are hyponormal, then generalized Weyl's theorem holds for $f(d_{A,B})$ for every $f \in \mathcal{H}(\sigma(d_{A,B}))$, where $\mathcal{H}(\sigma(d_{A,B}))$ is the set of all analytic functions defined on a neighborhood of $\sigma(d_{A,B})$, this result was extended to log-hyponormal or p-hyponormal operators in [15] and [19]. Also in [10] it is shown that if $A, B^* \in L(H)$ are w-hyponormal operators, then Weyl's theorem holds for $f(d_{A,B})$ for every $f \in \mathcal{H}(\sigma(d_{A,B}))$. In the next result we can give more, before that we recall the following definitions.

**Definition 3.1.** *For $T \in L(X)$ and a closed subset $F$ of $\mathbb{C}$ the glocal spectral subspace $\mathcal{X}_T(F)$ defined as the set of all $x \in X$ such that there is an analytic $X$-valued function $f : \mathbb{C}\backslash F \to X$ for which $(T - \lambda I)f(\lambda) = x$ for all $\lambda \in \mathbb{C}\backslash F$. $T \in L(X)$ is said to have Dunford property $(C)$ if every glocal spectral subspace is* *closed for every closed set $F \subseteq \mathbb{C}$ and $T \in L(X)$ is said to be decomposable if $T$ has both property $(C)$ and property $(\delta)$, where the last property means that for every open covering $(U, V)$ of $\mathbb{C}$ we have $X = \mathcal{X}_T(\overline{U}) + \mathcal{X}_T(\overline{V})$.*

**Definition 3.2.** *An operator $T \in L(X)$ has Bishop's property $(\beta)$ if for every open set $U \subset \mathbb{C}$ and every sequence of analytic functions $f_n : U \to X$, with the property that $(T - \lambda I)f_n(\lambda) \to 0$ uniformly on every compact subset of $U$, it follows that $f_n \to 0$, again locally uniformly on $U$.*

Bishop's property $(\beta)$ implies Dunford property $(C)$, also $T$ satisfies property $(\beta)$ if and only if $T^*$ satisfies property $(\delta)$ [18, Theorem 2.5.5]. For more information on property $(\beta)$, property $(\delta)$ and Dunford's condition $(C)$ we refer the interested reader to [18].

**Definition 3.3.** *An operator $T \in L(X)$ is said to have the single valued extension property at $\lambda_0 \in \mathbb{C}$ (abbreviated SVEP at $\lambda_0$), if for every open disc $\mathbb{D}$ centered at $\lambda_0$, the only analytic function $f : \mathbb{D} \to X$ which satisfies the equation $(T - \lambda I)f(\lambda) = 0$ for all $\lambda \in \mathbb{D}$ is the function $f \equiv 0$. An operator $T \in L(X)$ is said to have SVEP if $T$ has SVEP at every $\lambda \in \mathbb{C}$.*

Evidently, every operator $T$, as well as its dual $T^*$, has SVEP at every point in $\partial\sigma(T)$, where $\partial\sigma(T)$ is the boundary of the spectrum $\sigma(T)$, in particular at every isolated point of $\sigma(T)$.

**Lemma 3.4.** *Suppose that $A, B^* \in L(H)$ are paranormal operators, then $d_{A,B}$ has SVEP.*

*Proof.* From [20] we have $A$ satisfies property $(\beta)$ and $B$ satisfies property $(\delta)$. Hence both $L_A$ and $R_B$ satisfy condition $(C)$ by [18, Corollary 3.6.11]. Since $L_A$ and $R_B$ commute, it follows by [18, Theorem 3.6.3] and [18, Note 3.6.19] that $L_A - R_B$ and $L_A R_B$ have SVEP, then SVEP holds for $d_{A,B}$. □

**Theorem 3.5.** *Suppose that $A, B^* \in L(H)$ are paranormal operators, then*

$$\sigma_{BW}(\delta_{A,B}) = (\sigma_{BW}(A) - \sigma(B)) \cup (\sigma(A) - \sigma_{BW}(B)),$$

*and*

$$\sigma_{BW}(\Delta_{A,B}) = \sigma(A)\sigma_{BW}(B) \cup \sigma_{BW}(A)\sigma(B) - \{1\}.$$

*Proof.* We know from [13, Proposition 2.1] and [17, P. 229] that paranormal operators are polaroid. Since a Hilbert space operator is polaroid if and only if its adjoint is polaroid, it follows that $B$ is polaroid and from [14, Theorem 4.2] that generalized Weyl's theorem holds for $A$, $A^*$, $B^*$ and $B$.
Let $\lambda \notin (\sigma_{BW}(A) - \sigma(B)) \cup (\sigma(A) - \sigma_{BW}(B))$, then for every $\lambda = \mu - \nu$ such that $\mu \in \sigma(A)$ and $\nu \in \sigma(B)$, we have $\mu \notin \sigma_{BW}(A)$ and $\nu \notin \sigma_{BW}(B)$ which implies that $\mu \in E(A) = \Pi(A)$ and $\nu \in E(B) = \Pi(B)$. We argue as in the proof of Theorem 2.5, we get $\lambda \in \Pi(\delta_{A,B})$, it follows from [6, Theorem 2.3] that $\delta_{A,B} - \lambda I$ is B-Fredholm of index zero. Hence

$$\sigma_{BW}(\delta_{A,B}) \subset (\sigma_{BW}(A) - \sigma(B)) \cup (\sigma(A) - \sigma_{BW}(B)).$$

For the reverse inclusion, let $\lambda \in \sigma(\delta_{A,B})$ and $\lambda \notin \sigma_{BW}(\delta_{A,B})$. Since $d_{A,B}$ has SVEP from Lemma 3.4, then from [8, Theorem 3.3] $\sigma_{BW}(\delta_{A,B}) = \sigma_D(\delta_{A,B})$ and $\delta_{A,B}$ is polaroid from Lemma 2.2, therefore $\sigma_D(\delta_{A,B}) = acc\sigma(\delta_{A,B})$, it follows that $\lambda \in iso\sigma(\delta_{A,B})$, then there exist finite sequences $\{\mu_i\}_{i=1}^n$ and $\{\nu_i\}_{i=1}^n$ of values

$\mu_i \in iso\sigma(A)$ and $\nu_i \in iso\sigma(B)$ such that $\lambda = \mu_i - \nu_i$ for all $1 \leq i \leq n$, so $\mu_i \in \Pi(A) = E(A)$ and $\nu_i \in \Pi(A) = E(B)$, for all $1 \leq i \leq n$, which implies that $\mu_i \notin \sigma_{BW}(A)$ and $\nu_i \notin \sigma_{BW}(B)$, for all $1 \leq i \leq n$, consequently $\lambda \notin (\sigma_{BW}(A) - \sigma(B)) \cup (\sigma(A) - \sigma_{BW}(B))$. Hence

$$\sigma_{BW}(\delta_{A,B}) = (\sigma_{BW}(A) - \sigma(B)) \cup (\sigma(A) - \sigma_{BW}(B)).$$

The equality $\sigma_{BW}(\Delta_{A,B}) = \sigma(A)\sigma_{BW}(B) \cup \sigma_{BW}(A)\sigma(B) - \{1\}$, will be proved similarly □

**Corollary 3.6.** *Suppose that $A, B^* \in L(H)$ are paranormal operators, then generalized Weyl's theorem holds for $f(d_{A,B})$ and $f(d^*_{A,B})$ for every $f \in \mathcal{H}(\sigma(d_{A,B}))$, where $d^*_{A,B}$ is the dual of $d_{A,B}$.*

*Proof.* By Theorem 3.5 we get generalized Weyl's theorem holds for $d_{A,B}$. To show that generalized Weyl's theorem holds for $f(d_{A,B})$, observe first from Lemma 2.2 that $d_{A,B}$ is polaroid, then it is isoloid, i.e. every isolated point of the spectrum is an eigenvalue of $d_{A,B}$. From [22, Theorem 2.2] it follows that generalized Weyl's theorem holds for $f(d_{A,B})$.
Since $d_{A,B}$ is polaroid and has SVEP, then from [4, Theorem 2.10] and [4, Theorem 2.4] $d^*_{A,B}$ (the dual of $d_{A,B}$) satisfies generalized Weyl's theorem. Since $d_{A,B}$ is polaroid, then by [2, Lemma 2.3] $d^*_{A,B}$ is polaroid, hence $d^*_{A,B}$ is isoloid, From [7, Lemma 2.9] it follows that $f(d^*_{A,B})$ satisfies generalized Weyl's theorem for every $f \in \mathcal{H}(\sigma(d_{A,B}))$.
□

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