Bulletin of Mathematical Analysis and Applications
ISSN: 1821-1291, URL: http://www.bmathaa.org
Volume 3 Issue 4(2011), Pages 15-30.

CURVATURE AND RIGIDITY THEOREMS OF SUBMANIFOLDS IN A UNIT SPHERE

(COMMUNICATED BY UDAY CHAND DE)

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Abstract. In this paper, we investigate n-dimensional submanifolds with higher codimension in a unit sphere $S^{n+p}(1)$. We obtain some rigidity results of submanifolds in $S^{n+p}(1)$ with parallel mean curvature vector or with constant scalar curvature, which generalize some related rigidity results of hypersurfaces.

1. Introduction

Let M^n be an n-dimensional hypersurface in a unit sphere $S^{n+1}(1)$. It is well known that there are many rigidity results for hypersurfaces in $S^{n+1}(1)$ with constant mean curvature or constant scalar curvature (see [1], [4], [10]), but few of submanifolds with higher codimension in $S^{n+p}(1)$, especially, if the submanifolds are complete.

It is well known that H. Alencar, M. do Carmo [1] and H. Li [10] obtained some important results of compact hypersurface with constant mean curvature or constant scalar curvature in a unit sphere $S^{n+1}(1)$, respectively.

Theorem 1.1([1]). *Let M^n be an n-dimensional compact hypersurface in a unit sphere $S^{n+1}(1)$ with constant mean curvature. Assume that $|\phi|^2 \leq B_{H,n}$, then*

(1) $|\phi|^2 = 0$, M^n is totally umbilical; or

(2) $|\phi|^2 = B_{H,n}$ if and only if

(i) $H = 0$, M^n is a Clifford torus $S^k(\sqrt{k/n}) \times S^{n-k}(\sqrt{(n-k)/n})$, with $1 \leq k \leq n-1$;

(ii) $H \neq 0$, $n \geq 3$, and M^n is an $H(r)$-torus $S^{n-1}(r) \times S^1(\sqrt{1-r^2})$ with $r^2 < (n-1)/n$;

(iii) $H \neq 0$, $n = 2$, and M^n is an $H(r)$-torus $S^1(r) \times S^1(\sqrt{1-r^2})$ with $0 < r < 1$, $r^2 \neq \frac{1}{2}$.

Theorem 1.2([10]). *Let M^n be an n-dimensional $(n \geq 3)$ compact hypersurface in*

2000 *Mathematics Subject Classification.* 53C42, 53A10.

Key words and phrases. submanifolds; mean curvature; scalar curvature; rigidity theorems.

Submitted Jul 21, 2011. Published August 28, 2011.

a unit sphere $S^{n+1}(1)$ with constant scalar curvature $n(n-1)R$ and $\bar{R} = R-1 \geq 0$. If

$$n\bar{R} \leq S \leq \frac{n}{(n-2)(n\bar{R}+2)}\{n(n-1)\bar{R}^2 + 4(n-1)\bar{R} + n\}, \tag{1.1}$$

then either $S = n\bar{R}$ and M^n is totally umbilical, or $S = \frac{n}{(n-2)(n\bar{R}+2)}\{n(n-1)\bar{R}^2 + 4(n-1)\bar{R} + n\}$ and M^n is a product $S^1(\sqrt{1-r^2}) \times S^{n-1}(r)$, $r = \sqrt{\frac{n-2}{nR}}$.

Remark 1.1. We should notice that in Theorem 1.1, $|\phi|^2 = S - nH^2$ is the non-negative function on M^n, S and H the squared norm of the second fundamental form and mean curvature of M^n, $B_{H,n}$ the square of the positive real root of

$$P_{H,n}(x) = x^2 + \frac{n-2}{\sqrt{n(n-1)}}nHx - n(1+H^2) = 0.$$

We should notice that W. Santos [14], Cheng [5] obtained some important results of compact submanifolds with higher codimension and parallel mean curvature vector or constant scalar curvature in $S^{n+p}(1)$, but to our knowledge, the results of complete submanifolds in $S^{n+p}(1)$ are very few.

In this paper, we study n-dimensional compact or complete submanifolds with higher codimension and parallel mean curvature vector or constant scalar curvature in $S^{n+p}(1)$. In order to present our result, we define a function $Q_{\bar{R},p,n}(x)$ by

$$\begin{aligned} Q_{\bar{R},p,n}(x) =& n + n\bar{R} + [\frac{1}{n} - (2-\frac{1}{p})\frac{n-1}{n}](x - n\bar{R}) \\ &- \frac{n-2}{n}\sqrt{[x + n(n-1)\bar{R}](x - n\bar{R})}, \end{aligned} \tag{1.2}$$

then we may obtain the following result:

Theorem 1.3. *Let M^n be an n-dimensional compact submanifold in a unit sphere $S^{n+p}(1)$ with constant scalar curvature $n(n-1)R$ and $\bar{R} = R - 1 \geq 0$. If the normalized mean curvature vector is parallel and the squared norm S of the second fundamental form of M^n satisfies*

$$Q_{\bar{R},p,n}(S) \geq 0, \tag{1.3}$$

then

(1) *$S = n\bar{R}$ and M^n is totally umbilical; or*
(2) *$Q_{\bar{R},p,n}(S) = 0$. In the latter case, either*
(a) *$p = 1$ and M^n is a product $S^1(\sqrt{1-r^2}) \times S^{n-1}(r)$, $r = \sqrt{\frac{n-2}{nR}}$; or*
(b) *$n = 2$, $p = 2$ and M^n is Veronese surface in S^4.*

Remark 1.2. We note that if $p = 1$, Theorem 1.3 reduces to Theorem 1.2. We should notice that in [11], J.T. Li obtained some results of compact submanifold in $S^{n+p}(1)$ with constant scalar curvature and parallel normalized mean curvature vector, but his results are very different from us.

We define a polynomial $P_{H,p,n}(x)$ by

$$P_{H,p,n}(x) = (2-\frac{1}{p})x^2 + \frac{n-2}{\sqrt{n(n-1)}}nHx - n(1+H^2). \tag{1.4}$$

We easily know that $P_{H,p,n}(x) = 0$ has a positive real root, and denoted by $B_{H,p,n}$ the square of the positive real root.

If M^n is an n-dimensional complete submanifold with higher codimension in a unit sphere $S^{n+p}(1)$, we obtain the following results:

Theorem 1.4. *Let M^n be an n-dimensional complete submanifold in a unit sphere $S^{n+p}(1)$ with parallel mean curvature vector. Assume that $\sup|\phi|^2 \leq B_{H,p,n}$, then*

(1) $\sup|\phi|^2 = 0$, M^n is totally umbilical; or

(2) $\sup|\phi|^2 = B_{H,p,n}$. If the supremum $\sup|\phi|^2$ is attained on M^n, then either

(a) $p = 1$ and

(i) $H = 0$, M^n is an open piece of Clifford torus $S^k(\sqrt{k/n})\times S^{n-k}(\sqrt{(n-k)/n})$, with $1 \leq k \leq n-1$;

(ii) $H \neq 0$, $n \geq 3$, and M^n is an open piece of $H(r)$-torus $S^{n-1}(r)\times S^1(\sqrt{1-r^2})$ with $r^2 < (n-1)/n$;

(iii) $H \neq 0$, $n = 2$, and M^n is an open piece of $H(r)$-torus $S^1(r) \times S^1(\sqrt{1-r^2})$ with $0 < r < 1$, $r^2 \neq \frac{1}{2}$; or

(b) $n = 2$, $p = 2$ and M^n is an open piece of Veronese surface in S^4.

Theorem 1.5. *Let M^n be an n-dimensional complete submanifold in a unit sphere $S^{n+p}(1)$ with constant scalar curvature $n(n-1)R$ and $\bar{R} = R - 1 > 0$. If the normalized mean curvature vector is parallel and the squared norm S of the second fundamental form of M^n satisfies*

$$Q_{\bar{R},p,n}(\sup S) \geq 0, \tag{1.5}$$

then

(1) $\sup S = n\bar{R}$ and M^n is totally umbilical; or

(2) $Q_{\bar{R},p,n}(\sup S) = 0$. In the latter case, if the supremum $\sup S$ is attained on M^n, then either

(i) $p = 1$ and M^n is an open piece of $H(r)$-torus $S^{n-1}(r) \times S^1(\sqrt{1-r^2})$ with $0 < r < 1$; or

(ii) $n = 2$, $p = 2$ and M^n is an open piece of Veronese surface in S^4, where $Q_{\bar{R},p,n}(x)$ is defined by (1.2).

Remark 1.3. We note that Theorem 1.4 and Theorem 1.5 generalize the results of H. Alencar, M.do Carmo [1] and H. Li [10](Theorem 1.1 and Theorem 1.2) to complete submanifold with higher codimension.

2. Preliminaries

Let M^n be an n-dimensional submanifold in an $(n+p)$-dimensional unit sphere $S^{n+p}(1)$. Let $\{e_1, \ldots, e_n\}$ be a local orthonormal basis of M^n with respect to the induced metric, $\{\omega_1, \ldots, \omega_n\}$ are their dual form. Let $e_{n+1}, \ldots, e_{n+p}$ be the local unit orthonormal normal vector field. We make the following convention on the range of indices:

$$1 \leq i, j, k, \ldots \leq n; \quad n+1 \leq \alpha, \beta, \gamma, \ldots \leq n+p.$$

Then the structure equations are

$$d\omega_A = -\sum_B \omega_{AB} \wedge \omega_B, \quad \omega_{AB} + \omega_{BA} = 0, \tag{2.1}$$

$$d\omega_{AB} = -\sum_C \omega_{AC} \wedge \omega_{CB} + \frac{1}{2}\sum_{C,D} K_{ABCD}\omega_C \wedge \omega_D, \tag{2.2}$$

$$K_{ABCD} = \delta_{AC}\delta_{BD} - \delta_{AD}\delta_{BC}. \tag{2.3}$$

The Gauss equations are

$$R_{ijkl} = (\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}) + \sum_\alpha (h^\alpha_{ik}h^\alpha_{jl} - h^\alpha_{il}h^\alpha_{jk}), \tag{2.4}$$

$$n(n-1)(R-1) = n^2H^2 - S, \tag{2.5}$$

where $S = \sum\limits_{i,j,\alpha}(h^\alpha_{ij})^2$, $\vec{H} = \sum\limits_\alpha H^\alpha e_\alpha$, $H^\alpha = \frac{1}{n}\sum\limits_k h^\alpha_{kk}$, $H = |\vec{H}|$, R is the normalized scalar curvature of M^n.

The first covariant derivative $\{h^\alpha_{ijk}\}$ and the second covariant derivative $\{h^\alpha_{ijkl}\}$ of h^α_{ij} are defined by

$$\sum_k h^\alpha_{ijk}\omega_k = dh^\alpha_{ij} - \sum_k h^\alpha_{kj}\omega_{ki} - \sum_k h^\alpha_{ik}\omega_{kj} - \sum_\beta h^\beta_{ij}\omega_{\beta\alpha}, \tag{2.6}$$

$$\sum_l h^\alpha_{ijkl}\omega_l = dh^\alpha_{ijk} - \sum_l h^\alpha_{ljk}\omega_{li} - \sum_l h^\alpha_{ilk}\omega_{lj} - \sum_l h^\alpha_{ijl}\omega_{lk} - \sum_\beta h^\beta_{ijk}\omega_{\beta\alpha}. \tag{2.7}$$

Then, we have the Codazzi equations and the Ricci identities

$$h^\alpha_{ijk} = h^\alpha_{ikj}, \tag{2.8}$$

$$h^\alpha_{ijkl} - h^\alpha_{ijlk} = \sum_m h^\alpha_{mj}R_{mikl} + \sum_m h^\alpha_{im}R_{mjkl} + \sum_\beta h^\beta_{ij}R_{\beta\alpha kl}. \tag{2.9}$$

The Ricci equations are

$$R_{\alpha\beta ij} = \sum_k (h^\alpha_{ik}h^\beta_{kj} - h^\beta_{ik}h^\alpha_{kj}). \tag{2.10}$$

From (2.8) and (2.9), we have

$$\Delta h^\alpha_{ij} = \sum_k h^\alpha_{kkij} + \sum_{k,m} h^\alpha_{km}R_{mijk} + \sum_{k,m} h^\alpha_{mi}R_{mkjk} + \sum_{k,\beta} h^\beta_{ki}R_{\beta\alpha jk}. \tag{2.11}$$

Denote by $|\phi|^2 = S - nH^2$ the non-negative function $|\phi|$ on M^n. We know that $|\phi|^2 = 0$ exactly at the umbilical points of M^n. Define the first, second covariant derivatives and Laplacian of the mean curvature vector field $\vec{H} = \sum\limits_\alpha H^\alpha e_\alpha$ in the normal bundle $N(M^n)$ as follows

$$\sum_i H^\alpha_{,i}\theta_i = dH^\alpha + \sum_\beta H^\beta\theta_{\beta\alpha}, \tag{2.12}$$

$$\sum_j H^\alpha_{,ij}\theta_j = dH^\alpha_{,i} + \sum_j H^\alpha_{,j}\theta_{ji} + \sum_\beta H^\beta_{,i}\theta_{\beta\alpha}, \tag{2.13}$$

$$\Delta^\perp H^\alpha = \sum_i H^\alpha_{,ii}, \qquad H^\alpha = \frac{1}{n}\sum_k h^\alpha_{kk}. \tag{2.14}$$

Let f be a smooth function on M^n. The first, second covariant derivatives $f_i, f_{i,j}$ and Laplacian of f are defined by

$$df = \sum_i f_i\theta_i, \quad \sum_j f_{i,j}\theta_j = df_i + \sum_j f_j\theta_{ji}, \quad \Delta f = \sum_i f_{i,i}. \tag{2.15}$$

For the fix index $\alpha(n+1 \le \alpha \le n+p)$, we introduce an operator $\Box^\alpha$ due to Cheng-Yau [4] by

$$\Box^\alpha f = \sum_{i,j}(nH^\alpha \delta_{ij} - h^\alpha_{ij}) f_{i,j}. \tag{2.16}$$

Since M^n is compact, the operator $\Box^\alpha$ is self-adjoint (see[4]) if and only if

$$\int_M (\Box^\alpha f) g dv = \int_M f(\Box^\alpha g) dv, \tag{2.17}$$

where f and g are any smooth functions on M^n.

In general, for a matrix $A = (a_{ij})$ we denote by $N(A)$ the square of the norm of A, that is,

$$N(A) = \mathrm{tr}(A \cdot A^t) = \sum_{i,j}(a_{ij})^2.$$

Clearly, $N(A) = N(T^t A T)$ for any orthogonal matrix T.

We need the following Lemmas due to Chern-Do Carmo-Kobayashi [7], Cheng [5] and the author [15].

Lemma 2.1([7]). *Let A and B be symmetric $(n \times n)$-matrices. Then*

$$N(AB - BA) \le 2N(A)N(B), \tag{2.18}$$

and the equality holds for nonzero matrices A and B if and only if A and B can be transformed simultaneously by on orthogonal matrix into multiples of $\tilde{A}$ and $\tilde{B}$ respectively, where

$$\tilde{A} = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix} \qquad \tilde{B} = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & -1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

Moreover, if A_1, A_2 and A_3 are $(n \times n)$-symmetric matrices and if

$$N(A_\alpha A_\beta - A_\beta A_\alpha) = 2N(A_\alpha)N(A_\beta), 1 \le \alpha, \beta \le 3$$

then at least one of the matrices A_α must be zero.

Lemma 2.2([5]). *Let b_i for $i = 1, \cdots, n$ be real numbers satisfying $\sum_{i=1}^n b_i = 0$ and $\sum_{i=1}^n b_i^2 = B$. Then*

$$\sum_{i=1}^n b_i^4 - \frac{B^2}{n} \le \frac{(n-2)^2}{n(n-1)} B^2. \tag{2.19}$$

Lemma 2.3 ([5], [15]). *Let a_i and b_i for $i = 1, \cdots, n$ be real numbers satisfying $\sum_{i=1}^n a_i = 0$ and $\sum_{i=1}^n a_i^2 = a$. Then*

$$\left|\sum_{i=1}^n a_i b_i^2\right| \le \sqrt{\sum_{i=1}^n b_i^4 - \frac{(\sum_{i=1}^n b_i^2)^2}{n}}\sqrt{a}. \tag{2.20}$$

If $a_i = b_i$ for $i = 1, \cdots, n$, then Lemma 2.3 becomes to the well-known Lemma of M. Okumura [12].

Lemma 2.4 ([12]). *Let $\{a_i\}_{i=1}^n$ be a set of real numbers satisfying $\sum_i a_i = 0, \sum_i a_i^2 = a$, where $a \geq 0$. Then we have*

$$-\frac{n-2}{\sqrt{n(n-1)}}a^{3/2} \leq \sum_i a_i^3 \leq \frac{n-2}{\sqrt{n(n-1)}}a^{3/2}, \tag{2.21}$$

and the equalities hold if and only if at least $(n-1)$ of the a_i are equal.

3. Proof of Theorem 1.3

Define tensors

$$\tilde{h}_{ij}^\alpha = h_{ij}^\alpha - H^\alpha \delta_{ij}, \tag{3.1}$$

$$\tilde{\sigma}_{\alpha\beta} = \sum_{i,j} \tilde{h}_{ij}^\alpha \tilde{h}_{ij}^\beta, \quad \sigma_{\alpha\beta} = \sum_{i,j} h_{ij}^\alpha h_{ij}^\beta. \tag{3.2}$$

Then the $(p \times p)$-matrix $(\tilde{\sigma}_{\alpha\beta})$ is symmetric and can be assumed to be diagonized for a suitable choice of $e_{n+1}, \ldots, e_{n+p}$. We set

$$\tilde{\sigma}_{\alpha\beta} = \tilde{\sigma}_\alpha \delta_{\alpha\beta}. \tag{3.3}$$

By a direct calculation, we have

$$\sum_k \tilde{h}_{kk}^\alpha = 0, \quad \tilde{\sigma}_{\alpha\beta} = \sigma_{\alpha\beta} - nH^\alpha H^\beta, \quad |\phi|^2 = \sum_\alpha \tilde{\sigma}_\alpha = S - nH^2, \tag{3.4}$$

$$\sum_{i,j,k,\alpha} h_{kj}^\beta h_{ij}^\alpha h_{ik}^\alpha = \sum_{i,j,k,\alpha} \tilde{h}_{kj}^\beta \tilde{h}_{ij}^\alpha \tilde{h}_{ik}^\alpha + 2\sum_{i,j,\alpha} H^\alpha \tilde{h}_{ij}^\alpha \tilde{h}_{ij}^\beta + H^\beta |\phi|^2 + nH^2 H^\beta. \tag{3.5}$$

Setting $f = nH^\alpha$ in (2.16), we have

$$\begin{aligned}\Box^\alpha(nH^\alpha) &= \sum_{i,j}(nH^\alpha \delta_{ij} - h_{ij}^\alpha)(nH^\alpha)_{i,j} \\ &= \sum_i (nH^\alpha)(nH^\alpha)_{i,i} - \sum_{i,j} h_{ij}^\alpha (nH^\alpha)_{i,j}.\end{aligned} \tag{3.6}$$

We also have

$$\begin{aligned}\frac{1}{2}\Delta(nH)^2 &= \frac{1}{2}\Delta\sum_\alpha (nH^\alpha)^2 = \frac{1}{2}\sum_\alpha \Delta(nH^\alpha)^2 \\ &= \frac{1}{2}\sum_{\alpha,i}[(nH^\alpha)^2]_{i,i} = \sum_{\alpha,i}[(nH^\alpha)_{,i}]^2 + \sum_{\alpha,i}(nH^\alpha)(nH^\alpha)_{i,i} \\ &= n^2|\nabla^\perp \vec{H}|^2 + \sum_{\alpha,i}(nH^\alpha)(nH^\alpha)_{i,i}.\end{aligned} \tag{3.7}$$

Therefore, from (2.5), (3.6), (3.7), we get

$$\begin{aligned}\sum_\alpha \Box^\alpha(nH^\alpha) &= \frac{1}{2}\Delta(nH)^2 - n^2|\nabla^\perp \vec{H}|^2 - \sum_{i,j,\alpha} h_{ij}^\alpha (nH^\alpha)_{i,j} \\ &= \frac{1}{2}n(n-1)\Delta R + \frac{1}{2}\Delta S - n^2|\nabla^\perp \vec{H}|^2 - \sum_{i,j,\alpha} h_{ij}^\alpha (nH^\alpha)_{i,j}.\end{aligned} \tag{3.8}$$

From (2.11), we have

$$\begin{aligned}\frac{1}{2}\Delta S=&\sum_{i,j,k,\alpha}(h^{\alpha}_{ijk})^2+\sum_{i,j,\alpha}h^{\alpha}_{ij}\Delta h^{\alpha}_{ij}\\=&|\nabla h|^2+\sum_{i,j,\alpha}h^{\alpha}_{ij}(nH^{\alpha})_{i,j}+\sum_{\alpha}\sum_{i,j,k,l}h^{\alpha}_{ij}(h^{\alpha}_{kl}R_{lijk}+h^{\alpha}_{li}R_{lkjk})\\&+\sum_{\alpha,\beta}\sum_{i,j,k}h^{\alpha}_{ij}h^{\beta}_{ki}R_{\beta\alpha jk}.\end{aligned}\tag{3.9}$$

Putting (3.9) into (3.8), we have

$$\begin{aligned}\sum_{\alpha}\Box^{\alpha}(nH^{\alpha})=&|\nabla h|^2-n^2|\nabla^{\perp}\vec{H}|^2+\frac{1}{2}n(n-1)\Delta R\\&+\sum_{\alpha}\sum_{i,j,k,l}h^{\alpha}_{ij}(h^{\alpha}_{kl}R_{lijk}+h^{\alpha}_{li}R_{lkjk})+\sum_{\alpha,\beta}\sum_{i,j,k}h^{\alpha}_{ij}h^{\beta}_{ki}R_{\beta\alpha jk}.\end{aligned}\tag{3.10}$$

Thus, if M^n is compact, from (2.17) and Stokes formula, we have

$$\begin{aligned}0=&\int_{M^n}\{|\nabla h|^2-n^2|\nabla^{\perp}\vec{H}|^2\}dv\\&+\int_{M^n}\{\sum_{\alpha}\sum_{i,j,k,l}h^{\alpha}_{ij}(h^{\alpha}_{kl}R_{lijk}+h^{\alpha}_{li}R_{lkjk})\}dv+\int_{M^n}\sum_{\alpha,\beta}\sum_{i,j,k}h^{\alpha}_{ij}h^{\beta}_{ki}R_{\beta\alpha jk}dv.\end{aligned}\tag{3.11}$$

From (2.10), we have

$$\sum_{\alpha,\beta,k}(R_{\beta\alpha jk})^2=\sum_{\alpha,\beta,i,j,k}(h^{\beta}_{ji}h^{\alpha}_{ik}-h^{\beta}_{ki}h^{\alpha}_{ij})R_{\beta\alpha jk}=-2\sum_{\alpha,\beta}\sum_{i,j,k}h^{\alpha}_{ij}h^{\beta}_{ki}R_{\beta\alpha jk}.$$

Thus, we have

$$\begin{aligned}\sum_{\alpha,\beta}\sum_{i,j,k}h^{\alpha}_{ij}h^{\beta}_{ki}R_{\beta\alpha jk}=&-\frac{1}{2}\sum_{\alpha,\beta,k}(R_{\beta\alpha jk})^2\\=&-\frac{1}{2}\sum_{\alpha,\beta,j,k}(\sum_{l}h^{\beta}_{jl}h^{\alpha}_{lk}-\sum_{l}h^{\alpha}_{jl}h^{\beta}_{lk})^2\\=&-\frac{1}{2}\sum_{\alpha,\beta,j,k}(\sum_{l}\tilde{h}^{\beta}_{jl}\tilde{h}^{\alpha}_{lk}-\sum_{l}\tilde{h}^{\alpha}_{jl}\tilde{h}^{\beta}_{lk})^2\\=&-\frac{1}{2}\sum_{\alpha,\beta}N(\tilde{A}_{\alpha}\tilde{A}_{\beta}-\tilde{A}_{\beta}\tilde{A}_{\alpha}),\end{aligned}\tag{3.12}$$

where $\tilde{A}_{\alpha}:=(\tilde{h}^{\alpha}_{ij})=(h^{\alpha}_{ij}-H^{\alpha}\delta_{ij})$.

From (2.4), (2.10), (3.2), (3.4), (3.5) and (3.12), we have

$$\begin{aligned}&\sum_{\alpha}\sum_{i,j,k,l}h_{ij}^{\alpha}(h_{kl}^{\alpha}R_{lijk}+h_{li}^{\alpha}R_{lkjk}) \qquad (3.13)\\
=&n|\phi|^2-\sum_{\alpha,\beta}\sum_{i,j,k,l}h_{ij}^{\alpha}h_{ij}^{\beta}h_{lk}^{\alpha}h_{lk}^{\beta}+n\sum_{\alpha,\beta}\sum_{i,j,k}H^{\beta}h_{kj}^{\beta}h_{ij}^{\alpha}h_{ik}^{\alpha}+\sum_{\alpha,\beta,i,j,k}h_{ji}^{\alpha}h_{ik}^{\beta}R_{\beta\alpha jk}\\
=&n|\phi|^2-\sum_{\alpha,\beta}\sigma_{\alpha\beta}^2+n\sum_{\alpha,\beta}\sum_{i,j,k}H^{\beta}\tilde{h}_{kj}^{\beta}\tilde{h}_{ij}^{\alpha}\tilde{h}_{ik}^{\alpha}+2n\sum_{\alpha,\beta}\sum_{i,j}H^{\alpha}H^{\beta}\tilde{h}_{ij}^{\alpha}\tilde{h}_{ij}^{\beta}\\
&+n\sum_{\beta}(H^{\beta})^2|\phi|^2+n^2H^2\sum_{\beta}(H^{\beta})^2-\frac{1}{2}\sum_{\alpha,\beta}N(\tilde{A}_{\alpha}\tilde{A}_{\beta}-\tilde{A}_{\beta}\tilde{A}_{\alpha})\\
=&n|\phi|^2-\sum_{\alpha,\beta}\tilde{\sigma}_{\alpha\beta}^2+nH^2|\phi|^2+n\sum_{\alpha,\beta}\sum_{i,j,k}H^{\beta}\tilde{h}_{kj}^{\beta}\tilde{h}_{ij}^{\alpha}\tilde{h}_{ik}^{\alpha}\\
&-\frac{1}{2}\sum_{\alpha,\beta}N(\tilde{A}_{\alpha}\tilde{A}_{\beta}-\tilde{A}_{\beta}\tilde{A}_{\alpha}).\end{aligned}$$

Let $\sum_i(\tilde{h}_{ii}^{\beta})^2=\tau_{\beta}$. Then $\tau_{\beta}\le\sum_{i,j}(\tilde{h}_{ij}^{\beta})^2=\tilde{\sigma}_{\beta}$. Since $\sum_i\tilde{h}_{ii}^{\beta}=0$, $\sum_i\mu_i^{\alpha}=0$ and $\sum_i(\mu_i^{\alpha})^2=\tilde{\sigma}_{\alpha}$. We have from Lemma 2.2 and Lemma 2.3 that

$$\begin{aligned}\sum_{\alpha,\beta}\sum_{i,j,k}H^{\alpha}\tilde{h}_{ij}^{\alpha}\tilde{h}_{kj}^{\beta}\tilde{h}_{ik}^{\beta}=&\sum_{\beta,\alpha}\sum_{i,j,k}H^{\beta}\tilde{h}_{ij}^{\beta}\tilde{h}_{kj}^{\alpha}\tilde{h}_{ik}^{\alpha} \qquad (3.14)\\
=&\sum_{\alpha,\beta}H^{\beta}\sum_i\tilde{h}_{ii}^{\beta}(\mu_i^{\alpha})^2\ge-\frac{n-2}{\sqrt{n(n-1)}}\sum_{\alpha,\beta}|H^{\beta}|\tilde{\sigma}_{\alpha}\sqrt{\tau_{\beta}}\\
\ge&-\frac{n-2}{\sqrt{n(n-1)}}\sum_{\alpha}\tilde{\sigma}_{\alpha}\sum_{\beta}|H^{\beta}|\sqrt{\tilde{\sigma}_{\beta}}\\
\ge&-\frac{n-2}{\sqrt{n(n-1)}}|\phi|^2\sqrt{\sum_{\beta}(H^{\beta})^2\sum_{\beta}\tilde{\sigma}_{\beta}}\\
=&-\frac{n-2}{\sqrt{n(n-1)}}|H||\phi|^3.\end{aligned}$$

From Lemma 2.1, (3.3), (3.4), we have

$$\begin{aligned}-\sum_{\alpha,\beta}\tilde{\sigma}_{\alpha\beta}^2-\sum_{\alpha,\beta}N(\tilde{A}_{\alpha}\tilde{A}_{\beta}-\tilde{A}_{\beta}\tilde{A}_{\alpha})=&-\sum_{\alpha}\tilde{\sigma}_{\alpha}^2-\sum_{\alpha,\beta}N(\tilde{A}_{\alpha}\tilde{A}_{\beta}-\tilde{A}_{\beta}\tilde{A}_{\alpha}) \qquad (3.15)\\
\ge&-\sum_{\alpha}\tilde{\sigma}_{\alpha}^2-2\sum_{\alpha\neq\beta}\tilde{\sigma}_{\alpha}\tilde{\sigma}_{\beta}\\
=&-2(\sum_{\alpha}\tilde{\sigma}_{\alpha})^2+\sum_{\alpha}\tilde{\sigma}_{\alpha}^2\\
\ge&-2|\phi|^4+\frac{1}{p}(\sum_{\alpha}\tilde{\sigma}_{\alpha})^2\\
=&-(2-\frac{1}{p})|\phi|^4.\end{aligned}$$

Therefore, from (3.11), (3.12)-(3.15), we have the following:

Proposition 3.1. *Let M^n be an n-dimensional compact submanifolds in a unit sphere $S^{n+p}(1)$. Then there holds the following*

$$0 \geq \int_{M^n} \{|\nabla h|^2 - n^2|\nabla^\perp \vec{H}|^2\}dv + \int_{M^n} |\phi|^2\{n + nH^2 - \frac{n-2}{\sqrt{n(n-1)}}n|H||\phi| - (2 - \frac{1}{p})|\phi|^2\}dv. \tag{3.16}$$

Proof of Theorem 1.3. Since $R \geq 1$ and the normalized mean curvature vector is parallel, we easily know that

$$|\nabla h|^2 \geq n^2|\nabla^\perp \vec{H}|^2.$$

In fact, from (2.5) and $R \geq 1$, we have $S \leq n^2H^2$. Taking covariant derivative on (2.5), we get

$$n^2HH_{,k} = \sum_{i,j,\alpha} h^\alpha_{ij}h^\alpha_{ijk}.$$

From Cauchy-Schwarz's inequality, we get

$$n^4H^2|\nabla^\perp \vec{H}|^2 = n^4H^2\sum_k (H_{,k})^2 = \sum_k(\sum_{i,j,\alpha} h^\alpha_{ij}h^\alpha_{ijk})^2 \leq S \sum_{i,j,k,\alpha} (h^\alpha_{ijk})^2, \tag{3.17}$$

and we conclude. Denote $\bar{R} = R - 1$, by (2.5) we have $S - nH^2 = \frac{n-1}{n}(S - n\bar{R})$. Thus, we obtain

$$\begin{aligned} n+nH^2 - &\frac{n-2}{\sqrt{n(n-1)}}n|H||\phi| - (2 - \frac{1}{p})|\phi|^2 \\ =&n + n\bar{R} + [\frac{1}{n} - (2 - \frac{1}{p})\frac{n-1}{n}](S - n\bar{R}) \\ &- \frac{n-2}{n}\sqrt{[S + n(n-1)\bar{R}](S - n\bar{R})}. \end{aligned}$$

From the assumption of Theorem 1.3 and the Proposition 3.1, we have

$$\begin{aligned} 0 \geq &\int_{M^n} |\phi|^2\{n + nH^2 - \frac{n-2}{\sqrt{n(n-1)}}n|H||\phi| - (2 - \frac{1}{p})|\phi|^2\}dv \\ = &\int_{M^n} \frac{n-1}{n}(S - n\bar{R})\{n + n\bar{R} + [\frac{1}{n} - (2 - \frac{1}{p})\frac{n-1}{n}](S - n\bar{R}) \\ &- \frac{n-2}{n}\sqrt{[S + n(n-1)\bar{R}](S - n\bar{R})}\}dv \geq 0. \end{aligned} \tag{3.18}$$

Therefore, we have

(1) $S = n\bar{R}$, that is, M^n is totally umbilical;

(2) or

$$\begin{aligned} n+n\bar{R} + &[\frac{1}{n} - (2 - \frac{1}{p})\frac{n-1}{n}](S - n\bar{R}) \\ &- \frac{n-2}{n}\sqrt{[S + n(n-1)\bar{R}](S - n\bar{R})} = 0. \end{aligned} \tag{3.19}$$

In this case, the equalities in (3.18) (3.16), (3.15), (3.14) and Lemma 2.1 hold. Thus, we have

$$n + nH^2 - \frac{n-2}{\sqrt{n(n-1)}}n|H||\phi| - (2 - \frac{1}{p})|\phi|^2 = 0. \tag{3.20}$$

We see that M^n is not totally umbilical and the equalities in (3.16), (3.15), (3.14) and Lemma 2.1 hold. Thus, we have $\nabla h = 0$,

$$p\sum_\alpha \tilde{\sigma}_\alpha^2 = (\sum_\alpha \tilde{\sigma}_\alpha)^2,$$

that is

$$\tilde{\sigma}_{n+1} = \cdots = \tilde{\sigma}_{n+p}, \tag{3.21}$$

$$N(\tilde{A}_\alpha\tilde{A}_\beta - \tilde{A}_\beta\tilde{A}_\alpha) = 2N(\tilde{A}_\alpha)N(\tilde{A}_\beta), \quad \alpha \neq \beta, \tag{3.22}$$

and

$$\sum_\beta |H^\beta|\sqrt{\tilde{\sigma}_\beta} = |H||\phi|. \tag{3.23}$$

We may consider the case $p = 1$ and $p \geq 2$ separately.

Case (i). If $p = 1$, from (3.19), we have $n \neq 2$ and

$$S = \frac{n}{(n-2)(n\bar{R}+2)}\{n(n-1)\bar{R}^2 + 4(n-1)\bar{R} + n\}.$$

Thus, by the Theorem 1.1 of H. Li [10], we know that M^n is a product $S^1(\sqrt{1-r^2}) \times S^{n-1}(r)$, $r = \sqrt{\frac{n-2}{nR}}$.

Case (ii). If $p \geq 2$, from (3.21) and (3.23), we have

$$\sqrt{\tilde{\sigma}_{n+1}}\sum_\beta |H^\beta| = \sqrt{\sum_\beta (H^\beta)^2}\sqrt{\sum_\beta \tilde{\sigma}_\beta} = \sqrt{p\tilde{\sigma}_{n+1}}\sqrt{\sum_\beta (H^\beta)^2}.$$

Since M^n is not totally umbilical, we have $\tilde{\sigma}_{n+1} \neq 0$. Thus, we have

$$(\sum_\beta |H^\beta|)^2 = p\sum_\beta (H^\beta)^2,$$

that is,

$$|H^{n+1}| = \cdots = |H^{n+p}|. \tag{3.24}$$

From Lemma 2.1, we know that at most two of $\tilde{A}_\alpha = (\tilde{h}_{ij}^\alpha), \alpha = n+1, \ldots, n+p$, are different from zero. If all of $\tilde{A}_\alpha = (\tilde{h}_{ij}^\alpha)$ are zero, which is contradiction with M^n is not totally umbilical. If only one of them, say $\tilde{A}_\alpha$, is different from zero, which is contradiction with (3.21). Therefore, we may assume that

$$\tilde{A}_{n+1} = \lambda\tilde{A}, \quad \tilde{A}_{n+2} = \mu\tilde{B}, \quad \lambda, \mu \neq 0,$$
$$\tilde{A}_\alpha = 0, \qquad \alpha \geq n+3,$$

where $\tilde{A}$ and $\tilde{B}$ are defined in Lemma 2.1.

From (3.23), we have

$$(\sqrt{2}\lambda|H^{n+1}| + \sqrt{2}\mu|H^{n+2}|)^2 = H^2|\phi|^2 = \sum_\alpha (H^\alpha)^2(2\lambda^2 + 2\mu^2).$$

Thus, from (3.24), we have

$$(H^{n+1})^2(\lambda+\mu)^2 = p(H^{n+1})^2(\lambda^2+\mu^2),$$

that is,

$$(H^{n+1})^2[(p-1)\lambda^2 - 2\lambda\mu + (p-1)\mu^2] = 0.$$

Since $\lambda, \mu \neq 0$, we infer that $H^{n+1} = 0$. Thus, from (3.24), we have $H^\alpha = 0, n+1 \leq \alpha \leq n+p$, that is, $\vec{H} = 0$, M^n is a minimal submanifold in $S^{n+p}(1)$ and from (3.20), we have $S = \frac{n}{2-1/p}$ on M^n. From the Theorem of Chern-Do Carmo-Kobayashi [7], we know that $n=2$, $p=2$ and M^n is Veronese surface in S^4. This completes the proof of Theorem 1.3. □

4. Proof of Theorem 1.4 and 1.5

The important maximum principle of Omori [13], Yau [16] and Cheng [6] are useful to us.

Proposition 4.1 ([13], [16]). *Let M^n be a complete Riemannian manifold whose Ricci curvature is bounded from below. If f is a C^2-function bounded from above on M^n, then for any $\varepsilon>0$, there is a point $x\in M^n$ such that*

$$\sup f-\varepsilon<f(x),\quad |\nabla f|(x)<\varepsilon,\quad \Delta f(x)<\varepsilon. \tag{4.1}$$

Proposition 4.2 ([6]). *Let M^n be a complete Riemannian manifold whose Ricci curvature is bounded from below. Let f be a C^2-function which bounded from above. Then there exists a sequence $\{x_k\}$ in M^n such that*

$$\lim_{m\to\infty} f(x_k)=\sup f,\quad \lim_{m\to\infty}|\nabla f(x_k)|=0,\quad \lim_{m\to\infty}\sup Lf(x_k)\le 0, \tag{4.2}$$

where $Lf=\sum_j b_j f_{j,j}$ is a differential operator, and $b_j\ge 0$ is bounded.

We need the following Lemma.

Lemma 4.3 ([2], [9]). *Let $A=(a_{ij}), i,j=1,\cdots,n$ be a symmetric $(n\times n)$ matrix, $n\ge 2$. Assume that $A_1=\mathrm{tr}A, A_2=\sum_{i,j}(a_{ij})^2$. Then*

$$\begin{aligned}\sum_i(a_{in})^2-A_1a_{nn}\le&\frac{1}{n^2}\{n(n-1)A_2\\ &+(n-2)\sqrt{n-1}|A_1|\sqrt{nA_2-(A_1)^2}-2(n-1)(A_1)^2\},\end{aligned} \tag{4.3}$$

the equality holds if and only if $n=2$ or $n>2, (a_{ij})$ is of the following form

$$\begin{pmatrix} a & & & 0\\ & \ddots & & \\ & & a & \\ 0 & & & A_1-(n-1)a\end{pmatrix},$$

where $(na-A_1)A_1\ge 0$.

Proof of Theorem 1.4. We assume that $\sup|\phi|^2\le B_{H,p,n}$, then $0\le|\phi|\le\sqrt{B_{H,p,n}}$, we have $P_{H,p,n}(|\phi|)\le 0$, that is

$$n+nH^2-\frac{n-2}{\sqrt{n(n-1)}}n|H||\phi|-(2-\frac{1}{p})|\phi|^2\ge 0.$$

Since the mean curvature vector is parallel, we know that the mean curvature is constant. From (3.9), (3.12)-(3.15), we have

$$\frac12\Delta|\phi|^2\ge|\nabla\phi|^2+|\phi|^2\{n+nH^2-\frac{n-2}{\sqrt{n(n-1)}}n|H||\phi|-(2-\frac1p)|\phi|^2\}\ge 0. \tag{4.4}$$

For any point and any unit vector $v \in T_pM^n$, we choose a local orthonormal frame field $e_1, \dots, e_n$ such that $e_n = v$, we have from Gauss equation (2.4) that the Ricci curvature $\mathrm{Ric}(v, v)$ of M^n with respect to v is expressed as

$$\mathrm{Ric}(v,v) = (n-1) + \sum_\alpha [(\mathrm{tr}H_\alpha)h^\alpha_{nn} - \sum_i (h^\alpha_{in})^2], \tag{4.5}$$

where H_α is the $(n \times n)$-matrix (h^α_{ij}). Assume that $T_\alpha = \mathrm{tr}H_\alpha, S_\alpha = \sum_{i,j}(h^\alpha_{ij})^2$, then we have $n^2H^2 = \sum_\alpha T^2_\alpha, S = \sum_\alpha S_\alpha$. By Lemma 4.3, we have

$$\begin{aligned}\mathrm{Ric}(v,v) \geq &(n-1) - \sum_\alpha \frac{1}{n^2}\{n(n-1)S_\alpha \\ &+ (n-2)\sqrt{n-1}|T_\alpha|\sqrt{nS_\alpha - T^2_\alpha} - 2(n-1)T^2_\alpha\} \\ =&(n-1) - \frac{n-1}{n}S - \frac{n-2}{n}\sqrt{\frac{n-1}{n}}\sum_\alpha |T_\alpha|\sqrt{S_\alpha - \frac{T^2_\alpha}{n}} + \frac{2(n-1)}{n^2}\sum_\alpha T^2_\alpha \\ \geq&\frac{n-1}{n}\{n + 2nH^2 - S - \frac{n-2}{\sqrt{n(n-1)}}\sqrt{(\sum_\alpha T^2_\alpha)[\sum_\alpha(S_\alpha - \frac{T^2_\alpha}{n})]}\} \\ =&\frac{n-1}{n}\{n + nH^2 - \frac{n-2}{\sqrt{n(n-1)}}n|H||\phi| - |\phi|^2\} \\ \geq&\frac{n-1}{n}\{n + nH^2 - \frac{n-2}{\sqrt{n(n-1)}}n|H||\phi| - (2-\frac{1}{p})|\phi|^2\} \geq 0.\end{aligned} \tag{4.6}$$

Therefore, we know that the Ricci curvature $\mathrm{Ric}(v, v)$ is bounded from below.

Now we consider the following smooth function on M^n defined by $f = -(|\phi|^2 + a)^{-1/2}$, where $a(> 0)$ is a real number. Obviously, f is bounded, so we can apply Proposition 4.1 to f. For any $\varepsilon > 0$, there is a point $x \in M^n$, such that at which f satisfies the (4.1). By a simple and direct calculation, we have

$$f\Delta f = 3|df|^2 - \frac{1}{2}f^4\Delta|\phi|^2. \tag{4.7}$$

From (4.1) and (4.7), we have

$$\frac{1}{2}\Delta|\phi|^2(x) = f^{-4}(x)[3|df|^2(x) - f(x)\Delta f(x)] < f^{-4}(x)[3\varepsilon^2 - \varepsilon f(x)]. \tag{4.8}$$

Thus, for any convergent sequence $\{\varepsilon_m\}$ with $\varepsilon_m > 0$ and $\lim_{m\to\infty}\varepsilon_m = 0$, there exists a point sequence $\{x_m\}$ such that the sequence $\{f(x_m)\}$ converges to f_0 (we can take a subsequence if necessary) and satisfies (4.1), hence, $\lim_{m\to\infty}\varepsilon_m[3\varepsilon_m - f(x_m)] = 0$. From the definition of supremum and (4.1), we have $\lim_{m\to\infty} f(x_m) = f_0 = \sup f$ and hence the definition of f gives rise to $\lim_{m\to\infty}|\phi|^2(x_m) = \sup|\phi|^2$.

From (4.4) and (4.8), we have

$$\begin{aligned}f^{-4}(x_m)[3\varepsilon^2_m - \varepsilon_m f(x_m)] &> \frac{1}{2}\Delta|\phi|^2(x_m) \\ &\geq |\phi|^2(x_m)\{n + nH^2 - \frac{n-2}{\sqrt{n(n-1)}}n|H||\phi|(x_m) - (2-\frac{1}{p})|\phi|^2(x_m)\} \geq 0.\end{aligned} \tag{4.9}$$

Putting $m \to \infty$ in (4.9), we have

$$\sup|\phi|^2\{n + nH^2 - \frac{n-2}{\sqrt{n(n-1)}}n|H|\sup|\phi| - (2-\frac{1}{p})\sup|\phi|^2\} = 0.$$

Thus, we have

(1) $\sup|\phi|^2 = 0$ and M^n is totally umbilical; or

(2)

$$n + nH^2 - \frac{n-2}{\sqrt{n(n-1)}}n|H|\sup|\phi| - (2-\frac{1}{p})\sup|\phi|^2 = 0. \tag{4.10}$$

From (4.4), we know that $|\phi|^2$ is a subharmonic function on M^n. Since the supremum $\sup|\phi|^2$ is attained at some point of M^n, by the maximum principle, we have $|\phi|^2 = const. = B_{H,p,n}$. Thus, (4.10) becomes

$$n + nH^2 - \frac{n-2}{\sqrt{n(n-1)}}n|H||\phi| - (2-\frac{1}{p})|\phi|^2 = 0. \tag{4.11}$$

and (4.4) becomes equality. We may consider the case $p = 1$ and $p \geq 2$ separately.

Case (i). If $p = 1$, from equality in (4.4), we obtain that $\nabla\phi = \nabla h = 0$, that is, the second fundamental form is parallel. If $H = 0$, then by a classical local rigidity result of Lawson (see Proposition 1 in Lawson [8]), we know that M^n is an open piece of a minimal Clifford torus of the form $S^k(\sqrt{k/n}) \times S^{n-k}(\sqrt{(n-k)/n})$ with $1 \leq k \leq n-1$. If $H \neq 0$, then from the equality in (4.4), we also obtain the equality in Lemma 2.4 of Okumura, which implies that M^n has exactly two constant principal curvatures, with multiplicities $n-1$ and 1. Then, by the classical result on isoparametric hypersurfaces of E. Cartan [3] we conclude that if $n \geq 3$, M^n must be an open piece of $H(r)$-torus $S^1(\sqrt{1-r^2}) \times S^{n-1}(r)$, with $0 < r < 1$, $r^2 < (n-1)/n$; if $n = 2$, M^n is an open piece of $H(r)$-torus $S^1(r) \times S^1(\sqrt{1-r^2})$ with $0 < r < 1$, $r^2 \neq \frac{1}{2}$.

Case (ii). If $p \geq 2$, since (4.11) holds, M^n is not totally umbilical and the equalities in (3.16), (3.15), (3.14) and Lemma 2.1 hold. Thus, we have $\nabla h = 0$, and (3.21), (3.22), (3.23) hold. By the same assertion in the proof of Theorem 1.3, we know that M^n is a minimal submanifold in $S^{n+p}(1)$ and $S = \frac{n}{2-1/p}$ on M^n. From the Theorem of Chern-Do Carmo-Kobayashi [7], we know that $n = 2$, $p = 2$ and (4.11) reduces to $S = \frac{4}{3}$, M^n is an open piece of Veronese surface in S^4. This completes the proof of Theorem 1.4. $\Box$

Proof of Theorem 1.5. Let

$$\Box f = \sum_\alpha \Box^\alpha f = \sum_{i,j}(\sum_\alpha(nH^\alpha\delta_{ij} - h_{ij}^\alpha))f_{i,j}.$$

We may prove that the operator $\Box$ is elliptic. In fact, for a fixed $\alpha, n+1 \leq \alpha \leq n+p$, we can take a local orthonormal frame field $\{e_1, \ldots, e_n\}$ such that $h_{ij}^\alpha = \lambda_i^\alpha\delta_{ij}$, then

$$\Box f = \sum_i(\sum_\alpha(nH^\alpha - \lambda_i^\alpha))f_{i,i}. \tag{4.12}$$

Since $R > 1$, from (2.5), we have $S < n^2H^2$. For a fixed $\alpha, n+1 \leq \alpha \leq n+p$, if there is a λ_i^α such that $nH^\alpha - \lambda_i^\alpha \leq 0$, then $n^2H^2 = \sum_\alpha(nH^\alpha)^2 \leq \sum_\alpha(\lambda_i^\alpha)^2 \leq S$, this is a contradiction. Thus, we have $nH^\alpha - \lambda_i^\alpha > 0$, then $\sum\limits_\alpha (nH^\alpha - \lambda_i^\alpha) > 0$ and the operator $\Box$ is elliptic.

From (1.5), we have

$$n + n\bar{R} + [\frac{1}{n} - (2 - \frac{1}{p})\frac{n-1}{n}](\sup S - n\bar{R}) \qquad (4.13)$$
$$- \frac{n-2}{n}\sqrt{[\sup S + n(n-1)\bar{R}](\sup S - n\bar{R})} \geq 0.$$

Since $\frac{1}{n} - (2 - \frac{1}{p})\frac{n-1}{n} \leq -\frac{n-2}{n} \leq 0$, (4.13) implies that $\sup S < +\infty$ and

$$n + n\bar{R} + [\frac{1}{n} - (2 - \frac{1}{p})\frac{n-1}{n}](S - n\bar{R}) \qquad (4.14)$$
$$- \frac{n-2}{n}\sqrt{[S + n(n-1)\bar{R}](S - n\bar{R})} \geq 0.$$

Thus, from (2.5) and (4.6), we get the Ricci curvature $\mathrm{Ric}(v, v)$ is bounded from below.

From (3.10), (3.12)-(3.15), (4.14) and $R > 1$, we have

$$\Box(nH^\alpha) = \sum_\alpha \Box^\alpha(nH^\alpha) = |\nabla h|^2 - n^2|\nabla^\perp \vec{H}|^2 \qquad (4.15)$$
$$+ |\phi|^2\{n + nH^2 - \frac{n-2}{\sqrt{n(n-1)}}n|H||\phi| - (2 - \frac{1}{p})|\phi|^2$$
$$\geq \frac{n-1}{n}(S - n\bar{R})\{n + n\bar{R} + [\frac{1}{n} - (2 - \frac{1}{p})\frac{n-1}{n}](S - n\bar{R})$$
$$- \frac{n-2}{n}\sqrt{[S + n(n-1)\bar{R}](S - n\bar{R})}\} \geq 0.$$

Putting $f = nH^\alpha$ in (4.12), by $H^2 = \sum\limits_\alpha (H^\alpha)^2$, we have $|nH^\alpha| \leq nH$. From (2.5) and $\sup S < +\infty$, we have $f = nH^\alpha$ is bounded from above. Since we know that $nH^\alpha - \lambda_i^\alpha > 0$ and $\sum\limits_\alpha (nH^\alpha - \lambda_i^\alpha) > 0$, we have

$$0 < \sum_\alpha (nH^\alpha - \lambda_i^\alpha) \leq \sum_{\alpha,i} (nH^\alpha - \lambda_i^\alpha) = n(n-1)\sum_\alpha H^\alpha$$

is bounded. We may use Proposition 4.2 to $f = nH^\alpha$. Thus, we have

$$\lim_{m\to+\infty} (nH^\alpha)(x_m) = \sup(nH^\alpha), \quad \lim_{m\to+\infty} \sup \Box(nH^\alpha)(x_m) \leq 0, \qquad (4.16)$$

where $\{x_m\}$ is a sequence on M^n. From (2.5) and (4.16), we have $\lim_{m\to\infty} S(x_m) = \sup S$.

From (4.13), (4.15) and (4.16), we have

$$0 \geq \lim_{m\to+\infty} \sup \Box(nH^\alpha)(x_m)$$
$$\geq \frac{n-1}{n}(\sup S - n\bar{R})\{n + n\bar{R} + [\frac{1}{n} - (2 - \frac{1}{p})\frac{n-1}{n}](\sup S - n\bar{R})$$
$$- \frac{n-2}{n}\sqrt{[\sup S + n(n-1)\bar{R}](\sup S - n\bar{R})}\} \geq 0.$$

Thus, we have

(i) $\sup S - n\bar{R} = 0$, that is, $\sup S = n\bar{R}$. From (2.5), we have $\sup(S - nH^2) = 0$, thus, $S = nH^2$ and M^n is totally umbilical; or

(ii)

$$n+n\bar{R}+[\frac{1}{n}-(2-\frac{1}{p})\frac{n-1}{n}](\sup S-n\bar{R}) - \frac{n-2}{n}\sqrt{[\sup S+n(n-1)\bar{R}](\sup S-n\bar{R})}=0. \quad (4.17)$$

(4.17) implies that (4.14) and (4.15) hold. From the assumption, we know that $\sup S$ is attained at some point of M^n. Thus, from (2.5), we have $\sup(nH)^2$ is attained at this point of M^n. By $\sup(nH)^2 = \sum_\alpha \sup(nH^\alpha)^2$, we have $\sup(nH^\alpha)$ is attained at this point of M^n. Since the operator $\Box$ is elliptic, we have nH^α is constant. Thus, the equalities in (4.15) hold and $|\nabla h|^2 = n^2|\nabla^\perp \vec{H}|^2$. From (2.5) and (3.17), we have

$$0 \le n^3(n-1)(R-1)|\nabla^\perp \vec{H}|^2 \le S(|\nabla h|^2 - n^2|\nabla^\perp \vec{H}|^2).$$

Since we assume that $R > 1$, we have $\nabla^\perp \vec{H} = 0$. Therefore, we know that M^n is a complete submanifold in $S^{n+p}(1)$ with parallel mean curvature vector.

From (4.17), we have

$$n+nH^2-\frac{n-2}{\sqrt{n(n-1)}}n|H|\sup|\phi|-(2-\frac{1}{p})\sup|\phi|^2=0. \quad (4.18)$$

Thus, we have $\sup|\phi|^2 = B_{H,p,n}$. Since $n^2H^2 > S$, we have $H > 0$. By the result of Theorem 1.4, we have (i) $p = 1$ and M^n is an open piece of $H(r)$-torus $S^{n-1}(r)\times S^1(\sqrt{1-r^2})$ with $0 < r < 1$; or (ii) $n = 2$, $p = 2$ and M^n is an open piece of Veronese surface in S^4. This completes the proof of the Theorem 1.5. $\Box$

Acknowledgments. Project supported by NSF of Shaanxi Province(SJ08A31)and NSF of Shaanxi Educational Committee(11JK0479).

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