Bulletin of Mathematical Analysis and Applications
ISSN: 1821-1291, URL: http://www.bmathaa.org
Volume 3 Issue 4 (2011), Pages 240-246.

FEKETE-SZEGÖ PROBLEM FOR A CLASS OF COMPLEX ORDER RELATED TO SĂLĂGEAN OPERATOR

(COMMUNICATED BY R.K RAINA)

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Abstract. For non-zero complex b, let $\mathcal{N}^b_{m,n}(\varphi)\,(m>n),\ m\in\mathbb{N},\ n\in\mathbb{N}_0$ denote the class of normalized univalent functions f satisfying $1+\frac{1}{b}\left(\frac{D^m f(z)}{D^n f(z)}-1\right)\prec\varphi(z)$, z in the open unit disk $\mathcal{U}$, where $D^n f$ denotes the Sălăgean derivative of f. In this paper, we obtain sharp bounds for the Fekete-Szegö functional $|a_3-\mu a_2^2|$.

1. Introduction

The problem called Fekete-Szegö arised from the estimate:

$$|a_3-\lambda a_2^2|\ \le\ 1+2\ \exp\left(-2\lambda/(1-\lambda)\right) \tag{1.1}$$

due to Fekete and Szegö [6]. Here a_2, a_3 are the coefficients of a normalized univalent function

$$f(z)=z+\sum_{k=2}^{\infty}a_k\,z^k \tag{1.2}$$

in the open unit disk $\mathcal{U}=\{z\in\mathbb{C}:|z|<1\}$ and λ is a real number such that $0\le\lambda<1$. In [6] authors stated that inequality (1.1) is sharp for each λ. We observe that the coefficient functional $a_3-\lambda a_2^2$ was not chosen incidentally, but it represents several geometric quantities. Let S_f denotes the Schwarzian derivative of normalized analytic functions f in the unit disk e.g. $S_f=(f''/f')'-(f''/f')^2/2$. Then $S_f(0)/6$ is just Fekete-Szegö functional for $\lambda=1$, and the hyperbolic sup-norm of the Schwarzian derivative

$$\|S_f\|=\sup_{z\in\mathcal{U}}(1-|z|^2)^2|S_f(z)|.$$

plays an important role in the theory of Teichmüller spaces. In connection with Teichmüller spaces, it is an interesting problem to consider estimation of the norm of the Schwarzian derivatives as well as Fekete-Szegö functional for typical subclasses

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2010 *Mathematics Subject Classification.* Primary 30C45; Secondary 30C50.

Key words and phrases. univalent functions, Fekete-Szegö problem, Sălăgean operator.

S.P.G is supported by CSIR, New Delhi, India for Emeritus Scientistship, under the scheme number 21(084)/10/EMR-II.

Submitted November 14, 2011. Published December 1, 2011.

of univalent functions. For that reason Fekete-Szegö functional was studied by many authors and some estimates were found in a many subclasses of normalized univalent functions (see, for example [1] - [12]).

Let $\mathcal{S}$ be the set of all functions normalized, analytic and univalent in the unit disk $\mathcal{U}$ of the form (1.2). By $\mathcal{P}$ we denote the well-known class of Carathéodory functions, so that $\mathcal{P} = \{p : p \text{ analytic in } \mathcal{U},\ \mathrm{Re}\{p(z)\} > 0\}$. Assume also that $\varphi \in \mathcal{P}$ is such that $\varphi(0) = 1$, $\varphi'(0) > 0$ and φ maps the unit disk $\mathcal{U}$ onto a region starlike with respect to 1, and symmetric with respect to the real axis. We note that φ has a real coefficients around the origin. By $\mathcal{ST}(\varphi)$, $\mathcal{CV}(\varphi)$, $\mathcal{CC}(\varphi)$ we denote the following classes of functions

$$\mathcal{ST}(\varphi) = \left\{ f(z) \in \mathcal{S} : \frac{zf'(z)}{f(z)} \prec \varphi(z),\ z \in \mathcal{U} \right\} \tag{1.3}$$

$$\mathcal{CV}(\varphi) = \left\{ f(z) \in \mathcal{S} : 1 + \frac{zf''(z)}{f'(z)} \prec \varphi(z),\ z \in \mathcal{U} \right\}. \tag{1.4}$$

Let

$$g(z) = z + \sum_{k=2}^{\infty} b_k z^k,\ z \in \mathcal{U}. \tag{1.5}$$

Consider the class

$$\mathcal{CC}(\varphi) = \left\{ f(z) \in \mathcal{S} : \frac{zf'(z)}{g(z)} \prec \varphi(z),\ g \in \mathcal{ST}, z \in \mathcal{U} \right\} \tag{1.6}$$

where " $\prec$ " denotes the subordination between analytic functions. The classes $\mathcal{ST}(\varphi)$ and $\mathcal{CV}(\varphi)$ are the extensions of a classical sets of a starlike and convex functions, and in a such form were defined and studied by Ma and Minda [13].

In this note we consider a complex generalization of the mentioned above classes, namely the class $\mathcal{ST}_b(\varphi)$ (resp. $\mathcal{CV}_b(\varphi)$, $\mathcal{CC}_b(\varphi)$) of all functions f as follows

$$\mathcal{ST}_b(\varphi) = \left\{ f(z) \in \mathcal{S} : 1 + \frac{1}{b}\left(\frac{zf'(z)}{f(z)} - 1 \right) \prec \varphi(z),\ z \in \mathcal{U} \right\}, \tag{1.7}$$

$$\mathcal{CV}_b(\varphi) = \left\{ f(z) \in \mathcal{S} : 1 + \frac{1}{b}\left(\frac{zf''(z)}{f'(z)} \right) \prec \varphi(z),\quad z \in \mathcal{U} \right\}. \tag{1.8}$$

$$\mathcal{CC}_b(\varphi) = \left\{ f(z) \in \mathcal{S} : 1 + \frac{1}{b}\left(\frac{zf'(z)}{g(z)} - 1 \right) \prec \varphi(z),\ g \in \mathcal{ST},\ z \in \mathcal{U} \right\}. \tag{1.9}$$

We note that the choice $b = 1$ reduces $\mathcal{ST}_b(\varphi)$ and $\mathcal{CV}_b(\varphi)$ to $\mathcal{ST}(\varphi)$ (resp. $\mathcal{CV}(\varphi)$).

Let a differential operator be defined (Sălăgean, [15]) on a class of analytic functions of the form (1.2) as follows

$$D^0 f(z) = f(z), \quad D^1 f(z) = Df(z) = z f'(z),$$

and in general

$$D^n f(z) = D\left(D^{n-1} f(z)\right) \quad (n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}).$$

We easily find that

$$D^n f(z) = z + \sum_{k=2}^{\infty} k^n a_k z^k \quad (n \in \mathbb{N}_0). \tag{1.10}$$

Definition 1.1. *Let b be a non zero complex number, and let f be an univalent function of the form (1.2), such that $D^n f(z) \neq 0$ for $z \in \mathcal{U} \setminus \{0\}$. Also, let φ be an analytic univalent function with positive real part in the unit disk $\mathcal{U}$ with $\varphi(0) = 1$, $\varphi'(0) > 0$ which maps the unit disk $\mathcal{U}$ onto a region starlike with respect to 1 which is symmetric with respect to real axis. We say that f belongs to $\mathcal{N}^b_{m,n}(\varphi)$ $(m > n)$, $m \in \mathbb{N}$, $n \in \mathbb{N}_0$, if*

$$1 + \frac{1}{b}\left(\frac{D^m f(z)}{D^n f(z)} - 1\right) \prec \varphi(z), \quad z \in \mathcal{U}. \tag{1.11}$$

For $m = n+1$, $n \in \mathbb{N}_0$, the condition (1.11) reduces to

$$1+\frac{1}{b}\left(\frac{z\,(D^n f(z))'}{D^n f(z)} - 1\right) \prec \varphi(z) \quad \Longleftrightarrow \quad 1+\frac{1}{b}\left(\frac{D^{n+1} f(z)}{D^n f(z)} - 1\right) \prec \varphi(z), \quad z \in \mathcal{U}, \tag{1.12}$$

which is a natural generalization of classes $\mathcal{ST}_b(\varphi)$ (resp. $\mathcal{CV}_b(\varphi)$) for a choice $n = 0$ (resp. $n = 1$).

Definition 1.2. *Let b be a non-zero complex number, and let f be a univalent function of the form (1.2), we say f belongs to $\mathcal{M}^c_{n,b}(\varphi)$ if*

$$1 + \frac{1}{b}\frac{z\,(D^n f(z))''}{(D^n f(z))'} \prec \varphi(z), \quad z \in \mathcal{U}, \tag{1.13}$$

and f is a member of $\mathcal{CN}^c_{n,b}(\varphi)$ if

$$1 + \frac{1}{b}\left(\frac{D^{n+1} f(z)}{D^n g(z)} - 1\right) \prec \varphi(z), \quad z \in \mathcal{U}, \tag{1.14}$$

where $g \in \mathcal{N}^b_{n+1,n}(\varphi)$.

2. Main Results

We shall require the following estimates for the Carathéodory class $\mathcal{P}$ during our investigation.

Lemma 2.1. *([13], see also [14]) If $p(z) = 1 + c_1 z + c_2 z^2 + \cdots$ is a function with positive real part, then for any complex number μ,*

$$|c_2 - \mu\, c_1^2| \ \leq \ 2\max\{1, |2\mu - 1|\},$$

and the result is sharp for the functions given by

$$p(z) = \frac{1+z^2}{1-z^2} \text{ or } \frac{1+z}{1-z}.$$

Lemma 2.2. *([13]) Let $p \in \mathcal{P}$ with $p(z) = 1 + c_1 z + c_2 z^2 + \ldots$. Then*

$$|c_2 - v c_1^2| \leq \begin{cases} 2 - 4v & if \quad v \leq 0, \\ 2 & if \quad 0 \leq v \leq 1, \\ 4v - 2 & if \quad v \geq 1. \end{cases}$$

When $v < 0$ or $v > 1$, inequality holds if and only if $p(z)$ is $(1+z)/(1-z)$ or one of its rotations. If $0 < v < 1$, then equality holds if and only if $p(z)$ is $(1+z^2)/(1-z^2)$ or one of its rotations. Inequality becomes equality when $v = 0$ if and only if

$$p(z) = \left(\frac{1}{2} + \frac{1}{2}\lambda\right)\frac{1+z}{1-z} + \left(\frac{1}{2} - \frac{1}{2}\lambda\right)\frac{1-z}{1+z}, \ 0 \leq \lambda \leq 1,$$

or one of its rotations. While for $v = 1$, equality holds if and only if $p(z)$ is the reciprocal of one of the functions such that equality holds in the case of $v = 0$. For $0 < v < 1$ the above can be improved by

$$|c_2 - vc_1^2| + v|c_1|^2 \leq 2,\ 0 < v \leq \frac{1}{2}$$

and

$$|c_2 - vc_1^2| + (1-v)|c_1|^2 \leq 2,\ \frac{1}{2} \leq v < 1.$$

Theorem 2.3. *Let b be a non zero complex number and $\varphi(z) = 1+\alpha_1 z+\alpha_2 z^2+\cdots$. If $f(z)$ given by (1.2) belongs to $\mathcal{N}^b_{m,n}(\varphi)$, then for $\mu \in \mathbb{C}$*

$$|a_3 - \mu a_2^2| \ \leq \ \frac{|b|\,\alpha_1}{(3^m - 3^n)} \max\left\{1, \left|\frac{\alpha_2}{\alpha_1} + \left[\frac{2^n(2^m-2^n) - \mu(3^m-3^n)}{(2^m-2^n)^2}\right] b\alpha_1\right|\right\}. \tag{2.1}$$

The result is sharp.

Proof. For a function f given by (1.2) and by the definition of the class $\mathcal{N}^b_{m,n}(\varphi)$ $(m > n)$ there exists $p \in \mathcal{P}$ such that

$$\frac{D^m f(z)}{D^n f(z)} = 1 + b\,\{p(z) - 1\} = 1 + bc_1 z + bc_2 z^2 + \cdots, \tag{2.2}$$

and $p \prec \varphi$. Then the function

$$\psi(z) = \frac{1 + \varphi^{-1}(p(z))}{1 - \varphi^{-1}(p(z))} = 1 + d_1 z + d_2 z^2 + \cdots \tag{2.3}$$

is analytic and has positive real part in $\mathcal{U}$. Hence

$$p(z) = \varphi\left(\frac{\psi(z) - 1}{\psi(z) + 1}\right). \tag{2.4}$$

Equating power series expansion of $\varphi(z) = 1 + \alpha_1 z + \alpha_2 z^2 + \cdots$, we express c_n in terms of d_n and a_n in terms of d_n and α_n. From the equation (2.4), we obtain

$$c_1 = \frac{1}{2}\alpha_1 d_1, \quad \text{and} \quad c_2 = \frac{1}{2}\alpha_1\left(d_2 - \frac{1}{2}d_1^2\right) + \frac{1}{4}\alpha_2 d_1^2. \tag{2.5}$$

From (2.2), we get

$$a_2 = \frac{b\,c_1}{2^m - 2^n}, \quad \text{and} \quad a_3 = \frac{b}{3^m - 3^n}\left[c_2 + \frac{2^n b c_1^2}{2^m - 2^n}\right]. \tag{2.6}$$

By the equations (2.5) and (2.6), we have

$$a_2 = \frac{b\,\alpha_1 d_1}{2(2^m - 2^n)}$$

and

$$a_3 = \frac{b\,\alpha_1 d_2}{2(3^m - 3^n)} + \frac{bd_1^2}{4(3^m - 3^n)}\left[\frac{2^n b\alpha_1^2}{2^m - 2^n} + \alpha_2 - \alpha_1\right].$$

Therefore we have

$$a_3 - \mu a_2^2 = \frac{b\alpha_1}{2(3^m - 3^n)}\{d_2 - \nu\, d_1^2\}$$

where (note that $\alpha_1 = \varphi'(0) > 0$)

$$\nu = \frac{1}{2}\left[1 - \frac{\alpha_2}{\alpha_1} + \left(\frac{\mu(3^m - 3^n) - 2^n(2^m - 2^n)}{(2^m - 2^n)^2}\right) b\alpha_1\right].$$

Our result now follows by an application of Lemma 2.1. The result is sharp for the function defined by

$$1+\frac{1}{b}\left(\frac{D^m f(z)}{D^n f(z)}-1\right)=\varphi(z^2), \quad 1+\frac{1}{b}\left(\frac{D^m f(z)}{D^n f(z)}-1\right)=\varphi(z).$$

We next consider the case, when both μ and b are real. Then we have

Theorem 2.4. *Let $b>0$ and $\varphi(z)=1+\alpha_1 z+\alpha_2 z^2+\cdots$. If f given by (1.2) belongs to $\mathcal{N}_{m,n}^b(\varphi)$, then for $\mu\in\mathbb{R}$, we have*

$$|a_3-\mu a_2^2| \leq \begin{cases} \frac{b\alpha_2}{3^m-3^n}-\frac{\mu b^2\alpha_1^2}{(2^m-2^n)^2}+\frac{2^n b^2\alpha_1^2}{(2^m-2^n)(3^m-3^n)} & \text{for} \quad \mu\leq\sigma_1, \\ \frac{b\alpha_1}{3^m-3^n} & \text{for} \quad \sigma_1\leq\mu\leq\sigma_2, \\ -\frac{b\alpha_2}{3^m-3^n}+\frac{\mu b^2\alpha_1^2}{(2^m-2^n)^2}-\frac{2^n b^2\alpha_1^2}{(2^m-2^n)(3^m-3^n)} & \text{for} \quad \mu\geq\sigma_2, \end{cases}$$

where

$$\sigma_1=\frac{2^m-2^n}{3^m-3^n}\left[\frac{2^n b\alpha_1^2+(\alpha_2-\alpha_1)(2^m-2^n)}{\alpha_1^2}\right]$$

and

$$\sigma_2=\frac{2^m-2^n}{3^m-3^n}\left[\frac{2^n b\alpha_1^2+(2^m-2^n)(\alpha_2+\alpha_1)}{b\alpha_1^2}\right].$$

The result is sharp.

Proof. By Theorem 2.3, we have

$$a_3-\mu a_2^2=\frac{b\alpha_1}{2(3^m-3^n)}\{d_2-\nu\, d_1^2\}$$

where

$$\nu=\frac{1}{2}\left[1-\frac{\alpha_2}{\alpha_1}+\left(\frac{\mu(3^m-3^n)-2^n(2^m-2^n)}{(2^m-2^n)^2}\right)b\alpha_1\right].$$

Our result now follows by an application of Lemma 2.2.

Remark 2.5. *If f given by (1.2) belongs to $\mathcal{N}_{m,n}^b(\varphi)$ and $\sigma_1<\mu<\sigma_2$, then, in view of Lemma 2.2, Theorem 2.4 can be improved. Let σ_3 be given by*

$$\sigma_3=\frac{2^m-2^n}{3^m-3^n}\left[\frac{2^n b\alpha_1^2+\alpha_2(2^m-2^n)}{b\alpha_1^2}\right].$$

If $\sigma_1<\mu\leq\sigma_3$, then

$$|a_3-\mu a_2^2|+\frac{(2^m-2^n)^2}{4b(3^m-3^n)}\left[(\alpha_1-\alpha_2)+b\alpha_1^2\frac{\mu(3^m-3^n)-2^n(2^m-2^n)}{(2^m-2^n)^2}\right]|a_2|^2\leq\frac{b\alpha_1}{3^m-3^n}.$$

If $\sigma_3\leq\mu<\sigma_2$, then

$$|a_3-\mu a_2^2|+\frac{(2^m-2^n)^2}{4b(3^m-3^n)}\left[(\alpha_1+\alpha_2)+b\alpha_1^2\frac{\mu(3^m-3^n)-2^n(2^m-2^n)}{(2^m-2^n)^2}\right]|a_2|^2\leq\frac{b\alpha_1}{3^m-3^n}.$$

Taking $m=n+1$ $(n\in\mathbb{N}_0)$, Theorem 2.3 and 2.4 reduce to

Corollary 2.6. *Let b be a nonzero complex number and $\varphi(z)=1+\alpha_1\, z+\alpha_2\, z^2+\cdots$. If $f(z)$ given by (1.2) belongs to $\mathcal{N}_{n+1,n}^b(\varphi)$, then for $\mu\in\mathbb{C}$, we have*

$$|a_3-\mu a_2^2| \leq \frac{|b|\alpha_1}{3^n\, 2}\max\left\{1,\left|\frac{\alpha_2}{\alpha_1}+\left[1-2\mu\left(\frac{3}{4}\right)^n\right]b\alpha_1\right|\right\}.$$

The result is sharp.

Corollary 2.7. *Let $b > 0$ and $\varphi(z) = 1 + \alpha_1 z + \alpha_2 z^2 + \cdots$. If f given by (1.2) belongs to $\mathcal{N}^b_{n+1,n}(\varphi)$, then for $\mu \in \mathbb{R}$, we have*

$$|a_3 - \mu a_2^2| \le \begin{cases} \frac{b\alpha_2}{3^n\,2} - \frac{\mu b^2 \alpha_1^2}{2^{2n}} + \frac{b^2\alpha_1^2}{3^n\,2} & \text{if} \quad \mu \le \sigma_1, \\ \frac{b\alpha_1}{3^n\,2} & \text{if} \quad \sigma_1 \le \mu \le \sigma_2, \\ -\frac{b\alpha_2}{3^n\,2} + \frac{\mu b^2 \alpha_1^2}{2^{2n}} - \frac{b^2\alpha_1^2}{3^n\,2} & \text{if} \quad \mu \ge \sigma_2, \end{cases}$$

where

$$\sigma_1 = \frac{1}{2}\left(\frac{4}{3}\right)^n \left[1 + \frac{\alpha_2 - \alpha_1}{b\alpha_1^2}\right].$$

and

$$\sigma_2 = \frac{1}{2}\left(\frac{4}{3}\right)^n \left[1 + \frac{\alpha_2 + \alpha_1}{b\alpha_1^2}\right].$$

The result is sharp.

Remark 2.8. *If $f(z)$ given by (1.2) belongs to $\mathcal{N}^b_{n+1,n}(\varphi)$ and $\sigma_1 < \mu < \sigma_2$, then, in view of Lemma 2.2, Corollary 2.7 can be improved. Let σ_3 be given by*

$$\sigma_3 = \frac{1}{2}\left(\frac{4}{3}\right)^n \left[1 + \frac{\alpha_2}{b\alpha_1^2}\right].$$

If $\sigma_1 < \mu \le \sigma_3$, then

$$|a_3 - \mu a_2^2| + \frac{1}{8b}\left(\frac{4}{3}\right)^n \left\{(\alpha_1 - \alpha_2) + b\alpha_1^2\left[2\mu\left(\frac{3}{4}\right)^n - 1\right]\right\}|a_2|^2 \le \frac{b\alpha_1}{2.3^n}.$$

If $\sigma_3 \le \mu < \sigma_2$, then

$$|a_3 - \mu a_2^2| + \frac{1}{8b}\left(\frac{4}{3}\right)^n \left\{(\alpha_1 + \alpha_2) + b\alpha_1^2\left[2\mu\left(\frac{3}{4}\right)^n - 1\right]\right\}|a_2|^2 \le \frac{b\alpha_1}{2.3^n}.$$

On taking $b = (1-\beta)e^{-i\delta}\cos\delta$ $\left(0 \le \beta < 1,\ |\delta| < \frac{\pi}{2}\right)$ and $\varphi(z) = \frac{1+z}{1-z}$ in Theorem 2.3 and Corollary 2.6, we get the following result

Corollary 2.9. *If $f(z)$ given by (1.2) belongs to $\mathcal{N}^b_{m,n}(\varphi)$, then for $\mu \in \mathbb{C}$, we have*

$$|a_3 - \mu a_2^2| \le \frac{2(1-\beta)\cos\delta}{(3^m - 3^n)} \max\left\{1, \left|e^{i\delta} + 2\left[\frac{2^n(2^m - 2^n) - \mu(3^m - 3^n)}{(2^m - 2^n)^2}\right](1-\beta)\cos\delta\right|\right\}.$$

The result is sharp.

Corollary 2.10. *If $f(z)$ given by (1.2) belongs to $\mathcal{N}^b_{n+1,n}(\varphi)$, then for $\mu \in \mathbb{C}$, we have*

$$|a_3 - \mu a_2^2| \le \frac{(1-\beta)\cos\delta}{3^n} \max\left\{1, \left|e^{i\delta} + 2\left[1 - 2\mu\left(\frac{3}{4}\right)^n\right](1-\beta)\cos\delta\right|\right\}.$$

The result is sharp.

Acknowledgment: The authors are thankful to Prof. S. Kanas, Rzeszow University of Technology, Poland, for his valuable suggestions.

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