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EXACT CONTROLLABILITY OF SEMILINEAR SYSTEMS WITH IMPULSES

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Abstract. The purpose of this paper is to investigate the controllability of the impulsive semilinear system

$$x'(t) = A(t)x(t) + f(t, x(t)) + B(t)u(t),\ t \geq t_0,\ t \neq t_k,$$
$$x(t_k^+) = x(t_k^-) + I_k(x(t_k^-)),\ k = 1, 2, 3, \ldots.$$

By making use of Schaefer's fixed point theorem we obtain some results under which the system is completely controllable. Two examples are also given to illustrate the importance of our results.

1. Introduction

Since many dynamic processes in nature can encounter the abrupt changes at certain moments, there have been quite a number of literatures to study the differential systems with impulses, see, e.g., [1, 3, 4, 5, 6, 9] and the references therein. Specially, Nieto et al. [5] considered the controllability of the impulsive system

$$\begin{cases} x'(t) + \lambda x(t) = f(t, x(t)) + u(t),\ t \geq t_0 \text{ and } t \neq t_k, \\ x(t_k^+) = x(t_k^-) + I_k(x(t_k^-)),\ k = 1, 2, 3, \ldots \end{cases}$$

under the assumptions that

$$|f(t, x)| \leq a_0 + b_0|x|^{\alpha_0}$$

and

$$|I_k(x)| \leq a_k + b_k|x|^{\alpha_k},$$

where λ is an $n \times n$ real matrix, a_j, b_j and α_j are constants for $j \in \{0, k\}$. We observe that the derived results in [5] are valid only for $\alpha_k \in (0, 1)$. Now a problem emerges that whether we can weaken the restriction. To answer this question, we let

$$t_0 < t_1 < t_2 < \ldots \text{ and } t_k \to \infty \text{ as } k \to \infty$$

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and consider a more general impulsive control system

$$\begin{cases} x'(t) = A(t)x(t) + f(t, x(t)) + B(t)u(t),\ t \geq t_0 \text{ and } t \neq t_k, \\ x(t_k^+) = x(t_k^-) + I_k(x(t_k^-)),\ k = 1, 2, 3, \ldots, \end{cases} \tag{1.1}$$

where $u(\cdot)$ is a continuous input function, $A(\cdot)$ and $B(\cdot)$ are, respectively, continuous $n \times n$ and $n \times m$ functions defined on $[t_0, \infty)$, $f(\cdot, \cdot) \in C([t_0, \infty) \times \mathbb{R}^n, \mathbb{R}^n)$, $I_k \in C(\mathbb{R}^n, \mathbb{R}^n)$ and

$$x(t_k) = x(t_k^+) = \lim_{t \to t_k^+} x(t)\ ,\ x(t_k^-) = \lim_{t \to t_k^-} x(t).$$

Set $t_0 < \zeta \leq \infty$ and $J = [t_0, \zeta) \setminus \{t_k : \ k = 1, 2, 3, ...\}$. Let

$$\begin{aligned} & PC[t_0, \zeta) \\ = \ & \left\{x : [t_0, \zeta) \to \mathbb{R}^n : x \in C(J),\ x(t_k^-) \text{ and } x(t_k^+) \text{ exist and } x(t_k) = x(t_k^+)\right\}. \end{aligned}$$

Specially, for any given $\tau \in (t_0, \zeta)$, let $J' = [t_0, \tau] \setminus \{t_k : \ k = 1, 2, 3, ...\}$. Then, the space

$$\begin{aligned} & PC[t_0, \tau] \\ = \ & \left\{x : [t_0, \tau] \to \mathbb{R}^n : x \in C(J'),\ x(t_k^-) \text{ and } x(t_k^+) \text{ exist and } x(t_k) = x(t_k^+)\right\} \end{aligned}$$

with the norm

$$||x|| = \sup_{t \in [t_0, \tau]} |x(t)|$$

is a Banach space, where the norm $|x|$ for $x = (x^{(1)}, x^{(2)}, ..., x^{(n)})^T \in \mathbb{R}^n$ is defined by $|x| = |x^{(1)}| + |x^{(2)}| + \ldots + |x^{(n)}|$. In the sequel, the norm of $n \times n$ matrix $U = (u_{ij})$ will be defined by

$$|U| = \max_j \sum_i |u_{ij}|.$$

As usual, for any given $u(\cdot) \in C([t_0, \infty), \mathbb{R}^m)$, by a solution of (1.1) we mean that there exists a number ζ with $t_0 < \zeta \leq \infty$ and a function $x \in PC[t_0, \zeta)$ such that x is differentiable on $[t_0, \zeta) \setminus \{t_k : \ k = 1, 2, 3, ...\}$ and renders (1.1) into identity.

Referring to [7, 8], the system (1) is said to be controllable if, for a preassigned time $\tau > t_0$ and the states $x_0, x_1 \in \mathbb{R}^n$, there exists a control $u \in C([t_0, \tau], \mathbb{R}^m)$ such that the solution $x(t)$ of (1), with initial condition $x(t_0) = x_0$, exists on $[t_0, \tau]$ and satisfies $x(\tau) = x_1$. If the system is controllable for any $\tau > t_0$ and any $x_0, x_1 \in \mathbb{R}^n$, it will be called completely controllable.

2. Preliminaries

Let E be an identity matrix and $\Phi(t, t_0)$ a principal matrix solution of the linear system

$$x'(t) = A(t)x(t),\ t \geq t_0,$$

i.e.,

$$\Phi'(t, t_0) = A(t)\Phi(t, t_0),\ \Phi(t_0, t_0) = E,\ t \geq t_0$$

and

$$\Phi(t, t_1)\Phi(t_1, t_0) = \Phi(t, t_0).$$

Then (1) with initial value $x(t_0) = x_0$ is equivalent to the following system

$$
\begin{aligned}
x(t) &= \Phi(t,t_0)x_0 + \Phi(t,t_0)\int_{t_0}^{t}\Phi^{-1}(s,t_0)\,[f(s,x(s)) + B(s)u(s)]\,\mathrm{d}s \\
&\quad + \sum_{k:\ t_k\in(t_0,t)} \Phi(t,t_k)I_k(x(t_k^-)),\ t \geq t_0. \qquad (2.1)
\end{aligned}
$$

To enter our discussions, we give a blanket assumption as follows.

(A1) Suppose that $a_i \in C([t_0,\infty),[0,\infty))$, constants $b_i^{(k)} \geq 0, \alpha > 0$ and $\alpha_k > 0$ for $i = 1, 2$ and $k = 1, 2, 3, ...$, such that

$$|f(t,x)| \leq a_1(t) + a_2(t)|x|^{\alpha} \qquad (2.2)$$

and

$$|I_k(x)| \leq b_1^{(k)} + b_2^{(k)}|x|^{\alpha_k}. \qquad (2.3)$$

(A2) Let P and Q be constants such that

$$\begin{cases} |\Phi(t,s)| \leq P \text{ for all } \ t,s \in [t_0,\tau], \\ \int_{t_0}^{\tau} a_i(s)\mathrm{d}s \leq Q \ \text{ for } \ i = 1, 2, \end{cases} \qquad (2.4)$$

where $\tau > t_0$ is some finite number.

We remark that when α in assumption (A1) equals 1, the solution $x(t)$ of (1.1) with respect to any continuous input function u and any initial condition $x(t_0) = x_0$ is well defined on $[t_0,\infty)$. See [12, Theorem 2.17] for the details. In general, we have the following result.

Lemma 2.1. *Under the condition (2.2) with $\alpha \in (0,1]$, the solution $x(t)$ of (1.1) with respect to the control u and the initial condition $x(t_0) = x_0$ is well defined on $[t_0,\infty)$.*

Proof. Suppose to the contrary that the solution $x(t)$ of (1.1) is defined on $[t_0,\zeta)$, here ζ is finite. Let

$$b = \sum_{k:\ t_k\in(t_0,\zeta)} \left|\Phi(t,t_k)I_k(x(t_k^-))\right|,\ t \in [t_0,\zeta). \qquad (2.5)$$

For simplicity, we set

$$\int_{t_0}^{\zeta} |B(s)u(s)|\mathrm{d}s \leq Q,$$

where Q is the same as the assumption (A2) for $\tau = \zeta$. Now, with the aid of (2.1)–(2.2) and (2.4)–(2.5) we have

$$
\begin{aligned}
|x(t)| &\leq P|x_0| + 2PQ + P\int_{t_0}^{t} a_2(s)|x(s)|^{\alpha}\mathrm{d}s + b \\
&= R + P\int_{t_0}^{t} a_2(s)|x(s)|^{\alpha}\mathrm{d}s,\ t \in [t_0,\zeta), \qquad (2.6)
\end{aligned}
$$

where $R = P|x_0| + 2PQ + b$ and the relation $\Phi(t,t_0)\Phi^{-1}(s,t_0) = \Phi(t,s)$ has been imposed. If we set

$$\beta(t) = R + P\int_{t_0}^{t} a_2(s)|x(s)|^{\alpha}\mathrm{d}s,\ t \in [t_0,\zeta), \qquad (2.7)$$

then, from (2.6) it follows that

$$\beta'(t) = Pa_2(t)|x(t)|^\alpha \leq Pa_2(t)\beta^\alpha(t),\ t \in [t_0, \zeta), \tag{2.8}$$

which, together with (2.4), produces

$$\int_{\beta(t_0)}^{\beta(t)} \frac{1}{v^\alpha}\mathrm{d}v \leq P\int_{t_0}^{t} a_2(s)\mathrm{d}s \leq PQ,\ t \in [t_0, \zeta). \tag{2.9}$$

In addition, by the assumption that $[t_0, \zeta)$ is the maximum existence interval of $x(t)$, we have

$$\lim_{t\to\zeta^-} |x(t)| = \infty \tag{2.10}$$

and hence (2.6) implies that

$$\lim_{t\to\zeta^-} \beta(t) = \infty, \tag{2.11}$$

here we refer the reader to [12, Corollary 2.16] for (2.10). Now invoking (2.11) we see that (2.9) results in a contradiction and this completes our proof.

In the light of Lemma 2.1, for any given $\tau > t_0$, the solution $x(t)$ of (1.1) corresponding to the control $u(t)$ and the initial condition $x(t_0) = x_0$ is well defined on $[t_0, \tau]$. Next we let the $n \times n$ matrix function W be defined by

$$W(t) = \int_{t_0}^{t} \Phi^{-1}(s, t_0)B(s)B^T(s)\Phi^{-1}(s, t_0)^T\mathrm{d}s,\ t > t_0, \tag{2.12}$$

where T denotes the transpose of matrices and $\Phi^{-1}(s, t_0)^T = \left[\Phi^{-1}(s, t_0)\right]^T$. Let us also assume that $W(t)$ is invertible for all $t > t_0$. Then, referring to [11, Chapter 3] we learn that, for any $x_0, x_1 \in \mathbb{R}^n$ and any $\tau > t_0$, the following linear time-varying system

$$x'(t) = A(t)x(t) + B(t)u(t) \tag{2.13}$$

is completely controllable.

Our aim in the remainder of this paper is to consider whether the system (1.1) can inherit the controllability property when (2.13) is controllable, or, to consider whether the system (1.1) can possess the controllability although the linear system (2.13) is not controllable. To this end, we need the following standard conclusion[2, 5].

Lemma 2.2. *(Schaefer) Let X be a Banach space and $\Psi : X \to X$ be a continuous compact map. If the set*

$$\Omega = \{x \in X : x = \lambda\Psi(x) \textit{ for some } \lambda \in (0, 1)\}$$

is bounded, then $\Psi(x)$ has a fixed point.

For any two matrices $A_k(t) = (a_{ij}^{(k)}(t))_{n\times n},\ k = 1, 2$, by $A_1(t) \geq 0$ for $t \geq t_0$ we mean that $a_{ij}^{(1)}(t) \geq 0$ for all i, j and $t \geq t_0$. By $A_1(t) \geq A_2(t)$ we mean that $A_1(t) - A_2(t) \geq 0$. The matrix $A_1(t)$ is said to be nondecreasing in t if all the entries $a_{ij}^{(1)}(t)$ of A_1 are nondecreasing in t.

3. Main Results

In this section we begin to consider the controllability of (1.1). The main idea is to transform the controllability problem to the existence of a fixed point. We observe that this approach has been invoked by many authors, such as [2, 5, 6, 8].

Theorem 3.1. *Suppose that the matrix $W(\tau)$ defined as in (2.12) is invertible and that the conditions (2.2)–(2.4) are fulfilled, where $\tau > t_0$ is a preassigned time. Then the system (1.1) is controllable follows from one of the following conditions*

(i) $\alpha < 1$, $\max_k\{\alpha_k\} = 1$ *and* $2P \sum_{k=1}^{\tau} b_2^{(k)} < 1$. *Moreover* , $\Phi(t,t_0)$, $\Phi^{-1}(\tau,t_0)$, $W(t)$ *and* $W^{-1}(\tau)$ *are nonnegative matrics for* $t \in [t_0,\tau]$, *and* $\Phi(t,t_0)$ *and* $W(t)$ *are nondecreasing in* t;
(ii) $\alpha = 1$, $\max_k\{\alpha_k\} < 1$ *and* $P\int_{t_0}^{\tau} a_2(s)\mathrm{d}s < 1$;
(iii) $\alpha < 1$ *and all the impulse functions* I_k *are bounded, that is* , $b_2^{(k)}$ *in* (2.3) *are all equal to zero*;
(iv) $\max_k\{\alpha,\alpha_k\} < 1$.

Proof. For fixed $x_0, x_1 \in \mathbb{R}^n$, let the two operators $\varphi_1 : PC[t_0,\tau] \times C[t_0,\tau] \to PC[t_0,\tau]$ and $\varphi_2 : PC[t_0,\tau] \to C[t_0,\tau]$ be defined, respectively, by

$$\begin{aligned}[\varphi_1(x,u)](t) &= \Phi(t,t_0)x_0 + \Phi(t,t_0)\int_{t_0}^{t} \Phi^{-1}(s,t_0)\left[f(s,x(s)) + B(s)u(s)\right]\mathrm{d}s \\ &\quad + \sum_{k:\ t_k\in(t_0,t)} \Phi(t,t_k)I_k(x(t_k^-)), \end{aligned} \tag{3.1}$$

$$\begin{aligned} &(\varphi_2 x)(t) \\ = \;& B^T(t)\Phi^{-1}(t,t_0)^T W^{-1}(\tau)\left(\Phi^{-1}(\tau,t_0)x_1 - x_0 - \int_{t_0}^{\tau} \Phi^{-1}(s,t_0)f(s,x(s))\mathrm{d}s\right) \\ & -B^T(t)\Phi^{-1}(t,t_0)^T W^{-1}(\tau)\Phi^{-1}(\tau,t_0) \sum_{k:\ t_k\in(t_0,\tau)} \Phi(\tau,t_k)I_k(x(t_k^-)). \end{aligned} \tag{3.2}$$

Then, it is easy to see that φ_2 is continuous on $PC[t_0,\tau]$. Next, let us consider the operator Ψ defined by

$$\Psi = \varphi_1(\cdot,\varphi_2(\cdot)) : PC[t_0,\tau] \to PC[t_0,\tau].$$

Then Ψ is continuous via the continuity of φ_2. Next we show that Ψ is compact. Let $U \subset PC[t_0,\tau]$ be any bounded set. Then there exists a constant M such that

$$\sum_{k:\ t_k\in(t_0,\tau)} |\Phi(t,t_k)I_k(x(t_k^-))| \le M \quad \text{for all} \quad x \in U \quad \text{and all} \quad t \in [t_0,\tau]. \tag{3.3}$$

Further, for any $x \in U$ and any two $\sigma_1, \sigma_2 \in [t_0, \tau]$ with $\sigma_2 \geq \sigma_1$, it follows that

$$
\begin{aligned}
& |(\Psi x)(\sigma_1) - (\Psi x)(\sigma_2)| \\
\leq \ & |\Phi(\sigma_2, t_0) - \Phi(\sigma_1, t_0)|\,|x_0| + \sum_{k:\ t_k \in [\sigma_1, \sigma_2)} \left|\Phi(\sigma_2, \sigma_1)\Phi(\sigma_1, t_k) I_k(x(t_k^-))\right| \\
& + |\Phi(\sigma_2, \sigma_1) - E| \sum_{k:\ t_k \in (t_0, \sigma_1)} \left|\Phi(\sigma_1, t_k) I_k(x(t_k^-))\right| \\
& + |\Phi(\sigma_2, t_0) - \Phi(\sigma_1, t_0)| \int_{t_0}^{\tau} \left|\Phi^{-1}(s, t_0)\left[f(s, x(s)) + B(s)(\varphi_2 x)(s)\right]\right| \mathrm{d}s \\
& + |\Phi(\sigma_1, t_0)| \int_{\sigma_1}^{\sigma_2} \left|\Phi^{-1}(s, t_0)\left[f(s, x(s)) + B(s)(\varphi_2 x)(s)\right]\right| \mathrm{d}s. \qquad (3.4)
\end{aligned}
$$

Now for any given $\varepsilon > 0$ we take $\delta > 0$ so that, when $|\sigma_1 - \sigma_2| < \delta$,

$$|\Phi(\sigma_2, \sigma_1)| < \varepsilon,\ |\Phi(\sigma_2, t_0) - \Phi(\sigma_1, t_0)| < \varepsilon \ \text{ and } \ |\Phi(\sigma_2, \sigma_1) - E| < \varepsilon.$$

On the other hands, since $[t_0, \tau]$ is a finite interval, for simpleness we may set that

$$|\Phi(\sigma_1, t_0)| \leq M,\ \Phi^{-1}(s, t_0)[f(s, x(s)) + B(s)(\varphi_2 x)(s)] \leq M \ \text{ for all } \ s, \sigma_1 \in [t_0, \tau]$$

as well as

$$\int_{t_0}^{\tau} \left|\Phi^{-1}(s, t_0)\left[f(s, x(s)) + B(s)(\varphi_2 x)(s)\right]\right| \mathrm{d}s \leq M,$$

where $x \in U$. Hence, from (3.3)–(3.4) it follows that

$$|(\Psi x)(\sigma_1) - (\Psi x)(\sigma_2)| < |x_0|\varepsilon + 3M\varepsilon + M^2|\sigma_1 - \sigma_2|$$

and this implies that $\Psi(U)$ is equi-continuous. Consequently, the Arzela-Ascoli theorem [10] implies that Ψ is a compact operator.

Now we suppose that $x \in PC[t_0, \tau]$ such that

$$x = \lambda \Psi(x) \ \text{ for some } \lambda \in (0, 1). \qquad (3.5)$$

(i) For the first case, without loss of generality we suppose that $\alpha_k = 1$ for all k. In this case we set

$$b = P \sum_{k=1}^{\tau} b_2^{(k)}.$$

By the assumptions for Φ and W, from (3.1)–(3.2) and assumption (A2) it follows that

$$\begin{aligned}
&|[\varphi_1(x,\varphi_2 x)](t)| \\
\le\ & P|x_0| + P\int_{t_0}^{\tau} a_1(s)\mathrm{d}s + P\int_{t_0}^{\tau} a_2(s)||x||^{\alpha}\mathrm{d}s + P\sum_{k:\ t_k\in(t_0,\tau)} b_1^{(k)} + b||x|| + \\
& \left|\Phi(t,t_0)\int_{t_0}^{t}\Phi^{-1}(s,t_0)B(s)B^T(s)\Phi^{-1}(s,t_0)^T\mathrm{d}sW^{-1}(\tau)\ \times\right. \\
& \Phi^{-1}(\tau,t_0)\big|\sum_{k:\ t_k\in(t_0,\tau)}\left|\Phi(\tau,t_k)I_k(x(t_k^-))\right| \\
\le\ & M_0 + PQ||x||^{\alpha} + b||x|| + \left|\Phi(\tau,t_0)W(\tau)W^{-1}(\tau)\Phi^{-1}(\tau,t_0)\right| \times \\
& \left[\sum_{k:\ t_k\in(t_0,\tau)} b_1^{(k)}\left|\Phi(\tau,t_k)\right| + \sum_{k:\ t_k\in(t_0,\tau)}\left|\Phi(\tau,t_k)\right| b_2^{(k)}||x||\right] \\
\le\ & M_1 + PQ||x||^{\alpha} + 2b||x|| \ \text{ for } t\in[t_0,\tau], \qquad (3.6)
\end{aligned}$$

where we have used the relation $\Phi(t,t_0)\Phi^{-1}(s,t_0) = \Phi(t,s)$ and the inequalities of matrices, and

$$M_0 = P|x_0| + PQ + P\sum_{k:\ t_k\in(t_0,\tau)} b_1^{(k)},\ M_1 = M_0 + \sum_{k:\ t_k\in(t_0,\tau)} b_1^{(k)}\left|\Phi(\tau,t_k)\right|.$$

Now we turn to consider (3.5). By (3.6) we have

$$||x|| = ||\lambda\Psi(x)|| \le ||\varphi_1(x,\varphi_2 x)|| \le M_1 + PQ||x||^{\alpha} + 2b||x||,$$

which produces

$$(1-2b)||x|| \le M_1 + PQ||x||^{\alpha}. \qquad (3.7)$$

Note that the hypothesis $1-2b>0$, we consider the function F defined by

$$F(s) = (1-2b)s - M^2 s^{\alpha} - M_1,\ s\ge 0.$$

Then, there exists a unique $s_*>0$ such that

$$F(s_*) = 0 \ \text{ and } \ F(s)\le 0 \ \text{ for } \ s\in[0,s_*],$$

which, together with (3.7), infers that $||x|| \le s_*$ for all x satisfying (3.7). Subsequently, the set Ω in Lemma 2.2 is bounded.

(ii) For this case, similarly to (3.6) we have

$$|[\varphi_1(x,\varphi_2 x)](t)| \le M_1 + P\int_{t_0}^{\tau} a_2(s)\mathrm{d}s||x|| + 2\sum_{k:\ t_k\in(t_0,\tau)}\left|\Phi(\tau,t_k)\right| b_2^{(k)}||x||^{\alpha_k} \qquad (3.8)$$

Suppose that $\gamma = \max_k\{\alpha_k\}$ and $||x||\ge 1$. Then, by (3.8) we have

$$||x|| = ||\lambda\Psi(x)|| \le ||\varphi_1(x,\varphi_2 x)|| \le M_1 + 2b||x||^{\lambda} + P\int_{t_0}^{\tau} a_2(s)\mathrm{d}s||x||, \qquad (3.9)$$

where b is the same meaning as the case 1. Since $P\int_{t_0}^{\tau} a_2(s)\mathrm{d}s < 1$ and $\gamma\in(0,1)$, it is easy to see that (3.9) produces the set Ω in Lemma 2.2 is bounded.

(iii) For the third case, note that (2.4), we may set

$$\sum_{k:\ t_k\in(t_0,t)}\left|\Phi(t,t_k)I_k(x)\right| \le b \ \text{ for all } \ x\in\mathbb{R}^n \ \text{ and all } \ t\in[t_0,\tau].$$

In this case, by (3.2) it follows that

$$|(B\varphi_2 x)(t)| \le m_1 + m_2||x||^\alpha \quad \text{for } t \in [t_0,\tau], \tag{3.10}$$

where m_i are positive constants. Then, from (3.1) and (3.10) we have

$$\begin{aligned} ||x|| &= ||\lambda\Psi(x)|| \\ &\le ||[\varphi_1(x,\varphi_2 x)]|| \\ &\le P|x_0| + PQ + PQ||x||^\alpha + P(\tau - t_0)m_1 + Pm_2(\tau - t_0)||x||^\alpha + b. \end{aligned}$$

Similarly to the arguments above, the set Ω in Lemma 2.2 is also bounded.

(iv) For the fourth case, there exist three positive constants m_i such that it follows from (3.2) that

$$|(B\varphi_2 x)(t)| \le m_1 + m_2||x||^\alpha + +m_3 \sum_{k:\ t_k\in[t_0,\tau]} ||x||^{\alpha_k},\ t \in [t_0,\tau]. \tag{3.11}$$

Now from (3.1)–(3.2) and (3.11) we have

$$\begin{aligned} & |\varphi_1(x,\varphi_2 x)(t)| \\ \le\ & P|x_0| + PQ + PQ||x||^\alpha + P\sum_{k:\ t_k\in(t_0,\tau)} b_1^{(k)} + P\sum_{k:\ t_k\in[t_0,\tau)} b_2^{(k)}||x||^{\alpha_k} \\ & +P(\tau - t_0)m_1 + Pm_2(\tau - t_0)||x||^\alpha + Pm_3(\tau - t_0)\sum_{k:\ t_k\in[t_0,\tau)} ||x||^{\alpha_k} \end{aligned}$$

and then, there exist constants M_i such that

$$\begin{aligned} ||x|| &= ||\lambda\Psi(x)|| \\ &\le ||[\varphi_1(x,\varphi_2 x)]|| \\ &\le M_1 + M_2||x||^\alpha + M_3\sum_{k:\ t_k\in[t_0,\tau]} ||x||^{\alpha_k}. \end{aligned} \tag{3.12}$$

Since $\max_k\{\alpha,\alpha_k\} < 1$, by the way similar to the first case we can readily show that x satisfying (3.12) have a same bound. Therefore the set Ω in Lemma 2.2 is bounded again.

In a word, by Lemma 2.2 we see that there exists an $x_* \in PC[t_0,\tau]$ such that $x_*(t) = (\Psi x_*)(t) = \varphi_1(x_*,\varphi_2(x_*))(t)$ for all $t \in [t_0,\tau]$. Hence, when the input function is set by

$$\begin{aligned} & u(t) \\ =\ & B^T(t)\Phi^{-1}(t,t_0)^T W^{-1}(\tau)\left(\Phi^{-1}(\tau,t_0)x_1 - x_0 - \int_0^\tau \Phi^{-1}(s,t_0)f(s,x_*(s))\mathrm{d}s\right) \\ & -B^T(t)\Phi^{-1}(t,t_0)^T W^{-1}(\tau)\Phi^{-1}(\tau,t_0)\sum_{k:\ t_k\in(t_0,\tau)} \Phi(\tau,t_k)I_k(x_*(t_k^-)), \end{aligned}$$

$x_*(t) = \varphi_1(x_*,u)(t)$ is a solution of (1.1) with the initial condition $x(t_0) = x_0$, which exists on $[t_0,\tau]$ and satisfies that $x_*(\tau) = x_1$. The proof is complete.

Note that Lemma 2.1 shows that the solution of (1.1) with respect to any input control u and any initial condition $x(t_0) = x_0$ exists on $[t_0,\infty)$. Hence, the time τ in Theorem 3.1 can be arbitrary. Note further that the states x_0 and x_1 in the proof of Theorem 3.1 are also arbitrary. Therefore the following result is clear and the proof will be skipped.

Theorem 3.2. *Suppose that the matrix function W defined as in (2.12) is invertible for all $t > t_0$. Suppose further that the conditions (2.2)–(2.3) are fulfilled and*

$$|\Phi(t,s)| \le P \quad \text{for} \quad t \ge s \ge t_0. \tag{3.13}$$

Then the system (1.1) is completely controllable follows from one of the following conditions

(*i*) $\alpha < 1$, $\max_k\{\alpha_k\} = 1$ *and* $2P\sum_{k=1}^{\infty} b_2^{(k)} < 1$. *Moreover* , $\Phi(t,t_0)$, $\Phi^{-1}(t,t_0)$, $W(t)$ *and* $W^{-1}(t)$ *are nonnegative matrics for* $t \ge t_0$, *and* $\Phi(t,t_0)$ *and* $W(t)$ *are nondecreasing in* t;
(*ii*) $\alpha = 1$, $\max_k\{\alpha_k\} < 1$ *and* $P\int_{t_0}^{\infty} a_2(s)\mathrm{d}s < 1$;
(*iii*) $\alpha < 1$ *and all the impulse functions* I_k *are bounded, that is* , $b_2^{(k)}$ *in* (2.3) *are all equal to zero*;
(*iv*) $\max_k\{\alpha, \alpha_k\} < 1$.

Note that the definition of operator φ_2 in (3.2) depends on the matrix function W. We now consider a special case, that is, $B(t)$ in (1.1) satisfies that $B(t)B^T(t) \equiv E$ (an identity matrix). In this case we can avoid to impose the function W for the definition of operator φ_2, and the controllability criteria are simpler than the results above.

Theorem 3.3. *Suppose that the conditions (2.2)–(2.4) are fulfilled for a preassigned time $\tau > t_0$ and $B(t)B^T(t) \equiv E$. Then the system (1.1) is controllable follows from one of the following conditions*

(*i*) $\alpha < 1$, $\max_k\{\alpha_k\} = 1$ *and* $2P\sum_{k=1}^{\tau} b_2^{(k)} < 1$;
(*ii*) $\alpha = 1$, $\max_k\{\alpha_k\} < 1$ *and* $P\int_{t_0}^{\tau} a_2(s)\mathrm{d}s < 1$;
(*iii*) $\alpha < 1$ *and all the impulse functions* I_k *are bounded, that is* , $b_2^{(k)}$ *in* (2.3) *are all equal to zero*;
(*iv*) $\max_k\{\alpha, \alpha_k\} < 1$.

Proof. For fixed $x_0, x_1 \in \mathbb{R}^n$, let the operator $\varphi_1 : PC[t_0,\tau] \times C[t_0,\tau] \to PC[t_0,\tau]$ be defined as in (3.1) and $\varphi_2 : PC[t_0,\tau] \to C[t_0,\tau]$ be defined by

$$\begin{aligned} &(\varphi_2 x)(t) \\ = \;& \frac{1}{\tau - t_0} B^T(t)\Phi(t,t_0)\left(\Phi^{-1}(\tau,t_0)x_1 - x_0 - \int_{t_0}^{\tau} \Phi^{-1}(s,t_0) f(s,x(s))\mathrm{d}s\right) \\ & - \frac{1}{\tau - t_0} B^T(t)\Phi(t,\tau) \sum_{k:\ t_k \in (t_0,\tau)} \Phi(\tau,t_k) I_k(x(t_k^-)). \end{aligned} \tag{3.14}$$

The rest of the proof is similar to those in the proof of Theorem 3.1 and we skip it. The proof is complete.

Similarly to the Theorem 3.2 we have our last result as follows.

Theorem 3.4. *Suppose that $B(t)B^T(t) \equiv E$ and the conditions (2.2)–(2.3) are fulfilled. Also, suppose that*

$$|\Phi(t,s)| \le P \quad \text{for} \quad t \ge s \ge t_0. \tag{3.15}$$

Then the system (1.1) is completely controllable follows from one of the following conditions

(*i*) $\alpha < 1$, $\max_k\{\alpha_k\} = 1$ *and* $2P\sum_{k=1}^{\infty} b_2^{(k)} < 1$;
(*ii*) $\alpha = 1$, $\max_k\{\alpha_k\} < 1$ *and* $P\int_{t_0}^{\infty} a_2(s)\mathrm{d}s < 1$;
(*iii*) $\alpha < 1$ *and all the impulse functions* I_k *are bounded, that is* , $b_2^{(k)}$ *in* (2.3) *are all equal to zero*;
(*iv*) $\max_k\{\alpha, \alpha_k\} < 1$.

Next we give two examples to end this section.

Example 3.5. *Suppose in (1.1) that* $t_0 = 0$,

$$A(t) = \begin{bmatrix} -1 & 0 \\ 0 & -t \end{bmatrix} \quad \text{and} \quad B(t) \equiv \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Then (2.13) is completely controllable [11, Chapter 3]. *Now by a straightforward computing we obtain that*

$$\Phi(t,0) = \begin{pmatrix} e^{-t} & 0 \\ 0 & e^{-t^2/2} \end{pmatrix}, \; W(t) = \begin{pmatrix} \int_0^t e^{2s}\mathrm{d}s & 0 \\ 0 & \int_0^t e^{s^2}\mathrm{d}s \end{pmatrix}.$$

Suppose further that $f(t,x)$ *satisfies the condition (2.2) and the impulse functions are defined by*

$$I_k(x) = \frac{x}{4^k} \quad \text{for} \quad x \in \mathbb{R}^2 \tag{3.16}$$

where $\alpha \in (0,1)$. *Then the conditions in Theorem 3.2 are verified. Hence, system (1.1) is completely controllable.*

Example 3.6. *Let*

$$A(t) = \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad B(t) \equiv \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

and consider the system

$$x' = Ax + Bu. \tag{3.17}$$

Then (3.17) is not completely controllable[8, 11]. *In this case we have*

$$\Phi(t,s) = \begin{pmatrix} e^{-(t-s)} & 0 \\ 0 & 1 \end{pmatrix}$$

and hence $|\Phi(t,s)| \le 1$ *for all* $t \ge s \ge 0$. *Now we introduce the impulsive functions as (3.16). Then, for any given* $\tau > 0$, *if there exists a impulsive moment* $t_1 \in (0,\tau)$, *then Theorem 3.4 implies that the system (3.17) with impulse (3.16) is completely controllable.*

4. Conclusion

We remark that our discussions are based on the assumption

$$|f(t,x)| \le a_1(t) + a_2(t)|x|^{\alpha},$$

where $\alpha \in (0,1]$. Now a natural problem is whether we can consider the same problems as above under condition $\alpha > 1$. This will be discussed elsewhere.

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