

## COMMON FIXED POINTS UNDER CONTRACTIVE CONDITIONS FOR THREE MAPS IN CONE METRIC SPACES

(COMMUNICATED BY DANIEL PELLEGRINO)

M.RANGAMMA AND K.PRUDHVI

ABSTRACT. The aim of this paper is to present coincidence point result and common fixed point theorems for three mappings satisfying generalized contractive conditions without exploiting the nature of continuity of any map involved there in a cone metric space. Our results generalize and extends some recent results.

### 1. INTRODUCTION AND PRELIMINARIES

In 2007 Huang and Zhang [3] have generalized the concept of a metric space, replacing the set of real numbers by an ordered Banach space and obtained some fixed point theorems for mapping satisfying different contractive conditions. Subsequently, Abbas and Jungck [1] and Abbas and Rhoades [2] have studied common fixed point theorems in cone metric spaces (see also [3],[4] and the references mentioned therein. Recently, Stojan Radenović [5] has obtained coincidence point result for two mappings in cone metric spaces which satisfies new contractive conditions. In this paper we extend coincidence point results for three maps which satisfy generalized contractive condition without exploiting the notion of continuity.

In all that follows,  $E$  is a real Banach space. For the mappings  $f, g : X \rightarrow X$ , let  $C(f, g)$  denotes set of coincidence points of  $f, g$  that is  $C(f, g) := \{z \in X : fz = gz\}$ .

We recall some definitions of cone metric spaces and some of their properties [3].

**Definition 1.1.** Let  $E$  be a real Banach space and  $P$  be a subset of  $E$ . The set  $P$  is called a cone if and only if

- (a)  $P$  is closed, nonempty and  $P \neq \{0\}$ ;
- (b)  $a, b \in R, a, b \geq 0, x, y \in P \implies ax + by \in P$ ;
- (c)  $x \in P$  and  $-x \in P \implies x = 0$ .

**Definition 1.2.** Let  $P$  be a cone in a Banach space  $E$  define partial ordering  $\leq$  with respect to  $p$  by  $x \leq y$  if and only if  $y - x \in P$ . We shall write  $x < y$  to indicate

---

2000 *Mathematics Subject Classification.* 47H10, 54H25.

*Key words and phrases.* Cone metric space, common fixed point, coincidence point.

©2012 Universiteti i Prishtinës, Prishtinë, Kosovë.

Submitted November 10, 2011. Published January 30, 2012.

$x \leq y$  but  $x \neq y$  while  $x \ll y$  will stand for  $y - x \in \text{int}P$ , where  $\text{int}P$  denotes the interior of the set  $P$ . This Cone  $P$  is called an order cone.

**Definition 1.3.** Let  $E$  be a Banach Space and  $P \subset E$  be an order cone. The order cone  $P$  is called normal if there exists  $K > 0$  such that for all  $x, y \in E$ ,

$$0 \leq x \leq y \text{ implies } \|x\| \leq K \|y\|.$$

The least positive number  $K$  satisfying the above inequality is called the normal constant of  $P$ .

**Definition 1.4.** Let  $X$  be a nonempty set of  $E$ . Suppose that the map  $d : X \times X \rightarrow E$  satisfies

- (d1)  $0 \leq d(x, y)$  for all  $x, y \in X$  and  $d(x, y) = 0$  if and only if  $x = y$ ;
- (d2)  $d(x, y) = d(y, x)$  for all  $x, y \in X$ ;
- (d3)  $d(x, y) \leq d(x, z) + d(y, z)$  for all  $x, y, z \in X$ .

Then  $d$  is called a cone metric on  $X$  and  $(X, d)$  is called a cone metric space. It is obvious that the cone metric spaces generalize metric spaces.

**Example 1.5.** Let  $E = \mathbb{R}^2$ ,  $P = \{(x, y) \in E \text{ such that } : x, y \geq 0\} \subset \mathbb{R}^2$ ,  $X = \mathbb{R}$  and  $d : X \times X \rightarrow E$  such that  $d(x, y) = (|x - y|, \alpha|x - y|)$ , where  $\alpha \geq 0$  is a constant. Then  $(X, d)$  is a cone metric space.

**Definition 1.6.** Let  $(X, d)$  be a cone metric space. We say that  $\{x_n\}$  is

- (i) a Cauchy sequence if for every  $c$  in  $E$  with  $0 \ll c$ , there is  $N$  such that for all  $n, m > N$ ,  $d(x_n, x_m) \ll c$ ;
- (ii) a convergent sequence if for any  $0 \ll c$ , there is an  $N$  such that for all  $n > N$ ,  $d(x_n, x) \ll c$ , for some fixed  $x$  in  $X$ . We denote this  $x_n \rightarrow x$  ( $n \rightarrow \infty$ ). A cone metric space  $X$  is said to be complete if every Cauchy sequence in  $X$  is convergent in  $X$ . It is known that  $\{x_n\}$  converges to  $x$  if and only if  $\|d(x_n, x)\| \rightarrow 0$  as  $n \rightarrow \infty$ .

**Definition 1.7.** Let  $f, g : X \rightarrow X$ . Then the pair  $(f, g)$  is said to be (IT)-Commuting at  $z \in X$  if  $f(g(z)) = g(f(z))$  with  $f(z) = g(z)$ .

**Lemma 1.8.** ([3]). Let  $(X, d)$  be a cone metric space, and let  $P$  be a normal cone with normal constant  $K$ . Let  $\{x_n\}$  be a sequence in  $X$ . Then

- (i).  $\{x_n\}$  converges to  $x$  if and only if  $d(x_n, x) \rightarrow 0$  as  $(n \rightarrow \infty)$ .
- (ii).  $\{x_n\}$  is a Cauchy sequence if and only if  $d(x_n, x_m) \rightarrow 0$  as  $(n, m \rightarrow \infty)$ .

## 2. COMMON FIXED POINT THEOREMS

In this section we obtain coincidence points and common fixed point theorems for three maps in cone metric spaces.

The following theorem extends and improves Theorem 2.1 of [5].

**Theorem 2.1.** Let  $(X, d)$  be a cone metric space, and  $P$  a normal cone with normal constant  $L$ . Suppose the self maps  $f, g, h$  on  $X$  satisfy the condition

$$\|d(fx, gy)\| \leq \lambda \|d(hx, hy)\| \text{ for all } x, y \in X. \quad (2.1)$$

where  $\lambda \in (0, 1)$  is a constant . If  $f(X) \cup g(X) \subset h(X)$  and  $h(X)$  is a complete subspace of  $X$  .Then the maps  $f, g$  and  $h$  have a coincidence point  $p$  in  $X$  . Moreover if  $(f, h)$  and  $(g, h)$  are (IT)-Commuting at  $p$  , then  $f, g$  and  $h$  have a unique common fixed point.

*Proof.* Suppose  $x_0$  is an arbitrary point of  $X$ , and define the sequence  $\{y_n\}$  in  $X$  such that

$$y_{2n} = fx_{2n} = hx_{2n+1} \text{ and } y_{2n+1} = gx_{2n+1} = hx_{2n+2},$$

for all  $n= 0,1,2,\dots$  . By (2.1), we have

$$\begin{aligned} \|d(y_{2n}, y_{2n+1})\| &= \|d(fx_{2n}, gx_{2n+1})\| \\ &\leq \lambda(\|d(hx_{2n}, hx_{2n+1})\|) \\ &\leq \lambda(\|d(y_{2n}, y_{2n-1})\|). \end{aligned}$$

Similarly , it can be shown that  $\|d(y_{2n+1}, y_{2n+2})\| \leq \lambda \|d(y_{2n}, y_{2n+1})\|$  . Therefore, for all  $n$ ,

$$\|d(y_{n+1}, y_{n+2})\| \leq \lambda \|d(y_n, y_{n+1})\| \leq \dots \leq \lambda^{n+1} \|d(y_0, y_1)\| .$$

Now, for any  $m > n$ ,

$$\begin{aligned} \|d(y_n, y_m)\| &\leq \|d(y_n, y_{n+1})\| + \|d(y_{n+1}, y_{n+2})\| + \dots + \|d(y_{m-1}, y_m)\| . \\ &\leq [\lambda^n + \lambda^{n+1} + \dots + \lambda^{m-1}] \|d(y_1, y_0)\| \\ &\leq \frac{\lambda^n}{1-\lambda} \|d(y_1, y_0)\| . \end{aligned}$$

From (1.3), we have

$$\|d(y_n, y_m)\| \leq \frac{\lambda^n}{1-\lambda} K \|d(y_1, y_0)\| .$$

That is  $\|d(y_n, y_m)\| \rightarrow 0$  as  $n, m \rightarrow \infty$  , (since  $0 < \lambda < 1$ ).

Hence  $\{y_n\}$  is a Cauchy sequence , where  $y_n = \{hx_n\}$ .

Therefore  $\{hx_n\}$  is a Cauchy sequence. Since  $h(X)$  is complete, there exists  $q$  in  $h(X)$  such that  $\{hx_n\} \rightarrow q$  as  $n \rightarrow \infty$  . Consequently , we can find  $p$  in  $X$  such that  $h(p) = q$ . We shall show that  $hp = fp = gp$ .

Note that  $\|d(hp, fp)\| = \|d(q, fp)\|$  . Let us estimate  $\|d(hp, fp)\|$  .

We have , by the triangle inequality

$$\|d(hp, fp)\| \leq \|d(hp, hx_{2n+2})\| + \|d(hx_{2n+2}, fp)\|$$

$$= \|d(q, hx_{2n+2})\| + \|d(fp, gx_{2n+1})\| .$$

By the contractive condition , we get

$$\begin{aligned} \|d(fp, gx_{2n+1})\| &\leq \lambda(\|d(hp, hx_{2n+1})\|) . \\ &= \lambda(\|d(q, hx_{2n+1})\|) \rightarrow 0 \text{ (as } n \rightarrow \infty \text{)} . \end{aligned}$$

Therefore , for large  $n$  , we get

$$d(hp, fp) \leq d(q, hx_{2n+2}) \leq d(q, q) = 0$$

which leads to  $d(hp, fp) = 0$  and hence

$$hp = q = fp. \quad (2.2)$$

Similarly , we can show

$$hp = q = gp. \quad (2.3)$$

From (2.2) and (2.3) , it follows that

$$q = hp = fp = gp , p \text{ is a coincidence point of } f, g, h. \quad (2.4)$$

Since ,  $(f, h)$  ,  $(g, h)$  are  $(IT)$ -commuting at  $p$ . We get by (2.4) and contraction condition,

$$\begin{aligned} \| d(ffp, fp) \| = \| d(ffp, gp) \| &\leq \lambda(\| d(hfp, hp) \|) \\ &< \| d(hfp, hp) \| \\ &= \| d(fh p, fp) \| \\ &= \| d(ffp, fp) \|, \end{aligned}$$

$$\Rightarrow \| d(ffp, fp) \| < \| d(ffp, fp) \|,$$

a contradiction, (since  $\lambda < 1$  and  $fp = hp$ ).

Therefore ,  $ffp = fp \cdot fp = ffp = fh p = hfp$ ,

$$\Rightarrow ffp = hfp = fp = q.$$

Therefore,  $fp(= q)$  is a common fixed point of  $f$  and  $h$ . (2.5)

Similarly , we get

$$gp = ggp = ghp = hgp,$$

$$\Rightarrow ggp = hgp = gp = q.$$

Therefore,  $gp = (fp)(= q)$  is a common fixed point of  $g$  and  $h$ . (2.6)

In view of (2.5) and (2.6) it follows that  $f$ ,  $g$  and  $h$  have a common fixed point namely  $q$ . The uniqueness of the common fixed point of  $q$  follows equation (2.1). Indeed , let  $q_1$  be another common fixed point of  $f$ ,  $g$  and  $h$ . Consider,

$$\begin{aligned} \| d(q, q_1) \| &= \| d(fq, gq_1) \| \\ &\leq \lambda(\| d(hq, hq_1) \|) \\ &= \lambda(\| d(q, q_1) \|). \end{aligned}$$

As  $0 < \lambda < 1$  , it follows that  $\| d(q, q_1) \| = 0$ , that is  $q = q_1$ .

Therefore  $f$  ,  $g$  and  $h$  have a unique common fixed point . □

**Remark 2.2.** *If we take  $g = f$  and  $h = g$  in Theorem 2.1 , then we obtain Theorem 2.1 of [5].*

The following theorem extends and improves Theorem 2.3 of [5].

**Theorem 2.3.** *Let  $(X, d)$  be a cone metric space, and  $P$  a normal cone with normal constant  $L$ . Suppose the self maps  $f, g, h$  on  $X$  satisfy the contractive condition*

$$\| d(fx, gy) \| \leq \lambda (\| d(fx, hx) \| + \| d(gy, hy) \|), \text{ for all } x, y \in X. \quad (2.7)$$

where  $\lambda \in [0, 1/2)$  is a constant. If  $f(X) \cup g(X) \subset h(X)$  and  $h(X)$  is a complete subspace of  $X$ . Then the maps  $f, g$  and  $h$  have a coincidence point  $p$  in  $X$ . Moreover if  $(f, h)$  and  $(g, h)$  are (IT)-Commuting at  $p$ , then  $f, g$  and  $h$  have a unique common fixed point.

*Proof.* Suppose  $x_0$  is an arbitrary point of  $X$ , and define the sequence  $\{y_n\}$  in  $X$ , such that

$$y_{2n} = fx_{2n} = hx_{2n+1} \text{ and } y_{2n+1} = gx_{2n+1} = hx_{2n+2},$$

for all  $n = 0, 1, 2, \dots$ . By (2.7), we have

$$\begin{aligned} \| d(y_{2n}, y_{2n+1}) \| &= \| d(fx_{2n}, gx_{2n+1}) \| \\ &\leq \lambda (\| d(fx_{2n}, hx_{2n}) \| + \| d(gx_{2n+1}, hx_{2n+1}) \|) \\ &= \lambda (\| d(y_{2n}, y_{2n-1}) \| + \| d(y_{2n+1}, y_{2n}) \|) \\ (1 - \lambda) \| d(y_{2n}, y_{2n+1}) \| &\leq \lambda \| d(y_{2n}, y_{2n-1}) \| \\ \implies \| d(y_{2n}, y_{2n+1}) \| &\leq \delta \| d(y_{2n-1}, y_{2n}) \|, \text{ where } \delta = \frac{\lambda}{1 - \lambda}. \end{aligned}$$

Similarly, it can be shown that

$$\| d(y_{2n+1}, y_{2n+2}) \| \leq \delta \| d(y_{2n}, y_{2n+1}) \|.$$

Therefore, for all  $n$ ,

$$\| d(y_{n+1}, y_{n+2}) \| \leq \delta \| d(y_n, y_{n+1}) \| \leq \dots \leq \delta^{n+1} \| d(y_0, y_1) \|.$$

Now, for any  $m > n$ ,

$$\begin{aligned} \| d(y_n, y_m) \| &\leq \| d(y_n, y_{n+1}) \| + \| d(y_{n+1}, y_{n+2}) \| + \dots + \| d(y_{m-1}, y_m) \| \\ &\leq [\delta^n + \delta^{n+1} + \dots + \delta^{m-1}] d(y_1, y_0) \\ &\leq \frac{\delta^n}{1 - \delta} \| d(y_1, y_0) \|. \end{aligned}$$

From (1.3), we have

$$\| d(y_n, y_m) \| \leq \frac{\delta^n}{1 - \delta} K \| d(y_1, y_0) \|,$$

since  $\delta \in (0, 1)$ . According to Lemma (1.8),  $\{y_n\}$  is a Cauchy sequence, where  $y_n = \{hx_n\}$ . Therefore  $\{hx_n\}$  is a Cauchy sequence. Since  $h(X)$  is complete, there exists  $q$  in  $h(X)$  such that  $hx_n \rightarrow q$  as  $n \rightarrow \infty$ . Consequently, we can find  $p$  in  $X$  such that  $h(p) = q$ . We shall show that  $hp = fp = gp$ .

Consider,

$$\|d(fp, gx_{2n+1})\| \leq \lambda(\|d(fp, hp)\| + \|d(gx_{2n+1}, hx_{2n+1})\|).$$

Letting  $n \rightarrow \infty$ , we get

$$\begin{aligned} \|d(fp, q)\| &\leq \lambda(\|d(fp, hp)\| + \|d(q, q)\|) \\ &\leq \lambda \|d(fp, q)\| \\ \|d(fp, q)\| &< 1/2 (\|d(fp, q)\|), \text{ a contradiction.} \end{aligned}$$

$$\text{Therefore, } fp=q=hp. \quad (2.8)$$

$$\text{similarly, } \|d(gp, fx_{2n})\| \leq \lambda(\|d(gp, hp)\| + \|d(fx_{2n}, hx_{2n})\|).$$

Letting  $n \rightarrow \infty$ ,

$$\begin{aligned} \|d(gp, q)\| &\leq \lambda(\|d(gp, hp)\| + \|d(hp, hp)\|) \\ &\leq \lambda \|d(gp, hp)\| \\ &< \|d(gp, q)\|, \text{ with } \lambda < 1, \text{ a contradiction.} \end{aligned}$$

$$\text{Therefore, } g(p)=q=h(p). \quad (2.9)$$

From (2.8) and (2.9), it follows that

$$q = hp = fp = gp, \text{ } p \text{ is a coincidence point of } f, g, h. \quad (2.10)$$

Since,  $(f, h)$ ,  $(g, h)$  are  $(IT)$ -commuting at  $p$ . we get by (2.10) and contractive condition ,

$$\begin{aligned} \|d(ffp, fp)\| = \|d(ffp, gp)\| &\leq \lambda(\|d(ffp, hfp)\| + \|d(gp, hp)\|) \\ &\leq \lambda(\|d(ffp, fhp)\|) = 0, \text{ (since } fp = hp) \end{aligned}$$

which implies  $ffp = fp$ .  $fp = ffp = fhp = hfp$ ,

$$\Rightarrow ffp = hfp = fp = q. \quad (2.11)$$

$$\text{Therefore, } fp(=q) \text{ is a common fixed point of } f \text{ and } h. \quad (2.12)$$

similarly, we get,  $gp = ggp = ghp = hgp$ ,

$$\Rightarrow ggp = hgp = gp = q. \quad (2.13)$$

$$\text{Therefore, } gp = (fp)(=q) \text{ is a common fixed point of } g \text{ and } h \quad (2.14)$$

In view of (2.12) and (2.14) it follows that  $f$ ,  $g$  and  $h$  have a common fixed point namely  $q$ . The uniqueness of the common fixed point of  $q$  follows equation (2.7). Indeed, let  $q_1$  be another common fixed point of  $f$ ,  $g$  and  $h$ . Consider,

$$\begin{aligned} \|d(q, q_1)\| = \|d(fq, gq_1)\| &\leq \lambda(\|d(fq, hq) + d(gq_1, hq_1)\|) \\ &= \lambda(\|d(hqh) + \|d(hq_1, hq_1)\|) \end{aligned}$$

$$\Rightarrow \|d(q, q_1)\| \leq 0. \text{ Thus } q = q_1.$$

Therefore  $f$ ,  $g$  and  $h$  have a unique common fixed point .  $\square$

**Remark 2.4.** If we take  $g = f$  and  $h = g$  in Theorem 2.3, then we obtain Theorem 2.3 of [5].

**Acknowledgments.** The authors are grateful to the referee and Daniel Pellegrino for careful reading of the article and giving valuable suggestions.

#### REFERENCES

- [1] M.Abbas and G.Jungck, Common fixed point results for non commuting mappings without continuity in cone metric spaces. *J. Math. Anal.Appl.* **341(2008)**, 416-420.
- [2] M.Abbas and B.E.Rhoades, Fixed and periodic point results in cone metric Spaces. *Appl. Math. Lett.* **22(2009)**, 511-515.
- [3] L.G.Huang, X.Zhang, Cone metric spaces and fixed point theorems of contractive mappings, *J. Math. Anal. Appl.* **332(2)(2007)**, 1468-1476.
- [4] S.Rezapour and Halbarani, Some notes on the paper "cone metric spaces and fixed point theorem of contractive mappings " , *J. Math. Anal. Appl.* **345(2008)**, 719-724.
- [5] Stojan Radenović , Common fixed points under contractive conditions in cone metric spaces, *Computers and Mathematics with Applications* 58 (2009) 1273- 1278.

M.RANGAMMA AND K.PRUDHVI

DEPARTMENT OF MATHEMATICS ,OSMANIA UNIVERSITY,HYDERABAD, ANDHRA PRADESH,INDIA

*E-mail address:* rangamma1999@gmail.com ; prudhvikasani@rocketmail.com