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INEQUALITIES CONCERNING POLYNOMIALS HAVING ZEROS IN CLOSED INTERIOR OF A CIRCLE

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Abstract. In this paper, we have obtained certain inequalities for polynomials having zeros in closed interior of a circle. Our result gives the generalization of the known result.

1. Introduction and Statement of Results

Let $P(z)$ be a polynomial of degree n and let $M(P,R) = Max_{|z|=R}|P(z)|$, $m(P,k) = Min_{|z|=k}|P(z)|$, then by maximum modulus principle [4, p. 158 problem III 267 and 269], we have

$$M(P,r) \geq r^n M(P,1), \ for\ r < 1, \tag{1.1}$$

with equality only for $P(z) = \alpha z^n$, $|\alpha| = 1$.

Rivlin [5] obtained stronger inequality and proved that if $P(z)$ is a polynomial of degree n having all its zeros in the disk $|z| \geq 1$, then

$$M(P,r) \geq (\frac{1+r}{2})^n \ M(P,1) \ for\ r < 1. \tag{1.2}$$

Here equality holds for $P(z) = (\alpha + \beta z)^n$, $|\alpha| = |\beta|$.

For the polynomials of degree n not vanishing in $|z| < k$, $k > 0$, Aziz [1] obtained the following generalization of (1.2).

Theorem 1.1 *Let $P(z)$ be a polynomial of degree n, having no zeros in the disk $|z| < k$, $k > 0$, then*

$$M(P,r) \geq (\frac{r+k}{1+k})^n M(P,1), \ for\ k \geq 1\ and\ r < 1\ or\ k < 1\ and\ r \leq k^2. \tag{1.3}$$

Here equality holds for $P(z) = (z+k)^n$.

By using Theorem 1.1 to the polynomial $z^n P(\frac{1}{z})$, Aziz [1] obtained the following :

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Theorem 1.2 *Let $P(z)$ be a polynomial of degree n, having all its zeros in the disk $|z| \leq k$, $k > 0$, then*

$$M(P,R) \geq (\frac{R+k}{1+k})^n M(P,1), \ for\ k \leq 1\ and\ R > 1\ or\ k > 1\ and\ R \geq k^2. \quad (1.4)$$

Here equality holds for the polynomial $P(z) = (z+k)^n$.

For the polynomials having all their zeros in $|z| \leq k$, $k > 1$, Jain [3] proved the following:

Theorem 1.3 *Let $P(z)$ be a polynomial of degree n, having all its zeros in the disk $|z| \leq k$, $k > 1$, then for $k < R < k^2$,*

$$M(P,R) \geq R^s(\frac{R+k}{1+k})M(P,1), \ for\ s < n. \quad (1.5)$$

where s is the order of a possible zero of $P(z)$ at z=0.

In this paper, we have obtained the following generalization of Theorem 1.3 by involving the coefficients of the polynomial $P(z) := \sum_{j=0}^{n} a_j z^j$ of degree n having all its zeros in the disk $|z| \leq k$, $k > 1$ with s-fold zeros at the origin. In fact we prove:

Theorem 1.4 *Let $P(z) := \sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n having all its zeros in $|z| \leq k$, $k > 1$, then for $k < R < k^2$,*

$$M(P,R) \geq \frac{R^n[(n-s)(k^2+R^2)|a_n|+2R|a_{n-1}|]}{(n-s)(R^{n-s}k^2+R^2)|a_n|+R(R^{n-s}+1)|a_{n-1}|} Max_{|z|=1}|P(z)| + \frac{R^{s+1}(R^{n-s}-1)[(n-s)R|a_n|+|a_{n-1}|]}{k^s[(n-s)(R^{n-s}k^2+R^2)|a_n|+R(R^{n-s}+1)|a_{n-1}|]} m(P,k),$$

where s is the order of a possible zeros of P(z) at z=0.

2. Lemmas

The following lemma is due to Dewan, Singh and Yadav [2].

Lemma 2.1 *If $P(z) := \sum_{j=0}^{n} a_j z^j$ is a polynomial of degree n having no zeros in the disk $|z| < k$, $k \geq 1$, then*

$$Max_{|z|=1}|P'(z)| \leq n\frac{n|a_0|+k^2|a_1|}{n(1+k^2)|a_0|+2k^2|a_1|} Max_{|z|=1}|P(z)| - \{1 - \frac{n|a_0|+k^2|a_1|}{n(1+k^2)|a_0|+2k^2|a_1|}\}\frac{m(P,k)n}{k^n},$$

where $m(P,k) = Min_{|z|=k}|P(z)|$.

Lemma 2.2 *If $P(z) := \sum_{j=0}^{n} a_j z^j$ is a polynomial of degree n having all its zeros in the disk $|z| \geq k$, $k > 0$, then for $r \leq k \leq R$,*

$M(P,r)\geq$

$$\frac{nr^{n-1}(r^2+k^2)|a_0|+2k^2r^n|a_1|}{n(rR^n+r^{n-1}k^2)|a_0|+k^2(R^n+r^n)|a_1|}M(P,R) + \frac{r^{n-1}(R^n-r^n)(n|a_0|+r|a_1|)}{k^{n-2}[n(rR^n+r^{n-1}k^2)|a_0|+k^2(R^n+r^n)|a_1|]}m(P,k),$$

where $m = Min_{|z|=k}|P(z)|$.

Proof of lemma 2.2 Let $r \leq k \leq R$, then the polynomial $G(z) = P(rz)$ has no zeros in $|z| < \frac{k}{r}$. As $\frac{k}{r} \geq 1$, we have by lemma 2.1,

$$M(G',1) \leq n\frac{n|a_0|+\frac{k^2}{r^2}r|a_1|}{n(1+\frac{k^2}{r^2})|a_0|+2\frac{k^2}{r^2}r|a_1|}M(G,1) - \{1-\frac{n|a_0|+\frac{k^2}{r^2}r|a_1|}{n(1+\frac{k^2}{r^2})|a_0|+2\frac{k^2}{r^2}r|a_1|}\}Min_{|z|=\frac{k}{r}}|G(z)|\frac{n}{\frac{k^n}{r^n}},$$

or

$$M(P',r) \leq n\frac{nr|a_0|+k^2|a_1|}{n(r^2+k^2)|a_0|+2k^2r|a_1|}M(P,r) - \{1-\frac{nr^2|a_0|+k^2r|a_1|}{n(r^2+k^2)|a_0|+2k^2r|a_1|}\}\frac{m(P,k)nr^{n-1}}{k^n}, \quad (2.1)$$

Since $P'(z)$ is a polynomial of degree (n-1), we have by maximum modulus principle [4],

$$\frac{M(P',t)}{t^{n-1}} \leq \frac{M(P',r)}{r^{n-1}},\ t\geq r \quad (2.2)$$

Combining (2.1) and (2.2), we get

$$M(P',t) \leq \frac{t^{n-1}}{r^{n-1}}[n\frac{nr|a_0|+k^2|a_1|}{n(r^2+k^2)|a_0|+2k^2r|a_1|}M(P,r) - \{1-\frac{nr^2|a_0|+k^2r|a_1|}{n(r^2+k^2)|a_0|+2k^2r|a_1|}\}\frac{m(P,k)nr^{n-1}}{k^n}],\ t\geq r. \quad (2.3)$$

Now we have, for $0\leq\theta<2\pi$

$$|P(Re^{i\theta})-P(re^{i\theta})| \leq \int_r^R |P'(te^{i\theta})|dt$$

$$\leq n\frac{nr|a_0|+k^2|a_1|}{nr^{n-1}(r^2+k^2)|a_0|+2k^2r^n|a_1|}M(P,r)\int_r^R t^{n-1}dt - \frac{n}{k^n}\{1-\frac{nr^2|a_0|+k^2r|a_1|}{n(r^2+k^2)|a_0|+2k^2r|a_1|}\}m(P,k)\int_r^R t^{n-1}dt$$

$$= (R^n-r^n)\frac{nr|a_0|+k^2|a_1|}{nr^{n-1}(r^2+k^2)|a_0|+2k^2r^n|a_1|}M(P,r) - \frac{(R^n-r^n)}{k^n}\{1-\frac{nr^2|a_0|+k^2r|a_1|}{n(r^2+k^2)|a_0|+2k^2r|a_1|}\}m(P,k),$$

which is equivalent to

$$M(P,R) \leq (R^n-r^n)\frac{nr|a_0|+k^2|a_1|}{nr^{n-1}(r^2+k^2)|a_0|+2k^2r^n|a_1|}M(P,r) - \frac{(R^n-r^n)}{k^n}\{1-\frac{nr^2|a_0|+k^2r|a_1|}{n(r^2+k^2)|a_0|+2k^2r|a_1|}\}m(P,k)+M(P,r).$$

After by simple calculation, we get

$$M(P,r) \geq \frac{nr^{n-1}(r^2+k^2)|a_0|+2k^2r^n|a_1|}{n(rR^n+r^{n-1}k^2)|a_0|+k^2(R^n+r^n)|a_1|}M(P,R)$$
$$+\frac{r^{n-1}(R^n-r^n)(n|a_0|+r|a_1|)}{k^{n-2}[n(rR^n+r^{n-1}k^2)|a_0|+k^2(R^n+r^n)|a_1|]}m(P,k).$$

This completes the proof of Lemma 2.2.

3. Proof of the Theorem 1.4

The polynomial $Q(z) = z^n\overline{P(\frac{1}{\bar{z}})}$ has all its zeros in $|z| \geq \frac{1}{k}$, $\frac{1}{k} < 1$ and is of degree $n-s$. By applying Lemma 2.2 to the polynomial $Q(z)$ with R=1, we have

$$M(Q,r) \geq \frac{(n-s)r^{n-s-1}(r^2+\frac{1}{k^2})|a_n|+\frac{2}{k^2}r^{n-s}|a_{n-1}|}{(n-s)(r+\frac{r^{n-s-1}}{k^2})|a_n|+\frac{1}{k^2}(1+r^{n-s})|a_{n-1}|}M(Q,1)$$

$$+k^{n-s-2}\frac{r^{n-s-1}(1-r^{n-s})((n-s)|a_n|+r|a_{n-1}|)}{(n-s)(r+\frac{r^{n-s-1}}{k^2})|a_n|+\frac{1}{k^2}(1+r^{n-s})|a_{n-1}|}Min_{|z|=\frac{1}{k}}|Q(z)|,\ \frac{1}{k^2} < r < \frac{1}{k},$$

which is equivalent to

$$r^nMax_{|z|=\frac{1}{r}}|P(z)| \geq \frac{(n-s)r^{n-s-1}(r^2+\frac{1}{k^2})|a_n|+\frac{2}{k^2}r^{n-s}|a_{n-1}|}{(n-s)(r+\frac{r^{n-s-1}}{k^2})|a_n|+\frac{1}{k^2}(1+r^{n-s})|a_{n-1}|}Max_{|z|=1}|P(z)|$$
$$+k^{n-s-2}\frac{r^{n-s-1}(1-r^{n-s})((n-s)|a_n|+r|a_{n-1}|)}{(n-s)(r+\frac{r^{n-s-1}}{k^2})|a_n|+\frac{1}{k^2}(1+r^{n-s})|a_{n-1}|}\frac{1}{k^n}Min_{|z|=k}|P(z)|,\ \frac{1}{k^2} < r < \frac{1}{k},$$

which on simplification and by replacing r by $\frac{1}{R}$, we get

$$M(P,R) \geq \frac{R^n[(n-s)(k^2+R^2)|a_n|+2R|a_{n-1}|]}{(n-s)(R^{n-s}k^2+R^2)|a_n|+R(R^{n-s}+1)|a_{n-1}|}Max_{|z|=1}|P(z)|$$

$$+\frac{R^{s+1}(R^{n-s}-1)[(n-s)R|a_n|+|a_{n-1}|]}{k^s[(n-s)(R^{n-s}k^2+R^2)|a_n|+R(R^{n-s}+1)|a_{n-1}|]}Min_{|z|=k}|P(z)|,\ k>1 \ and\ k<R<k^2.$$

This completes the proof of the Theorem 1.4.

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