Bulletin of Mathematical Analysis and Applications
ISSN: 1821-1291, URL: http://www.bmathaa.org
Volume 4 Issue 1(2012), Pages 208-213.

AN EXISTENCE THEOREM FOR A CLASS OF NONLINEAR DIRICHLET SYSTEMS

(COMMUNICATED BY VICENTIU RADULESCU)

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Abstract. In this article, we discuss the existence of weak solution for the nonlinear system

$$\begin{cases} -\operatorname{div}\Big(h_1(|\nabla u|^p)\ |\nabla u|^{p-2}\nabla u\Big) &= f(x,u,v) \quad \text{in } \Omega,\\ -\operatorname{div}\Big(h_2(|\nabla v|^p)\ |\nabla v|^{p-2}\nabla v\Big) &= g(x,u,v) \quad \text{in } \Omega,\\ u=v &= 0 \quad \text{on } \partial\Omega, \end{cases}$$

where Ω is a bounded smooth open set in R^n, $p \geq 2$ and $h_1, h_2 \in C(R,R)$. Using variational methods, under suitable assumptions on the nonlinearities, we show the existence of weak solution.

1. Introduction

In this paper, we study the existence of weak solution of the following Dirichlet system

$$\begin{cases} -\operatorname{div}\Big(h_1(|\nabla u|^p)\ |\nabla u|^{p-2}\nabla u\Big) &= f(x,u,v) \quad \text{in } \Omega,\\ -\operatorname{div}\Big(h_2(|\nabla v|^p)\ |\nabla v|^{p-2}\nabla v\Big) &= g(x,u,v) \quad \text{in } \Omega,\\ u=v &= 0 \quad \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded smooth open set in R^n, $2 \leq p$ and $h_1, h_2 \in C(R,R)$.

Elliptic systems have several practical applications. For example they can describe the multiplicative chemical reaction catalyzed by grains under constant or variant temperature, a correspondence of the stable station of dynamical system determined by the reaction-diffusion system. In recent years, many publications have appeared concerning quasilinear elliptic systems which have been used in a great variety of applications, we refer the readers to [2, 3, 4, 5, 6, 7] and the references therein. J. Zhang and Z. Zhang [8] used variational methods to obtain weak

2000 *Mathematics Subject Classification.* 35B30, 35J60, 35P15.
Key words and phrases. Weak solution; Nonlinear system; Variational methods.

Submitted August 26, 2011. Published December 1, 2011.

solution of the nonlinear elliptic system (1.1) with $p = 2$.

Motivated by [8], in this paper, we will discuss problem (1.1). Through this paper for $(u, v) \in R^2$, denote $|(u, v)|^2 = |u|^2 + |v|^2$. We assume that $F : \Omega \times R^2 \to R$ is of C^1 class such that $F(x, 0, 0) = 0$ for all $x \in \overline{\Omega}$ and $(f, g) = (\frac{\partial F}{\partial u}, \frac{\partial F}{\partial v})$, f and g are caratheodory functions satisfying the following growth conditions:

(**i**) $\lim_{|u|\to\infty} \frac{|f(x,u,v)|}{|u|^{p-1}} = 0,\ \lim_{|v|\to\infty} \frac{|g(x,u,v)|}{|v|^{p-1}} = 0.$
uniformly in $(x, v) \in \overline{\Omega} \times R$ and $(x, u) \in \overline{\Omega} \times R$

(**ii**) Let h_1 and $h_2 \in C(R, R)$. We assume that h_1 and h_2 are the continuous and nondecreasing functions satisfying the following growth conditions: There exist $\alpha_1, \alpha_2, \beta_1$ and $\beta_2 \in R$ such that

$$0 < \alpha_1 \le h_1(t) \le \beta_1,$$

$$0 < \alpha_2 \le h_2(t) \le \beta_2.$$

The main result of this paper is the following:

Theorem 1.1. *Assume that* (**i**) $-$ (**ii**) *hold. Then system (1.1) has at least one weak solution.*

The plan of this paper is as follows. In section 2, we give some notations and recall some relevant lemmas. The main result is proved in section 3.

2. Notations and preliminary lemmas

Let the product space $H = H_0^{1,p}(\Omega) \times H_0^{1,p}(\Omega)$ with the norm $\|(u, v)\|_H = \|u\|_{1,p} + \|v\|_{1,p} = (\int_\Omega |\nabla u|^p\ dx)^{\frac{1}{p}} + (\int_\Omega |\nabla v|^p\ dx)^{\frac{1}{p}}$.
Let us define the mappings

$$h(u, v) = \frac{1}{p}\int_0^u h_1(s)\ ds + \frac{1}{q}\int_0^v h_2(s)\ ds$$
$$J(u, v) = \int_\Omega h(|\nabla u|^p, |\nabla v|^p)\ dx$$

and $J' : H \to H^*$ by

$$\langle J'(u, v), (\xi, \eta)\rangle = \int_\Omega [h_1(|\nabla u|^p)|\nabla u|^{p-2}\nabla u \nabla \xi + h_2(|\nabla v|^p)|\nabla v|^{p-2}\nabla v \nabla \eta]\ dx$$

for any $(u, v), (\xi, \eta) \in H$.
Let us define the mapping

$$\widehat{W}(u, v) = \int_\Omega F(x, u, v)\ dx$$

and $\widehat{W}' : H \to H^*$ by

$$\langle \widehat{W}'(u, v), (\xi, \eta)\rangle = \int_\Omega [f(x, u, v)\xi + g(x, u, v)\eta]\ dx$$

for any $(u,v),(\xi,\eta)\in H$.

As usual, a weak solution of system (1.1) is any $(u,v)\in H$ such that

$$\langle J'(u,v),(\xi,\eta)\rangle = \langle \widehat{W}'(u,v),(\xi,\eta)\rangle$$

for any $(\xi,\eta)\in H$.

We need certain properties of functional $J: H\to R$ defined by

$$J(u,v)=\frac{1}{p}\int_\Omega\int_0^{|\nabla u|^p} h_1(s)\ ds+\frac{1}{p}\int_\Omega\int_0^{|\nabla v|^p} h_2(s)\ ds \tag{2.1}$$

for all $(u,v)\in H$.

Lemma 2.1. *The functional J given by (2.1) is weakly lower semicontinuous.*

Proof. Let $(u_1,v_1)\in H$ and $\epsilon>0$ be fixed. Using the properties of lower semicontinuous function (see [1], section I.3) is enough to prove that there exists $\delta>0$ such that

$$J(u,v)\geq J(u_1,v_1)-\epsilon\ \forall\ (u,v)\in H\ \|(u,v)-(u_1,v_1)\|<\delta. \tag{2.2}$$

Using hypotheses (**ii**), it is easy to check that J is convex. Hence we have

$$J(u,v)\geq J(u_1,v_1)+\langle J'(u_1,v_1),(u-u_1,v-v_1)\rangle\ \forall\ (u,v)\in H.$$

Using condition (**ii**) and Holder's inequality we deduce there exists a positive constant $c>0$ such that

$$\begin{aligned}J(u,v)&\geq J(u_1,v_1)-\int_\Omega |h_1(|\nabla u_1|^p)|\ |\nabla u_1|^{p-2}|\nabla u_1|\ |\nabla u-\nabla u_1|dx\\&\quad-\int_\Omega |h_2(|\nabla v_1|^p)|\ |\nabla v_1|^{p-2}|\nabla v_1|\ |\nabla v-\nabla v_1|dx\\&\geq J(u_1,v_1)-\beta_1\|u_1\|_{1,p}^{p-1}\|u-u_1\|_{1,p}-\beta_2\|v_1\|_{1,p}^{p-1}\|v-v_1\|_{1,p}\\&\geq J(u_1,v_1)-c\|(u-u_1,v-v_1)\|_H\end{aligned}$$

for all $(u,v)\in H$.

It is clear that taking $\delta=\frac{\epsilon}{c}$ relation (2.2) holds true for all $(u,v)\in H$ with $\|(u,v)-(u_1,v_1)\|_H<\delta$. Thus we proved that J is strongly lower semicontinuous. Taking into account the fact that J is convex then by [1], corollary III.8, we conclude that J is weakly lower semicontinuous and the proof of Lemma (2.1) is complete. □

Lemma 2.2. *The Functional $\widehat{W}$ is weakly continuous.*

Proof. Let $\{w_n\}=\{(u_n,v_n)\}$ be a sequence converges weakly to $w=(u,v)$ in H. We will show that

$$\lim_{n\to\infty}\int_\Omega F(x,u_n,v_n)\ dx=\int_\Omega F(x,u,v)\ dx. \tag{2.3}$$

From (**i**) and the continuity of the potential F, for any $\epsilon > 0$, there exists a positive constant $M = M(\epsilon)$ such that

$$|f(x,u,v)| \leq \epsilon |u|^{p-1} + M_\epsilon \qquad |g(x,u,v)| \leq \epsilon |v|^{p-1} + M_\epsilon \tag{2.4}$$

for all $(x,u,v) \in \overline{\Omega} \times R^2$. Hence

$$\begin{aligned}
&\int_\Omega | \ F(x,u_n,v_n) - F(x,u,v)| \ dx \\
&\quad = \int_\Omega \nabla F(x, w + \theta_n (w_n - w)) \ (w_n - w) \ dx \\
&\quad = \int_\Omega F_u(x, u + \theta_{1,n}(u_n - u), v + \theta_{2,n}(v_n - v)) \ (u_n - u) \ dx \\
&\qquad\qquad + \int_\Omega F_v(x, u + \theta_{1,n}(u_n - u), v + \theta_{2,n}(v_n - v)) \ (v_n - v) \ dx
\end{aligned}$$

where $\theta_n = (\theta_{1,n}, \theta_{2,n})$ and $0 \leq \theta_{1,n}(x), \theta_{2,n}(x) \leq 1$ for all $x \in \Omega$. Now, using (2.4) and Holders inequality we conclude that

$$\begin{aligned}
&| \int_\Omega [F(x,u_n,v_n) - F(x,u,v)] \ dx| \\
&\quad \leq \int_\Omega |F_u(x, u + \theta_{1,n}(u_n - u), v + \theta_{2,n}(v_n - v)| \ |u_n - u| \ dx \\
&\qquad + \int_\Omega |F_v(x, u + \theta_{1,n}(u_n - u), v + \theta_{2,n}(v_n - v)| \ |v_n - v| \ dx \\
&\quad \leq \int_\Omega (\epsilon \ |u + \theta_{1,n}(u_n - u)|^{p-1} + M_\epsilon) \ |u_n - u| \ dx \\
&\qquad + \int_\Omega (\epsilon| \ v + \theta_{2,n}(v_n - v)|^{p-1} + M_\epsilon) \ |v_n - v| \ dx \\
&\quad \leq M_\epsilon |\Omega|^{\frac{p-1}{p}} \|u_n - u\|_{L^p(\Omega)} + \epsilon \|u + \theta_{1,n}(u_n - u)\|^{p-1}_{L^p(\Omega)} \ \|u_n - u\|_{L^p(\Omega)} \\
&\qquad + M_\epsilon |\Omega|^{\frac{p-1}{p}} \|v_n - v\|_{L^p(\Omega)} + \epsilon \|v + \theta_{2,n}(v_n - v)\|^{p-1}_{L^p(\Omega)} \ \|v_n - v\|_{L^p(\Omega)}
\end{aligned} \tag{2.5}$$

on the other hand, since $H \hookrightarrow L^i(\Omega) \times L^j(\Omega)$ is compact for all $i \in [p, p^*)$ and $j \in [p, p^*)$ the sequence $\{w_n\}$ converges to $w = (u, v)$ in the space $L^p(\Omega) \times L^p(\Omega)$, i.e., $\{u_n\}$ converges strongly to u in $L^p(\Omega)$ and $\{v_n\}$ converges strongly to v in $L^p(\Omega)$. Hence, it is easy to see that the sequences $\{\|u + \theta_{1,n}(u_n - u)\|_{L^p(\Omega)}\}$ and $\{\|v + \theta_{2,n}(v_n - v)\|_{L^p(\Omega)}\}$ are bounded. Thus, it follows from (2.5) that relation (2.3) holds true. □

3. Proof of main theorem

In this section we give the proof of theorem 1.1.

Proof. Let $J(u,v) = \int_\Omega h(|\nabla u|^p, |\nabla v|^p) \ dx$ as in section 2, and let the energy $E : H \to R$ given by

$$E(u,v) = J(u,v) - \int_\Omega F(x,u,v) \ dx$$

for any $(u, v) \in H$. Then a weak solution of system (1.1) is a critical point of $E(u, v)$ in H. Lemma 2.1 and 2.2 imply that E is weakly lower semicontinuous.

By Holder's inequality, (2.4), we have

$$
\begin{aligned}
F(x,u,v) &= \int_0^u \frac{\partial F}{\partial s}(x,s,v)\ ds + F(x,0,v) \\
&= \int_0^u \frac{\partial F}{\partial s}(x,s,v)\ ds + \int_0^v \frac{\partial F}{\partial s}(x,0,s)\ ds + F(x,0,0) \\
&\leq \int_0^u (\epsilon|u|^{p-1} + M_\epsilon)\ ds + \int_0^v (\epsilon|v|^{p-1} + M_\epsilon)\ ds \\
&= \frac{\epsilon}{p}|u|^p + M_\epsilon u + \frac{\epsilon}{p}|v|^p + M_\epsilon v
\end{aligned}
$$

so

$$
\begin{aligned}
|\int_\Omega F(x,u,v)\ dx| \leq \int_\Omega |F(x,u,v)|\ dx \\
&\leq \epsilon\ [\frac{1}{p}\int_\Omega |u|^p\ dx + \frac{1}{p}\int_\Omega |v|^p\ dx] + M_\epsilon[\int_\Omega udx + \int_\Omega vdx] \\
&\leq \frac{\epsilon}{p}\ S_1^p\ \int_\Omega |\nabla u|^p\ dx + \frac{\epsilon}{p}\ S_1^p\ \int_\Omega |\nabla v|^p\ dx + M_\epsilon |\Omega|^{\frac{p-1}{p}} S_1 (\int_\Omega |\nabla u|^p\ dx)^{\frac{1}{p}} \\
&\quad + M_\epsilon |\Omega|^{\frac{p-1}{p}} S_1 (\int_\Omega |\nabla v|^p\ dx)^{\frac{1}{p}} \\
&\leq \frac{\epsilon}{p}\ S_1^p\ \int_\Omega |\nabla u|^p\ dx + \frac{\epsilon}{p}\ S_1^p\ \int_\Omega |\nabla v|^p\ dx \\
&\quad + A[\int_\Omega |\nabla u|^p\ dx)^{\frac{1}{p}} + \int_\Omega |\nabla v|^p\ dx)^{\frac{1}{p}}
\end{aligned}
$$

where S_1 is the embedding constant of $H_0^{1,p}(\Omega) \hookrightarrow L^p(\Omega)$ and $A = M_\epsilon |\Omega|^{\frac{p-1}{p}} S_1$. Hence

$$
E(u,v) \geq \frac{1}{p}(\alpha_1 - \epsilon S_1^p) \int_\Omega |\nabla u|^p\ dx + \frac{1}{p}(\alpha_2 - \epsilon S_1^p) \int_\Omega |\nabla v|^p\ dx - A\|(u,v)\|_H.
$$

Letting $\epsilon = \frac{1}{2}\min\{\frac{\alpha_1}{S_1^p}, \frac{\alpha_2}{S_1^p}\}$. Noting that $(a+b)^p \leq 2^{p-1}(a^p + b^p)$ for all $a, b > 0$ and $p > 1$. Hence

$$
\int_\Omega |\nabla u|^p\ dx + \int_\Omega |\nabla v|^q\ dx \geq \frac{1}{2^{p-1}}\ [(\int_\Omega |\nabla u|^p\ dx)^{\frac{1}{p}} + (\int_\Omega |\nabla v|^p\ dx)^{\frac{1}{p}}]^p
$$

so we obtain

$$
E(u,v) \geq \frac{1}{2p}\min\{\alpha_1, \alpha_2\} \times [\frac{1}{2^{p-1}}\|(u,v)\|_H^p] - A\|(u,v)\|_H
$$

it follows that E is coercive in H. By (**i**),(**ii**) E is continuously differentiable on H and

$$\begin{aligned}\langle E'(u,v),(\epsilon,\eta)\rangle &= \int_\Omega [h_1(|\nabla u|^p)|\nabla u|^{p-2}\nabla u\nabla\xi + h_2(|\nabla v|^q)|\nabla u|^{q-2}\nabla v\nabla\eta]dx \\ &\quad - \int_\Omega [f(x,u,v)\xi + g(x,u,v)\eta]\ dx \\ &= \langle J'(u,v),(\epsilon,\eta)\rangle - \langle \widehat{W}'(u,v),(\epsilon,\eta)\rangle\end{aligned}$$

for any $(u,v) \in H$. Therefore E has a minimum at some point $(u,v) \in H$ and $E'(u,v) = 0$. Thus, this implies that

$$\langle J'(u,v),(\epsilon,\eta)\rangle = \langle \widehat{W}'(u,v),(\epsilon,\eta)\rangle$$

for any $(u,v) \in H$, that is, (u,v) is a weak solution of system (1.1). This completes the proof of theorem 1.1. □

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