

A SHAPE RECONSTRUCTION PROBLEM WITH THE LAPLACE OPERATOR

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ABSTRACT. We discuss the inverse boundary value problem of reconstructing of an unknown part of boundary. We use shape optimization tools and maximum principle theory to reconstruct the unknown part of boundary from a knowledge of cauchy data. From necessary conditions, we estimate a Lagrange multiplier λ which appears by derivation with respect to the domain and we give a numerical value of λ .

1. INTRODUCTION

The inverse problem in this paper means the problem of reconstructing object from observation data. We restrict ourselves to the case when the observation data are given as a boundary of the Cauchy data of a solution of an elliptic diffusion equation and the unknown object is a boundary. Let Ω be a bounded domain in \mathbb{R}^N with smooth boundary $\partial\Omega$. Moreover, let us consider a partition of this boundary $\partial\Omega = \Gamma_0 \cup \Gamma$, $\Gamma_0 \cap \Gamma = \emptyset$ where Γ_0 is the accessible part, regular e.g \mathcal{C}^2 and satisfies the interior sphere condition see [14] and $\Gamma = \partial\Omega \setminus \Gamma_0$ is the unknown part of boundary.

Given Ω a bounded domain in \mathbb{R}^N , we consider the Dirichlet problem for a function u satisfying the following problem

$$\begin{cases} -\Delta u + u = 0 & \text{in } \Omega \\ u = g & \text{on } \Gamma_0 \\ u = 0 & \text{on } \Gamma \end{cases} \quad (1.1)$$

where g is a given positive continuous function prescribed on Γ_0 . One can show that there is a unique solution $u \in H^1(\Omega)$ to the direct problem (1.1) provided $\partial\Omega$ is Lipschitz. We have the well-posed direct problem.

Our inverse problem consists on finding a formula reconstructing Γ from the Cauchy data $(g|_{\Gamma_0}, h|_{\Gamma_0})$ of a weak solution u to the following problem

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$$\begin{cases} -\Delta u + u & = 0 & \text{in } \Omega \\ u & = g & \text{on } \Gamma_0 \\ u & = 0 & \text{on } \Gamma \\ \frac{\partial u}{\partial \nu} & = h & \text{on } \Gamma_0, \end{cases} \quad (1.2)$$

where g is a given positive continuous function prescribed on Γ_0 , h a given negative function, the corresponding Neumann data measured on Γ_0 and ν is outer normal vector unit. In this case, Ω and u are unknowns and we assume that the normal derivatives of the function u can be measured by h .

These types of problems have a lot of applications. In medical imaging, see [1], G.Bal studied the problem which consists of measuring the electric potential on the endocardial surface(the inside of the cardial wall). His method consists of sending a probe inside the heart and to measure the potential on the probe. The inverse problem consists then to reconstruct the potential on the endocardial surface from measurements.

In [20], A.Kirsch studied inverse problems in geological prospecting. This problem consists to determine the location, the shape and or some parameters (such conductivity) of geological anomalies in Earth's interior from measurements at its surface. He also studied inverse scattering problem and computer tomography.

In [18] V.Isakov studied the inverse spectral problems. This domain problem was formulated already by Sir A. Shuster, who in 1882 introduced spectroscopy as a way to find a shape of a bell by means of the sounds which it is capable of sending out. More rigorously, it has been posed by Bochner in the 1950s and then in the well-known lecture of Marc Kac see [19], Can one hear the shape of a drum? in 1966.He also studied inverse problem of gravimetry, inverse conductivity problem, tomography and the inverse seismic problem, and indicate their applications.

In [22], using conformal mapping technique, R. Kress studied mathematical modelling of electrostatic or thermal imaging methods in non destructive testing and evaluation. In these applications an unknown inclusion within a conducting host medium with constant conductivity is assessed from over determined Cauchy data on the accessible exterior boundary of the medium.

In this paper, the reconstructing Γ from the Cauchy data is one of our aim and the estimation of the Lagrange multiplier which appears by derivation with respect to the domain of the energy of system in a admissible set of domains is another interesting ones. We could note some applications as the cut of optical fibers, the detection of oil, the cut in the electricity or water network. And two main questions in this paper can be connected and we are going to establish results on the reconstruction of Γ and the monotonicity result of the Lagrange multiplier. The last question is not easy to prove in the general cases and the authors are not aware about a theory or new techniques and tools in order to estimate the Lagrange multiplier depending on the domain.

For simplicity of our presentation, we are going to consider the following problem:

find (Ω, u) such that

$$\begin{cases} -\Delta u + u &= 0 & \text{in } \Omega \\ u &= g & \text{on } \Gamma_0 \\ u &= 0 & \text{on } \Gamma \\ \frac{\partial u}{\partial \nu} &= h & \text{on } \Gamma_0, \end{cases} \quad (1.3)$$

And $\int_{\Omega} dx = V_0$ with V_0 is a given and fixed value.

Our aim is to situate Γ the boundary reconstruction from a knowledge of the Cauchy data. To do this, we study the shape optimization problem and prove the necessary conditions of optimality. These conditions leads to estimate the Lagrange multiplier.

This paper is organized as follow:

In section 2, we establish the shape optimization results. And we prove the necessary conditions of optimality that is the existence of Lagrange multiplier. Section 3 is devoted to auxiliary lemmas based on maximum principal theory [24]. These lemmas play a fundamental part in the following sections. In section 4, we give, under geometrical assumptions, an uniqueness results which ensure the statement of an algorithm and its convergence. In the last section, we discuss on a general elliptical operator and we end by some classical numerical illustrations. But this numerical analysis gives interesting information and allows us to talk on futures.

2. PRELIMINARIES

We study the existence result for the following shape optimization problem:

$$\inf\{J(\omega), \omega \in \Theta\}$$

where Θ is defined by

$$\Theta = \{\omega \subset \mathbb{R}^N \text{ uniformly lipschitzian open bounded domain, } \int_{\omega} dx = V_0\}.$$

V_0 is a positive given real number.

And the functional J is defined by

$$J(\omega) = \frac{1}{2} \int_{\omega} |\nabla u_{\omega}|^2 dx + \frac{1}{2} \int_{\omega} |u_{\omega}|^2 dx$$

where u_{ω} is the solution of the following problem

$$\begin{cases} -\Delta u_{\omega} + u_{\omega} &= 0 & \text{in } \omega \\ u_{\omega} &= g & \text{on } \Gamma_0 \\ u_{\omega} &= 0 & \text{on } \partial\omega \setminus \Gamma_0 \end{cases} \quad (2.1)$$

Suppose that Γ_0 is regular i.e. at least of class C^2 .

Let us introduce the energy of the system:

$$J(\omega) = \frac{1}{2} \int_{\omega} |\nabla u_{\omega}|^2 dx + \frac{1}{2} \int_{\omega} |u_{\omega}|^2 dx$$

We are going to minimize $J(\omega)$ on

$$\Theta = \{\omega \subset \mathbb{R}^N \text{ uniformly lipschitzian open bounded domain, } \int_{\omega} dx = V_0\}.$$

V_0 is given .

Remark. Let us note that ω uniformly Lipschitz means ω satisfies the ϵ -cone property, for details see [17].

We are going to consider a fixed and bounded domain D which contains all open subsets we used.

We have the following lemma

Lemma 2.1. Let $(\Omega_n)_{n \in \mathbb{N}}$ be a sequence of open set in \mathbb{R}^N having the ϵ -cône property with $\bar{\Omega}_n \subset F \subset D$, F a compact set. Then there exists an open set Ω including in F which satisfies the $\frac{\epsilon}{2}$ -cône property and a subsequence Ω_{n_k} such that

$$\begin{array}{ccc} \chi_{\Omega_{n_k}} & \xrightarrow{L^1} & \chi_{\Omega}, \quad \Omega_{n_k} \xrightarrow{H} \Omega \\ \partial\Omega_{n_k} & \xrightarrow{H} & \partial\Omega, \quad \bar{\Omega}_{n_k} \xrightarrow{H} \bar{\Omega}. \end{array}$$

Proof. see [17] for details. □

We have the existence and shape continuity results.

Proposition 2.2.

There exists $\Omega \in \Theta$ such that: $\inf\{J(\omega), \omega \in \Theta\} = J(\Omega)$ and

$$\begin{cases} -\Delta u_{\Omega} + u_{\Omega} & = 0 & \text{in } \Omega \\ u_{\Omega} & = g & \text{on } \Gamma_0 \\ u_{\Omega} & = 0 & \text{on } \partial\Omega \setminus \Gamma_0. \end{cases}$$

The above proposition is well known, but we give a complete proof. See for example [17], [5] and references therein.

Proof. Consider the function \tilde{u} defined by

$$\tilde{u} = \begin{cases} u & \text{if } x \in \Omega \\ 0 & \text{if } x \in \Omega^c \end{cases}$$

$$\nabla \tilde{u} = \begin{cases} \nabla u & \text{if } x \in \Omega \\ 0 & \text{if } x \in \Omega^c \end{cases}$$

Let E be a functional defined on $H^1(D)$ by

$$E(\tilde{u}_w) = \frac{1}{2} \int_D |\nabla \tilde{u}_w|^2 dx + \frac{1}{2} \int_D |\tilde{u}_w|^2 dx,$$

where \tilde{u}_w is the extension by 1 in $\bar{\Omega}$ of u_w solution of the problem

$$\begin{aligned} -\Delta u_w + u_w &= 0 & \text{in } w \\ u_w &= g & \text{on } \partial\Gamma_0 \\ u_w &= 0 & \text{on } \partial w \setminus \Gamma_0 \end{aligned}$$

Let $J(w) := E(\tilde{u}_w)$. Then $J(w) > 0$ this implies that $\inf\{J(w), w \in \mathcal{O}_{\epsilon}\} > -\infty$. Let $\alpha = \inf\{J(w), w \in \mathcal{O}_{\epsilon}\}$. Then, there exists a minimizing sequence $(\Omega_n)_{n \in \mathbb{N}} \subset \mathcal{O}_{\epsilon}$ such that $J(\Omega_n)$ converges on α .

Since the sequence $(\Omega_n)_{n \in \mathbb{N}}$ is bounded, there exists a compact set F such that $\bar{\Omega}_n \subset F \subset D$. By lemma (2.1), there is a subsequence $(\Omega_{n_k})_{k \in \mathbb{N}}$, and Ω verifying the ϵ -cone property such that $\Omega_{n_k} \xrightarrow{H} \Omega$ and $\chi_{\Omega_{n_k}} \rightarrow \chi_{\Omega}$ a.e. Let us set $u_{\Omega_n} = u_n$ and show that the sequence $(\tilde{u}_n)_{n \in \mathbb{N}}$ is bounded in $H^1(D)$. If not, for all s there exists

a subsequence denoted $\tilde{u}_n^s \in H^1(D)$ such that $\frac{1}{2} \int_D |\nabla \tilde{u}_n^s|^2 dx + \frac{1}{2} \int_D |\tilde{u}_n^s|^2 dx > s$ and

$$\frac{1}{2} \int_D |\nabla \tilde{u}_n^s|^2 dx + \frac{1}{2} \int_D |\tilde{u}_n^s|^2 dx = \frac{1}{2} \int_{\Omega_n} |\nabla \tilde{u}_n^s|^2 dx + \frac{1}{2} \int_{\Omega_n} |\tilde{u}_n^s|^2 dx$$

That is, $J(\Omega_n)$ converges on $+\infty$. Then, $\inf\{J(w), w \in \mathcal{O}_\epsilon\} = +\infty$ is a contradiction. Since $H^1(D)$ is a reflexive space, there exists a subsequence $(u_{n_k})_{k \in \mathbb{N}}$ and u^* such that u_{n_k} converges weakly on u^* in $H^1(D)$ and

$$\frac{1}{2} \int_{\Omega} |\nabla u^*|^2 dx + \frac{1}{2} \int_{\Omega} |u^*|^2 dx \leq \liminf \frac{1}{2} \int_{\Omega_{n_k}} |\nabla u_{n_k}|^2 dx + \frac{1}{2} \int_{\Omega} |u_{n_k}|^2 dx$$

From the above we get $J(\Omega) \leq J(\Omega_{n_k})$ and $J(\Omega) \leq \inf\{J(w), w \in \Theta\}$. Finally, we have $J(\Omega) = \min\{J(w), w \in \Theta\}$. \square

Remark. *On the one hand, it is easy to verify that u^* equals u_Ω and satisfies*

$$\begin{aligned} -\Delta u^* + u^* &= 0 \quad \text{in } \Omega \\ u^* &= g \quad \text{on } \partial\Gamma_0 \\ u^* &= 0 \quad \text{on } \partial\Omega \setminus \Gamma_0 \end{aligned}$$

On the other hand, we have a regularity of u_Ω solution to the problem (2.1); see [11, 23, 26].

Measure assumption:

We assume that it is possible to estimate the normal derivative of u_Ω on Γ_0 i.e there exists

$$\begin{aligned} h : \mathbb{R}^N &\longrightarrow \mathbb{R}_-^* \quad \text{such that} \\ \frac{\partial u_\Omega}{\partial \nu} &= h \quad \text{on } \Gamma_0. \end{aligned}$$

Where ν is the exterior unit normal vector field defined on Γ_0 .

We have the following necessary conditions of optimality.

Proposition 2.3. *If Ω is the solution of the shape optimization problem $\min\{J(\omega), \omega \in \Theta\}$, then there exists a Lagrange multiplier $\lambda(\Omega)$ such that $|\nabla u| = -\frac{\partial u}{\partial \nu} = (-2\lambda(\Omega))^{1/2}$ on Γ .*

Proof. As $J(\Omega) = \inf\{J(\omega), \omega \in \Theta\}$, using the derivative with respect to domain, in the direction of vector field, we show that there exists $\lambda(\Omega)$ Lagrange multiplier such that :

$$dJ(\Omega, V) = \lambda(\Omega)dJ_1(\Omega, V)$$

where $J_1(\Omega) = \int_{\Omega} dx - V_0$. Note that we perturb Ω only on $\partial\Omega \setminus \Gamma_0 = \Gamma$, Γ_0 is fixed.

$$dJ(\Omega, V) = \lambda(\Omega)dJ_1(\Omega, V) \iff \frac{1}{2}|\nabla u_\Omega|^2 = -\lambda(\Omega) \quad \text{on } \Gamma \quad (2.2)$$

For more details for the expression (2.2)see [27].

Let's take $u_\Omega = u$. To estimate $\lambda(\Omega)$, it suffices that to recognize Ω and if we suppose that $\partial\Omega \setminus \Gamma_0 = \Gamma$ is of class C^2 , and since $u = 0$ on Γ , then we have

$$\frac{1}{2}|\nabla u|^2 = \frac{1}{2}\left(\frac{\partial u}{\partial \nu}\right)^2 \quad \text{on } \Gamma,$$

where ν is the outer normal vector. Note that $\frac{1}{2}|\nabla u|^2 = -\lambda(\Omega)$ is an optimality condition and if we situate Γ , we will able to estimate $|\nabla u|$ on Γ . Therefore we deduce an approximation of Lagrange multiplier $\lambda(\Omega)$. \square

3. AUXILIARY LEMMAS

In this section, we sum up some fundamental lemmas for the algorithm which we will present in the next sections. These lemmas are based only on maximum principle theory. We assume also that $u_i \in C^2(\Omega) \cap C(\bar{\Omega} \setminus \Gamma_i)$, $i = 1, 2$

Lemma 3.1.

Let Ω_1, Ω_2 be two open subsets of \mathbb{R}^n , two open sets such that: $\Omega_2 \subset \Omega_1$ and $\Gamma_0 \subset \partial\Omega_1 \cap \partial\Omega_2$. We suppose that:

$$\begin{cases} -\Delta u_1 + u_1 = 0 & \text{in } \Omega_1 \\ u_1 = g & \text{on } \Gamma_0 \\ u_1 = 0 & \text{on } \Gamma_1 \end{cases} \quad (3.1)$$

$$\begin{cases} -\Delta u_2 + u_2 = 0 & \text{in } \Omega_2 \\ u_2 = g & \text{on } \Gamma_0 \\ u_2 = 0 & \text{on } \Gamma_2 \end{cases} \quad (3.2)$$

Where

$$\begin{aligned} \bar{\Gamma}_0 \cup \bar{\Gamma}_1 &= \partial\Omega_1, & \Gamma_0 \cap \Gamma_1 &= \emptyset \\ \bar{\Gamma}_0 \cup \bar{\Gamma}_2 &= \partial\Omega_2, & \Gamma_0 \cap \Gamma_2 &= \emptyset \end{aligned}$$

and g a positive given function. This inequality holds

$$-\frac{\partial u_1}{\partial \nu}(s) > -\frac{\partial u_2}{\partial \nu}(s) \quad \text{for all } s \in \Gamma_0.$$

Proof. Consider $u_1 - u_2$, we have

$$\begin{cases} -\Delta(u_1 - u_2) + (u_1 - u_2) = 0 & \text{in } \Omega_2 \\ u_1 - u_2 = 0 & \text{on } \Gamma_0 \\ u_1 - u_2 = u_1 & \text{on } \Gamma_2 \end{cases} \quad (3.3)$$

As $g > 0$ on $\partial\Omega_1$, by maximum principle, $u_1 \geq 0$ in Ω_1 .

And then, using again the maximum principle, $(u_1 - u_2)(x) \geq 0$ for all $x \in \Omega_2$ and even $\max_{\Gamma_1} u_1 > u_1 - u_2 > 0$, for all $x \in \Omega_2$.

Let $x_0 \in \Gamma_0$, and $x = x_0 - \nu h \in \Omega$ where ν is exterior normal on Γ_0 and $h > 0$. By Hopf Lemma, we have

$$-\frac{\partial u_1}{\partial \nu}(x_0) > -\frac{\partial u_2}{\partial \nu}(x_0).$$

\square

Lemma 3.2.

If (Ω_1, u_1) and (Ω_2, u_2) are two solutions of the following free boundary problem

$$\begin{cases} -\Delta u + u = 0 & \text{in } \Omega \\ u = g & \text{on } \Gamma_0 \\ u = 0 & \text{on } \partial\Omega \setminus \Gamma_0 \\ \frac{\partial u}{\partial \nu} = \lambda(\Omega) & \text{on } \partial\Omega \setminus \Gamma_0. \end{cases} \quad (3.4)$$

such that

- : a- $\lambda(\Omega)$ is the Lagrangian multiplier.
- : b- $\Omega_2 \subset \Omega_1$.
- : c- $(\partial\Omega_2 \cap \partial\Omega_1) \setminus \Gamma_0 \neq \emptyset$.

Then this inequality holds $\lambda(\Omega_1) < \lambda(\Omega_2)$.

Proof. We have

$$\begin{cases} \Delta(u_1 - u_2) = 0 & \text{in } \Omega_2 \\ u_1 - u_2 = 0 & \text{on } \Gamma_0 \\ u_1 - u_2 = u_1 & \text{on } \Gamma_2. \end{cases} \quad (3.5)$$

Thanks to maximum principle, $u_1 \geq 0$ on Γ_2 and, $u_1 - u_2 > 0$ on Ω_2 .

Let $x_0 \in \partial\Omega_2 \cap \partial\Omega_1 \setminus \Gamma_0$ then $(u_1 - u_2)(x_0) = 0$.

Hopf lemma implies

$$-\frac{\partial u_1}{\partial \nu}(x_0) > -\frac{\partial u_2}{\partial \nu}(x_0) \text{ i.e } \lambda(\Omega_1) < \lambda(\Omega_2).$$

□

4. UNIQUENESS AND CONVERGENCE RESULTS

4.1. Result. In this section, using results establishing in section 3, we are going to show the uniqueness of the domain Ω under some hypothesis.

Most of the time, in the inverse problems, it is a great challenge to get uniqueness results. Now we are not able to produce an uniqueness result in the case where the unknown domain Ω is supposed to be starshaped with respect to a fixed point x_0 in Ω . We think that it would be interesting to investigate this question. Our uniqueness result is obtained for any domains belonging to B_h , a class of admissible domains satisfying an inequality constraint on the accessible boundary Γ_0 , see below the definition of B_h . And another important hypothesis for our uniqueness result is the inclusion property in the following sense: One assumes that there are two domains in the class B_h and one of the two domains is included in the other. This is an interesting problem to weaken the inclusion property's hypothesis. The uniqueness result is started in the below theorem 4.2.

We have already assumed that one is able to estimate the normal derivative of u_Ω on Γ_0 i.e there exists

$$\begin{aligned} h : \mathbb{R}^N &\longrightarrow \mathbb{R}_+^* \text{ such that} \\ \frac{\partial u_\Omega}{\partial \nu} &= h \text{ on } \Gamma_0. \end{aligned}$$

Where ν is the exterior unit normal vector field defined on Γ_0 .

Let us define a class of geometrical sets in the Beurling sense see for instance [16],[25].

Let be \mathcal{O} the following class of domains.

$$\mathcal{O} = \{ \Omega \subset \mathbb{R}^N, \Omega, \text{ be a bounded domain and uniformly Lipschitz } \}$$

And let Γ_0 be a subset of \mathbb{R}^{N-1} , of class \mathcal{C}^2 . Let's take the following class of domains

$$B_h = \{ \Omega \in \mathcal{O}, \Gamma_0 \subset \partial\Omega, \frac{\partial u_\Omega}{\partial \nu} \geq h \text{ on } \Gamma_0 \}$$

and u_Ω is solution to the following problem

$$\begin{cases} -\Delta u_\Omega + u_\Omega &= 0 & \text{in } \Omega \\ u_\Omega &= g & \text{on } \Gamma_0 \\ u_\Omega &= 0 & \text{on } \partial\Omega \setminus \Gamma_0. \end{cases} \quad (4.1)$$

By construction B_h is a non empty set because of the existence of solution of the minimization see proposition (2.2) and measure assumption (see page 5).

We have the following lemma:

Lemma 4.1. *Let $\Omega_1, \Omega_2 \in B_h$ then $\Omega_1 \cap \Omega_2 \in B_h$.*

Proof. Let w be solution to the following problem

$$\begin{cases} -\Delta w + w &= 0 & \text{in } \Omega_1 \cap \Omega_2 \\ w &= g & \text{on } \Gamma_0 \\ w &= 0 & \text{on } \partial(\Omega_1 \cap \Omega_2) \setminus \Gamma_0. \end{cases} \quad (4.2)$$

We have

$$\begin{cases} -\Delta(u_{\Omega_1} - w) + u_{\Omega_1} - w &= 0 & \text{in } \Omega_1 \cap \Omega_2 \\ u_{\Omega_1} - w &= 0 & \text{on } \Gamma_0 \\ u_{\Omega_1} - w &= u_{\Omega_1}|_{\partial(\Omega_1 \cap \Omega_2) \setminus \Gamma_0} & \text{on } \partial(\Omega_1 \cap \Omega_2) \setminus \Gamma_0. \end{cases} \quad (4.3)$$

By the maximum principle, we show that

$$u_{\Omega_1} \geq 0 \quad \text{and} \quad u_{\Omega_1} - w \geq 0.$$

Let $x_0 \in \Gamma_0$ we get

$$\begin{aligned} -\frac{\partial u_{\Omega_1}}{\partial \nu}(x_0) &\geq -\frac{\partial w}{\partial \nu}(x_0) \quad \text{i.e} \\ \frac{\partial w}{\partial \nu}(x_0) &\geq -\frac{\partial u_{\Omega_1}}{\partial \nu}(x_0) \geq h \quad \text{for all } x_0 \in \Gamma_0. \end{aligned}$$

Then $\Omega_1 \cap \Omega_2 \in B_h$. □

Theorem 4.2.

Let Ω be a bounded domain and Γ_0, g, h, Γ defined as in the introduction. Consider the following Cauchy problem.

$$\begin{cases} -\Delta u + u &= 0 & \text{in } \Omega \\ u &= g & \text{on } \Gamma_0 \\ u &= 0 & \text{on } \partial\Omega \setminus \Gamma_0 \\ \frac{\partial u}{\partial \nu} &= h(x) & \text{on } \Gamma_0, \quad \Gamma_0 = \partial\Omega \setminus \bar{\Gamma}. \end{cases} \quad (4.4)$$

Assume also that there are two domains Ω_1 and Ω_2 , such that $\Omega_2 \subset \Omega_1$ for which (4.4) is verified. Then we have $\Omega_1 = \Omega_2$.

Proof. Let Ω_1, Ω_2 such that

$$\begin{cases} -\Delta u_1 + u_1 &= 0 & \text{in } \Omega_1 \\ u_1 &= g & \text{on } \Gamma_0 \\ u &= 0 & \text{on } \partial\Omega_1 \setminus \Gamma_0 \\ \frac{\partial u_1}{\partial \nu} &= h(x) & \text{on } \Gamma_0. \end{cases} \quad (4.5)$$

$$\begin{cases} -\Delta u_2 + u_2 & = 0 & \text{in } \Omega_2 \\ u_2 & = g & \text{on } \Gamma_0 \\ u_2 & = 0 & \text{on } \partial\Omega_2 \setminus \Gamma_0 \\ \frac{\partial u_2}{\partial \nu} & = h(x) & \text{on } \Gamma_0 \end{cases} \quad (4.6)$$

As Ω_1 and $\Omega_2 \in B_h$, we obtain $\Omega_1 \cap \Omega_2 \in B_h$.

Since $\Omega_2 \subset \Omega_1$, we have $\Omega_1 \cap \Omega_2 = \Omega_2$, by lemma (4.1) we have $\frac{\partial u_2}{\partial \nu} \geq h$ on Γ_0 . Consider the following problem

$$\begin{cases} -\Delta(u_1 - u_2) + u_1 - u_2 & = 0 & \text{in } \Omega_2 \\ u_1 - u_2 & = 0 & \text{on } \bar{\Gamma}_0 \\ u_1 - u_2 & = u_1 & \text{on } \partial\Omega_2 \setminus \Gamma_0. \end{cases} \quad (4.7)$$

We have

$$u_1 \geq 0 \text{ and } \max_{\Gamma_2} u_1 > u_1 - u_2 > 0.$$

As Γ_0 satisfies the interior sphere condition; by Hopf lemma we get

$$\frac{\partial u_1}{\partial \nu} < \frac{\partial u_2}{\partial \nu} \text{ on } \Gamma_0 \text{ i.e } h(x) < h(x) \forall x \in \Gamma_0$$

which is false, then $\Omega_1 = \Omega_2$. □

4.2. Algorithm and a convergence result in the convex case. In this part we give a method to reconstruct the unknown domain Ω . Consider the problem (1.3) and assume that the unknown domain is convex, we construct an increasing sequence of domains

$(\Omega_n)_{n \in \mathbb{N}}$ in the sense $\Omega_{n-1} \subset \Omega_n$ and $\partial\Omega_n = \Gamma_0 \cup \Gamma_n$, for any $n \geq 2$ where Γ_0 is given and of class \mathcal{C}^2 . Now let's write some steps of the algorithm: Let $\epsilon > 0$ be a small number, ϵ states the threshold such a construction is mainly based on the lemma 3.1.

Initialization

Let Ω_1 be a \mathcal{C}^2 convex domain such that $\Gamma_0 \subset \partial\Omega_1$ and $\int_{\Omega_1} dx = V_1 < V_0$. Given a positive, continuous function g , we solve at first the following value problem

$$\begin{cases} -\Delta u_1 + u_1 & = 0 & \text{in } \Omega_1 \\ u_1 & = g & \text{on } \Gamma_0 \\ u_1 & = 0 & \text{on } \partial\Omega_1 \setminus \Gamma_0. \end{cases} \quad (4.8)$$

We estimate $\frac{\partial u_1}{\partial \nu}|_{\Gamma_0}$ and for any fixed $x \in \Gamma_0$, we compute $\frac{\partial u_1}{\partial \nu}(x) - h(x)$.

If $\left| \frac{\partial u_1}{\partial \nu}(x) - h(x) \right| \simeq \epsilon$, then $\Omega_1 \simeq \Omega$.

If not we continue by taking $\Omega_1 \subset \Omega_2$ and $\int_{\Omega_2} dx = V_2 < V_0$.

We can continue the process until a step denoted by n which we will determine. Finally, we have constructed an increasing sequence of domain Ω_n solutions: $\Omega_0 \subset \Omega_1 \subset \Omega_2 \dots \subset \Omega_n$ converging to Ω in the Hausdorff sense and moreover $\lambda(\Omega_n)$ converges to $\lambda(\Omega)$ when n increases.

Some illustrations:

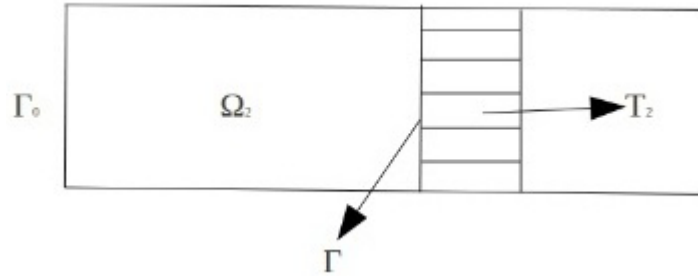


FIGURE 1. convex cylinder

- (1) If one works with a convex cylinder then we obtain the following figure 1
 $\Omega_2 = \Omega \cup T_2$ where T_2 is defined by

$$T_2 = \{y \in \mathbb{R}^N / y = x + \alpha\nu(x), x \in \Gamma, \alpha \in]0, 1[\}$$

where $\nu(x)$ is the exterior normal unit defined on Γ .

- (2) If we work with a convex sets then we obtain the figure 2, where $\Omega_2 = \Omega_1 \cup T_2$
and T_2 is defined by

$$T_2 = \{y \in \mathbb{R}^N / y = x + \alpha\nu(x), x \in \Gamma, \alpha \in]0, 1[\}$$

where $\nu(x)$ is the exterior unit normal defined on Γ .

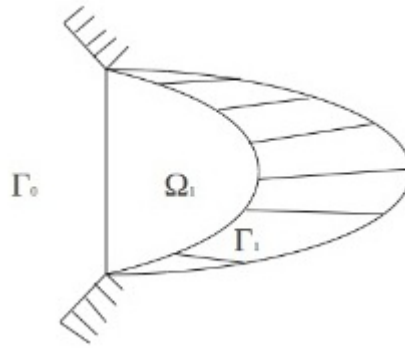


FIGURE 2. convex set

5. GENERALIZATION AND NUMERICAL ILLUSTRATIONS

5.1. **A general case.** In this section, we consider the following elliptic operator L defined:

$$\left\{ \begin{array}{ll} Lu &= \sum_{i,j=1}^N a_{ij}(x) \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^N b_i(x) \frac{\partial u}{\partial x_i} + c(x)u & \text{in } \Omega \\ Lu &= 0 & \text{in } \Omega \\ u|_{\Gamma_0} &= g, & g > 0 \\ \frac{\partial u}{\partial \nu}(x) &= h(x) & \text{for all } x \in \Gamma_0. \end{array} \right. \quad (5.1)$$

where $a_{ij} = a_{ji} \in C(\overline{\Omega})$, $b_i \in C(\overline{\Omega})$ and there exists c_0, C_0 , $0 < c_0 < C_0$ such that for all $x \in \Omega$ and $\xi \in \mathbb{R}^N$, we have

$$c_0|\xi|^2 \leq a_{ij}(x)\xi_i\xi_j \leq C_0|\xi|^2.$$

The uniqueness result is satisfied. Only, we have to use the following results which could be found in [24].

Lemma 5.1.

let Ω be a domain in \mathbb{R}^N and $u \in C(\Omega)$.

There exists $x_0 \in \Omega : u(x_0) = \mu = \max_{\overline{\Omega}} u$ and assume that $u \not\equiv \mu$ then there exists $B = B(a, \rho)$ a ball $\neq \emptyset$ such that $B \subset \overline{B} \subset \Omega$, $u < \mu$ in B and there exists $p \in \partial B \subset \Omega$ such that $u(p) = \mu$.

We have the following comparison principle result.

Theorem 5.2. [Strong maximum principle in boundary]

Let Ω be a domain, L as above with $c(x) \leq 0$. Let $u \in C^2(\Omega)$ and continuous at $p \in \partial\Omega$. we assume that $Lu \geq 0$ in Ω and $u(p) = \max_{\overline{\Omega}} u$.

We suppose also that Ω verify the interior sphere condition in p and if $c \not\equiv 0$, $u(p) \geq 0$ then one of the two assertions is true.

- : i- u is constant .
- : ii- $\frac{\partial u}{\partial \xi}(p) > 0$, where ξ is outer direction of the interior sphere.

Remark.

If $Lu \leq 0$ in Ω and there exists $x_0 \in \Omega : u(x_0) = \max_{\overline{\Omega}} u$ then

- : i- if $c(x) \equiv 0$ u is constant.
- : ii- if $c(x) \leq 0$ and if $u(x_0) \leq 0$ then u is constant.

With the above results, we obtain under the same hypothesis the uniqueness result of Ω . And if we suppose that $Lu = \text{div}(a(x)\nabla u) + c(x)u$.

With $a(x) = (a_{ij})$ as above. We have an estimate of Lagrange multiplier $\lambda(\Omega)$. In fact

$$\frac{1}{2}|a(x)\nabla u \nabla u(x)|^2 = \lambda(\Omega) \quad \text{on } \Gamma.$$

All the auxiliary lemmas are verified. The results are always true if $c(x)$ changes sign.

Remark.

To obtain the optimality condition $\frac{1}{2}|a(x)\nabla u \nabla u|^2 = \lambda(\Omega)$ on Γ , we introduce the shape functional:

$$J(\omega) = \frac{1}{2} \int_{\omega} |a(x)\nabla u \nabla u|^2 dx + \frac{1}{2} \int_{\omega} c(x)u^2 dx$$

and minimize $J(\omega)$ on Θ .

5.2. Some numerical illustrations. It is not hard to get such numerical results. But we present this in order to claim that it would be interesting to study such problem in a numerical point of view. We give some simulations for $N = 1$. We

consider

$$\begin{cases} -u'' + q(x)u = 0 & \text{in }]0, a[\\ u(0) = 1 \\ u(a) = 0 \\ u'(0) = -93 \end{cases} \quad (5.2)$$

See that by comparison principle $u'(0)$ is negative and $u'(0) = -93$ is an example for the simulations. We take $\epsilon = 0.06$ as an accuracy.

Our problem is to estimate a and then we give an estimation of the Lagrange multiplier ($\lambda(\Omega) = \frac{1}{2}u_x^2(a)$).

We use finite element method to calculate at first $u_h \simeq u$ and the test of the algorithm to obtain an approached numerical value of a .

q	1	$\exp x$	$1 + \sin x$	x	$1 + x^2$	$\frac{1}{1+x}$	$\frac{1}{1+x^2}$
$N[u'_h(0)]$	-92.9943	-93.0166	-93.004	-93.0227	-92.9813	-93.0111	-92.077
a	$\frac{1}{2} \log \left(\frac{u'_x(0)-1}{u'_x(0)+1} \right)$ =0.0107531						
a_{app}	0.0106963	0.0106457	0.0106452	0.0106963	0.0106483	0.0106483	0.0106483

For the different value of q , we observe by the computer that the Lagrange multiplier:

$\lambda(\Omega) = \frac{1}{2}u_x^2(s) := f(s)$ defined on $[0, a]$ conserves the same behavior if we represent the graphic. We see that for $N = 1$ we have $f(s)$ is decreasing.

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