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FOCK SPACES FOR THE q-BESSEL-STRUVE KERNEL

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Abstract. In this work, we introduce a class of Hilbert spaces $F_{q,\alpha}$ of entire functions on the disk $D(0, \frac{1}{1-q})$, $0 < q < 1$, with reproducing kernel given by the q-Bessel-Struve kernel $S_\alpha(z; q^2)$. The definition and properties of the space $F_{q,\alpha}$ extend naturally those of the well-known classical Fock space. Next, we study the bounded of some operators on the Fock spaces $F_{q,\alpha}$; and we give an application of the theory of reproducing kernels to the Tikhonov regularization, which gives the approximate solutions for bounded linear operator equation on the Fock spaces $F_{q,\alpha}$.

1. Introduction

Fock space F (called also Segal-Bargmann space [3]) is the Hilbert space of entire functions $f(z) = \sum_{n=0}^{\infty} a_n z^n$ on $\mathbb{C}$ such that

$$\|f\|_F^2 := \sum_{n=0}^{\infty} |a_n|^2 \, n! < \infty.$$

This space was introduced by Bargmann in [2] and it was the aim of many works [3]. Especially, the differential operator $D = d/dz$ and the multiplication operator by z are densely defined, closed and adjoint-operators on F (see [2]).

In [7], Gasmi and Soltani introduced a Hilbert space F_α of entire functions on $\mathbb{C}$, where the inner product is weighted by the modified Macdonald function. On F_α the Bessel-Struve operator

$$\ell_\alpha f(z) := D^2 f(z) + \frac{2\alpha+1}{z}\Big[Df(z) - Df(0)\Big], \quad \alpha > -1/2,$$

and the multiplication operator M by z^2 are densely defined, closed and adjoint-operators.

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In this paper, we consider the q-Bessel-Struve kernel:

$$S_\alpha(x;q^2) := \sum_{n=0}^{\infty} \frac{x^n}{c_n(\alpha;q^2)},$$

where $c_n(\alpha;q^2)$ are given later in section 2. We discuss some properties of a class of Fock spaces associated to the q-Bessel-Struve kernel and we give some applications.

The contents of the paper are as follows. In section 2, building on the ideas of Bargmann [2], Cholewinski [4] and as the same of paper [18], we define the q-Fock space $F_{q,\alpha}$ as the space of entire functions $f(z) = \sum_{n=0}^{\infty} a_n z^n$ on the disk $D(0, \frac{1}{1-q})$ of center 0 and radius $\frac{1}{1-q}$, and such that

$$\|f\|_{F_{q,\alpha}}^2 := \sum_{n=0}^{\infty} |a_n|^2 \, c_n(\alpha;q^2) < \infty.$$

Let f and g be in $F_{q,\alpha}$, such that $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum_{n=0}^{\infty} b_n z^n$, the inner product is given by

$$\langle f, g\rangle_{F_{q,\alpha}} = \sum_{n=0}^{\infty} a_n \overline{b_n} \, c_n(\alpha;q^2).$$

The q-Fock space $F_{q,\alpha}$ has also a reproducing kernel $K_{q,\alpha}$ given by

$$K_{q,\alpha}(w,z) = S_\alpha(\overline{w}z;q^2); \quad w,z \in D(0, \frac{1}{1-q}).$$

Then, if $f \in F_{q,\alpha}$, we have

$$\langle f, K_{q,\alpha}(w,.)\rangle_{F_{q,\alpha}} = f(w), \quad w \in D(0, \frac{1}{1-q}).$$

Using this property, we prove that the space $F_{q,\alpha}$ is a Hilbert space and we give an Hilbert basis.

In section 3, using the previous results, we consider the multiplication operator M by z^2 and the q-Bessel-Struve operator $\ell_{q,\alpha}$ on the Fock space $F_{q,\alpha}$, and we prove that these operators are continuous from $F_{q,\alpha}$ into itself, and satisfy:

$$\|\ell_{q,\alpha} f\|_{F_{q,\alpha}} \leq \frac{1}{1-q} \|f\|_{F_{q,\alpha}},$$

$$\|Mf\|_{F_{q,\alpha}} \leq \frac{1}{1-q} \|f\|_{F_{q,\alpha}}.$$

Then, we prove that these operators are adjoint-operators on $F_{q,\alpha}$:

$$\langle Mf, g\rangle_{F_{q,\alpha}} = \langle f, \ell_{q,\alpha} g\rangle_{F_{q,\alpha}}; \quad f,g \in F_{q,\alpha}.$$

These properties are not true on the classical Fock spaces [2, 4, 7, 15, 16]. For example on F_α, these properties are statued on $\mathcal{D}(\ell_\alpha) = \{f \in F_\alpha \ : \ \ell_\alpha f \in F_\alpha\}$, $\mathcal{D}(M) = \{f \in F_\alpha \ : \ Mf \in F_\alpha\}$ and $\mathcal{D}(\ell_\alpha) \cap \mathcal{D}(M)$, respectively.

Lastly, we define and study on the Fock space $F_{q,\alpha}$, the q-translation operators:

$$T_z f(w) := S_\alpha(z\ell_{q,\alpha};q^2) f(w); \quad w,z \in D(0, \frac{1}{1-q}),$$

and the generalized multiplication operators:

$$M_z f(w) := S_\alpha(zM; q^2) f(w); \quad w, z \in D(0, \frac{1}{1-q}).$$

Using the continuous properties of $\ell_{q,\alpha}$ and M we deduce also that the operators T_z and M_z, for $z \in D(0, \frac{1}{1-q})$, are continuous from $F_{q,\alpha}$ into itself, and satisfy:

$$\|T_z f\|_{F_{q,\alpha}} \leq S_\alpha(\frac{|z|}{1-q}; q^2) \|f\|_{F_{q,\alpha}},$$

$$\|M_z f\|_{F_{q,\alpha}} \leq S_\alpha(\frac{|z|}{1-q}; q^2) \|f\|_{F_{q,\alpha}}.$$

These properties are not true on the classical Fock spaces.

The last section of this paper is devoted to give an application of the theory of reproducing kernels to the Tikhonov regularization, which gives the approximate solutions for bounded linear operator equation on the Fock spaces $F_{q,\alpha}$.

Let $L : F_{q,\alpha} \to F_{q,\alpha}$ be a bounded operator from $F_{q,\alpha}$ into itself. For $\lambda > 0$, we define on the space $F_{q,\alpha}$, the new inner product by setting

$$\langle f, g \rangle_{\lambda, F_{q,\alpha}} = \lambda \langle f, g \rangle_{F_{q,\alpha}} + \langle Lf, Lg \rangle_{F_{q,\alpha}}.$$

Building on the ideas of Saitoh, Matsuura and Yamada [11, 12, 14, 19], and using the theory of reproducing kernels [1], we give best approximation of the operator L. More precisely, for all $\lambda > 0$, $h \in F_{q,\alpha}$, the infimum

$$\inf_{f \in F_{q,\alpha}} \left\{ \lambda \|f\|^2_{F_{q,\alpha}} + \|h - Lf\|^2_{F_{q,\alpha}} \right\},$$

is attained at one function $f^*_{\lambda,h}$, called the extremal function.
In particular for $f \in F_{q,\alpha}$ and $h = Lf$, the corresponding extremal functions $\{f^*_{\lambda,Lf}\}_{\lambda>0}$ converges to f as $\lambda \to 0^+$.

2. Preliminaries and the q-Fock spaces $F_{q,\alpha}$

Let a and q be real numbers such that $0 < q < 1$; the q-shifted factorial are defined by

$$(a; q)_0 := 1, \quad (a; q)_n := \prod_{i=0}^{n-1} (1 - aq^i), \;\; n = 1, 2, ..., \infty.$$

Jackson [8] defined the q-analogue of the Gamma function as

$$\Gamma_q(x) := \frac{(q; q)_\infty}{(q^x; q)_\infty} (1-q)^{1-x}, \quad x \neq 0, -1, -2, ...$$

It satisfies the functional equation

$$\Gamma_q(x+1) = \frac{1-q^x}{1-q} \Gamma_q(x), \quad \Gamma_q(1) = 1,$$

and tends to $\Gamma(x)$ when q tends to 1^-. In particular, for $n = 1, 2, ...$, we have

$$\Gamma_q(n+1) = \frac{(q; q)_n}{(1-q)^n}.$$

The q-derivative $D_q f$ of a suitable function f (see [10]) is given by

$$D_q f(x) := \frac{f(x) - f(qx)}{(1-q)x}, \quad x \neq 0,$$

and $D_qf(0)=f'(0)$ provided $f'(0)$ exists.
If f is differentiable then $D_qf(x)$ tends to $f'(x)$ as $q\to 1^-$.

Taking account of the paper [6, 7] and the same way, we define the q-Bessel-Struve kernel by

$$S_\alpha(x;q^2):=j_\alpha(ix;q^2)-ih_\alpha(ix;q^2),$$

where $j_\alpha(x;q^2)$ is the q-normalized Bessel function [5, 17] given by

$$j_\alpha(x;q^2):=\Gamma_{q^2}(\alpha+1)\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{(1+q)^{2n}\Gamma_{q^2}(n+1)\Gamma_{q^2}(n+\alpha+1)},$$

and $h_\alpha(x;q^2)$ is the q-normalized Struve function given by

$$h_\alpha(x;q^2):=\Gamma_{q^2}(\alpha+1)\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{(1+q)^{2n+1}\Gamma_{q^2}(n+\frac{3}{2})\Gamma_{q^2}(n+\alpha+\frac{3}{2})}.$$

Furthermore, the q-Bessel-Struve kernel $S_\alpha(x;q^2)$ can be expanded in a power series in the form

$$S_\alpha(x;q^2):=\sum_{n=0}^{\infty}\frac{x^n}{c_n(\alpha;q^2)},$$

where

$$c_n(\alpha;q^2):=\frac{(1+q)^n\Gamma_{q^2}(\frac{n}{2}+1)\Gamma_{q^2}(\frac{n}{2}+\alpha+1)}{\Gamma_{q^2}(\alpha+1)}. \tag{1}$$

If we put $U_n:=\frac{1}{c_n(\alpha;q^2)}$, then

$$\frac{U_n}{U_{n+1}}\to\frac{1}{(1-q)^2},\qquad q\to 1^-.$$

Thus, the q-Bessel-Struve kernel $S_\alpha(x;q^2)$ is defined on $D(0,\frac{1}{(1-q)^2})$ and tends to the Bessel-Struve kernel $S_\alpha(x)$ as $q\to 1^-$.

We consider the q-Bessel-Struve operator $\ell_{q,\alpha}$ defined by

$$\ell_{q,\alpha}f(x):=D_q^2f(x)+\frac{[2\alpha+1]_q}{x}\Big[D_qf(qx)-D_qf(0)\Big],$$

where

$$[2\alpha+1]_q:=\frac{1-q^{2\alpha+1}}{1-q}.$$

The q-Bessel-Struve operator tends to the Bessel-Struve operator ℓ_α as $q\to 1^-$ (see [6, 7]).

Lemma 2.1. *The function $S_\alpha(\lambda.;q^2)$, $\lambda\in D(0,\frac{1}{1-q})$, is the unique analytic solution of the q-problem:*

$$\ell_{q,\alpha}y(x)=\lambda^2y(x), \tag{2}$$

$$D_qy(0)=\frac{\lambda\Gamma_{q^2}(\alpha+1)}{(1+q)\Gamma_{q^2}(\frac{3}{2})\Gamma_{q^2}(\alpha+\frac{3}{2})},$$

$$y(0)=1.$$

Proof. Searching a solution of (2) in the form $y(x)=\sum_{n=0}^{\infty}a_nx^n$. Then

$$D_qy(x)=\sum_{n=1}^{\infty}a_n[n]_qx^{n-1},$$

and

$$D_q^2y(x)=D_q(D_qy)(x)=\sum_{n=2}^{\infty}a_n[n]_q[n-1]_qx^{n-2}.$$

Replacing in (2), we obtain

$$\sum_{n=2}^{\infty}a_n[n]_q([n-1]_q+q^{n-1}[2\alpha+1]_q)x^n=\lambda^2\sum_{n=2}^{\infty}a_{n-2}x^n.$$

Thus,

$$a_n[n]_q([n-1]_q+q^{n-1}[2\alpha+1]_q)=\lambda^2a_{n-2},\quad n=2,3,...$$

Using the fact that $[n-1]_q+q^{n-1}[2\alpha+1]_q=[n+2\alpha]_q$, we deduce that

$$a_n=\frac{\lambda^2}{[n]_q[n+2\alpha]_q}a_{n-2},\quad n=2,3,...$$

Since $[2n]_q=(1+q)[n]_{q^2}$, we deduce

$$a_{2n}=\frac{\lambda^2}{(1+q)^2[n]_{q^2}[n+\alpha]_{q^2}}a_{2n-2},$$

$$a_{2n+1}=\frac{\lambda^2}{(1+q)^2[n+\frac{1}{2}]_{q^2}[n+\alpha+\frac{1}{2}]_{q^2}}a_{2n-1}.$$

This proves that

$$a_{2n}=\frac{\lambda^{2n}\Gamma_{q^2}(\alpha+1)}{(1+q)^{2n}\Gamma_{q^2}(n+1)\Gamma_{q^2}(n+\alpha+1)},$$

$$a_{2n+1}=\frac{\lambda^{2n+1}\Gamma_{q^2}(\alpha+1)}{(1+q)^{2n+1}\Gamma_{q^2}(n+\frac{3}{2})\Gamma_{q^2}(n+\alpha+\frac{3}{2})}.$$

Therefore,

$$\begin{aligned}y(x)&=\Gamma_{q^2}(\alpha+1)\sum_{n=0}^{\infty}\frac{(\lambda x)^{2n}}{(1+q)^{2n}\Gamma_{q^2}(n+1)\Gamma_{q^2}(n+\alpha+1)}\\&+\Gamma_{q^2}(\alpha+1)\sum_{n=0}^{\infty}\frac{(\lambda x)^{2n+1}}{(1+q)^{2n+1}\Gamma_{q^2}(n+\frac{3}{2})\Gamma_{q^2}(n+\alpha+\frac{3}{2})}\\&=j_\alpha(i\lambda x;q^2)-ih_\alpha(i\lambda x;q^2),\end{aligned}$$

which completes the proof of the lemma. □

Lemma 2.2. *The constants $b_n(\alpha;q^2)$, $n\in\mathbb{N}$ satisfy the following relation:*

$$c_{n+2}(\alpha;q^2)=[n+2]_q[n+2\alpha+2]_qc_n(\alpha;q^2).$$

Lemma 2.3. *For $n\in\mathbb{N}$, we have*

$$\ell_{q,\alpha}z^n=\frac{c_n(\alpha;q^2)}{c_{n-2}(\alpha;q^2)}z^{n-2},\quad n\geq 2.$$

Proof. Since

$$S_\alpha(\lambda z;q^2):=\sum_{k=0}^{\infty}\frac{(\lambda z)^k}{c_k(\alpha;q^2)},$$

then from equation (2) we obtain

$$\sum_{k=2}^{\infty}\frac{\ell_{q,\alpha}z^k}{c_k(\alpha;q^2)}\lambda^k=\sum_{k=2}^{\infty}\frac{z^{k-2}}{c_{k-2}(\alpha;q^2)}\lambda^k.$$

This clearly yields the result. □

Definition 2.1. *Let $\alpha\geq -1/2$. The q-Fock space $F_{q,\alpha}$ is the prehilbertian space of entire functions $f(z)=\sum_{n=0}^{\infty}a_nz^n$ on $D(0,\frac{1}{1-q})$, such that*

$$\|f\|^2_{F_{q,\alpha}}:=\sum_{n=0}^{\infty}|a_n|^2\,c_n(\alpha;q^2)<\infty, \tag{3}$$

where $c_n(\alpha;q^2)$ is given by (1).

The inner product in $F_{q,\alpha}$ is given for $f(z)=\sum_{n=0}^{\infty}a_nz^n$ and $g(z)=\sum_{n=0}^{\infty}b_nz^n$ by

$$\langle f,g\rangle_{F_{q,\alpha}}=\sum_{n=0}^{\infty}a_n\overline{b_n}\,c_n(\alpha;q^2). \tag{4}$$

Remark 1. *If $q\to 1^-$, the space $F_{q,\alpha}$ agrees with the generalized Fock space associated to the Bessel-Struve operator (see* [7]*).*

The following theorem prove that $F_{q,\alpha}$ is a reproducing kernel space.

Theorem 2.1. *The function $K_{q,\alpha}$ given for $w,z\in D(0,\frac{1}{1-q})$, by*

$$K_{q,\alpha}(w,z)=S_\alpha(\overline{w}z;q^2),$$

is a reproducing kernel for the q-Fock space $F_{q,\alpha}$, that is:

(*i*) *for all $w\in D(0,\frac{1}{1-q})$, the function $z\to K_{q,\alpha}(w,z)$ belongs to $F_{q,\alpha}$.*

(*ii*) *For all $w\in D(0,\frac{1}{1-q})$ and $f\in F_{q,\alpha}$, we have*

$$\langle f,K_{q,\alpha}(w,.)\rangle_{F_{q,\alpha}}=f(w).$$

Proof. (i) Since

$$K_{q,\alpha}(w,z)=\sum_{n=0}^{\infty}\frac{\overline{w}^n}{c_n(\alpha;q^2)}z^n;\quad z,\,w\in D(0,\frac{1}{1-q}), \tag{5}$$

then from (3), we deduce that

$$\|K_{q,\alpha}(w,.)\|^2_{F_{q,\alpha}}=\sum_{n=0}^{\infty}\frac{|w|^{2n}}{c_n(\alpha;q^2)}=S_\alpha(|w|^2;q^2)<\infty,$$

which proves (i).

(ii) If $f(z)=\sum_{n=0}^{\infty}a_nz^n\in F_{q,\alpha}$, from (4) and (5), we deduce

$$\langle f,K_{q,\alpha}(w,.)\rangle_{F_{q,\alpha}}=\sum_{n=0}^{\infty}a_nw^n=f(w),\quad w\in D(0,\frac{1}{1-q}).$$

This completes the proof of the theorem. □

Remark 2. *From Theorem 2.1 (ii), for* $f \in F_{q,\alpha}$ *and* $w \in D(0, \frac{1}{1-q})$*, we have*

$$|f(w)| \leq \|K_{q,\alpha}(w,.)\|_{F_{q,\alpha}} \|f\|_{F_{q,\alpha}} = [S_\alpha(|w|^2; q^2)]^{1/2} \|f\|_{F_{q,\alpha}}. \tag{6}$$

Proposition 2.1. *The space* $F_{q,\alpha}$ *equipped with the inner product* $\langle .,. \rangle_{F_{q,\alpha}}$ *is an Hilbert space; and the set* $\left\{\xi_n(.; q^2)\right\}_{n\in\mathbb{N}}$ *given by*

$$\xi_n(z; q^2) = \frac{z^n}{\sqrt{c_n(\alpha; q^2)}}, \quad z \in D(0, \frac{1}{1-q}),$$

forms an Hilbert basis for the space $F_{q,\alpha}$.

Proof. Let $\{f_n\}_{n\in\mathbb{N}}$ be a Cauchy sequence in $F_{q,\alpha}$. We put

$$f = \lim_{n\to\infty} f_n, \quad \text{in} \quad F_{q,\alpha}.$$

From (6), we have

$$|f_{n+p}(w) - f_n(w)| \leq [S_\alpha(|w|^2; q^2)]^{1/2} \|f_{n+p} - f_n\|_{F_{q,\alpha}}.$$

This inequality shows that the sequence $\{f_n\}_{n\in\mathbb{N}}$ is pointwise convergent to f. Since the function $w \to [S_\alpha(|w|^2; q^2)]^{1/2}$ is continuous on $D(0, \frac{1}{1-q})$, then $\{f_n\}_{n\in\mathbb{N}}$ converges to f uniformly on all compact set of $D(0, \frac{1}{1-q})$. Consequently, f is an entire function on $D(0, \frac{1}{1-q})$, then f belongs to the space $F_{q,\alpha}$.

On the other hand, from the relation (4), we get

$$\langle \xi_n(.; q^2), \xi_m(.; q^2) \rangle_{F_{q,\alpha}} = \delta_{n,m},$$

where $\delta_{n,m}$ is the Kronecker symbol.
This shows that the family $\left\{\xi_n(.; q^2)\right\}_{n\in\mathbb{N}}$ is an orthonormal set in $F_{q,\alpha}$.

Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be an element of $F_{q,\alpha}$ such that

$$\langle f, \xi_n(.; q^2) \rangle_{F_{q,\alpha}} = 0, \quad \forall\, n \in \mathbb{N}.$$

From the relation (4), we deduce that

$$a_n = 0, \quad \forall\, n \in \mathbb{N}.$$

This completes the proof. □

Remark 3. (*a*) *The set* $\left\{S_\alpha(\overline{w}.; q^2), \quad w \in D(0, \frac{1}{1-q})\right\}$ *is dense in* $F_{q,\alpha}$.

(*b*) *For all* $z, w \in D(0, \frac{1}{1-q})$*, we have*

$$S_\alpha(w\overline{z}; q^2) = \langle S_\alpha(\overline{z}.; q^2), S_\alpha(\overline{w}.; q^2) \rangle_{F_{q,\alpha}}.$$

3. Multiplication and translation operators on $F_{q,\alpha}$

On $F_{q,\alpha}$, we consider the multiplication operators M and N_q given by

$$Mf(z) := z^2 f(z),$$

$$N_q f(z) := z D_q f(z) = \frac{f(z) - f(qz)}{1-q}.$$

We denote also by $\ell_{q,\alpha}$ the q-Bessel-Struve operator defined for entire functions on $D(0, \frac{1}{1-q})$.
We write

$$[\ell_{q,\alpha}, M] = \ell_{q,\alpha} M - M \ell_{q,\alpha}.$$

Then by straightforward calculation we obtain.

Lemma 3.1. $[\ell_{q,\alpha}, M] = (1+q)[2\alpha+2]_q B_{q^2} + W_{q,\alpha}$,
where

$$B_q f(z) := f(qz),$$

$$W_{q,\alpha} f(z) := (1+q)(1+q^{2\alpha}) qz D_q f(qz) + [2\alpha+1]_q z D_q f(0). \tag{7}$$

Remark 4. *The constant $(1+q)[2\alpha+2]_q$ equals to $c_2(\alpha;q^2)$, and the Lemma 3.1 is the analogous commutation rule of [7]. When $q \to 1^-$, then $[\ell_{q,\alpha}, M]$ tends to $4(\alpha+1)I + W$, where I is the identity operator and W is the operator given by*

$$Wf(z) := 4zDf(z) + (2\alpha+1)zDf(0).$$

Lemma 3.2. *If $f \in F_{q,\alpha}$ then $B_q f$, $N_q f$ and $W_{q,\alpha} f$ belong to $F_{q,\alpha}$, and*

(i) $\|B_q f\|_{F_{q,\alpha}} \le \|f\|_{F_{q,\alpha}}$,
(ii) $\|N_q f\|_{F_{q,\alpha}} \le \frac{1}{1-q} \|f\|_{F_{q,\alpha}}$,
(iii) $\|W_{q,\alpha} f\|_{F_{q,\alpha}} \le \frac{(1+q)(1+q^{2\alpha})}{1-q} \|f\|_{F_{q,\alpha}}$.

Proof. Let $f(z) = \sum_{n=0}^{\infty} a_n z^n \in F_{q,\alpha}$, then

$$B_q f(z) = f(qz) = \sum_{n=0}^{\infty} a_n q^n z^n, \tag{8}$$

$$N_q f(z) = \frac{f(z) - f(qz)}{1-q} = \sum_{n=0}^{\infty} a_n [n]_q z^n, \tag{9}$$

and from (3), we obtain

$$\|B_q f\|^2_{F_{q,\alpha}} = \sum_{n=0}^{\infty} |a_n|^2 q^{2n} c_n(\alpha;q^2) \le \sum_{n=0}^{\infty} |a_n|^2 c_n(\alpha;q^2) = \|f\|^2_{F_{q,\alpha}},$$

and

$$\|N_q f\|^2_{F_{q,\alpha}} = \sum_{n=0}^{\infty} |a_n|^2 ([n]_q)^2 c_n(\alpha;q^2).$$

Using the fact that $[n]_q \le \frac{1}{1-q}$, we deduce

$$\|N_q f\|^2_{F_{q,\alpha}} \le \frac{1}{(1-q)^2} \sum_{n=0}^{\infty} |a_n|^2 c_n(\alpha;q^2) = \frac{1}{(1-q)^2} \|f\|^2_{F_{q,\alpha}}.$$

On the other hand from (7), we have

$$W_{q,\alpha} f(z) = [2\alpha+1]_q a_1 z + (1+q)(1+q^{2\alpha}) \sum_{n=1}^{\infty} a_n [n]_q q^n z^n, \tag{10}$$

and

$$\|W_{q,\alpha} f\|^2_{F_{q,\alpha}} = (q+q^2+[2\alpha+3]_q)^2 |a_1|^2 c_1(\alpha;q^2) + [(1+q)(1+q^{2\alpha})]^2 \sum_{n=2}^{\infty} |a_n|^2 ([n]_q)^2 q^{2n} c_n(\alpha;q^2).$$

Using the fact that $[n]_q \le \frac{1}{1-q}$, we deduce that

$$\|W_{q,\alpha} f\|^2_{F_{q,\alpha}} \le \frac{[(1+q)(1+q^{2\alpha})]^2}{(1-q)^2} \sum_{n=1}^{\infty} |a_n|^2 c_n(\alpha;q^2).$$

Therefore, we conclude that

$$\|W_{q,\alpha}f\|_{F_{q,\alpha}} \leq \frac{(1+q)(1+q^{2\alpha})}{1-q}\|f\|_{F_{q,\alpha}}.$$

which completes the proof of the Lemma. □

In the classical Fock spaces F_α, if $f \in F_\alpha$, the functions $\ell_\alpha f$ and Mf are not necessarily elements of F_α. For thus, the authors statued these operators on $\mathcal{D}(\ell_\alpha) = \{f \in F_\alpha \ : \ \ell_\alpha f \in F_\alpha\}$ and $\mathcal{D}(M) = \{f \in F_\alpha \ : \ Mf \in F_\alpha\}$, respectively. But on $F_{q,\alpha}$, we can study the continuous property of the operators $\ell_{q,\alpha}$ and M from $F_{q,\alpha}$ into itself.

Theorem 3.1. *If $f \in F_{q,\alpha}$ then $\ell_{q,\alpha}f$ and Mf belong to $F_{q,\alpha}$, and we have*

(*i*) $\|\ell_{q,\alpha}f\|_{F_{q,\alpha}} \leq \frac{1}{1-q}\|f\|_{F_{q,\alpha}}$,

(*ii*) $\|Mf\|_{F_{q,\alpha}} \leq \frac{1}{1-q}\|f\|_{F_{q,\alpha}}$.

Proof. Let $f(z) = \sum_{n=0}^{\infty} a_n z^n \in F_{q,\alpha}$.

(i) From Lemma 2.3,

$$\ell_{q,\alpha}f(z) = \sum_{n=2}^{\infty} a_n \frac{c_n(\alpha;q^2)}{c_{n-2}(\alpha;q^2)} z^{n-2} = \sum_{n=0}^{\infty} a_{n+2} \frac{c_{n+2}(\alpha;q^2)}{c_n(\alpha;q^2)} z^n. \tag{11}$$

Then from (11), we get

$$\|\ell_{q,\alpha}f\|^2_{F_{q,\alpha}} = \sum_{n=0}^{\infty} |a_{n+2}|^2 \frac{c_{n+2}(\alpha;q^2)}{c_n(\alpha;q^2)}\, c_{n+2}(\alpha;q^2).$$

Using Lemma 2.2, we obtain

$$\|\ell_{q,\alpha}f\|^2_{F_{q,\alpha}} = \sum_{n=0}^{\infty} |a_{n+2}|^2 [n+2]_q [n+2\alpha+2]_q\, c_{n+2}(\alpha;q^2),$$

and consequently,

$$\|\ell_{q,\alpha}f\|^2_{F_{q,\alpha}} = \sum_{n=2}^{\infty} |a_n|^2 [n]_q [n+2\alpha]_q\, c_n(\alpha;q^2). \tag{12}$$

Using the fact that $[n]_q[n+2\alpha]_q \leq \frac{1}{(1-q)^2}$, we obtain

$$\|\ell_{q,\alpha}f\|_{F_{q,\alpha}} \leq \frac{1}{1-q}\Big[\sum_{n=0}^{\infty} |a_n|^2\, c_n(\alpha;q^2)\Big]^{1/2} = \frac{1}{1-q}\|f\|_{F_{q,\alpha}}.$$

(ii) On the other hand, since

$$Mf(z) = \sum_{n=2}^{\infty} a_{n-2}\, z^n, \tag{13}$$

then

$$\|Mf\|^2_{F_{q,\alpha}} = \sum_{n=2}^{\infty} |a_{n-2}|^2\, c_n(\alpha;q^2) = \sum_{n=0}^{\infty} |a_n|^2\, c_{n+2}(\alpha;q^2).$$

By Lemma 2.2, we deduce

$$\|Mf\|^2_{F_{q,\alpha}} = \sum_{n=0}^{\infty} |a_n|^2 [n+2]_q [n+2\alpha+2]_q\, c_n(\alpha;q^2). \tag{14}$$

Using the fact that $[n+2]_q[n+2\alpha+2]_q \le \frac{1}{(1-q)^2}$, we obtain

$$\|Mf\|_{F_{q,\alpha}} \le \frac{1}{1-q}\|f\|_{F_{q,\alpha}},$$

which completes the proof of the theorem. □

We deduce also the following norm equalities.

Theorem 3.2. *If $f \in F_{q,\alpha}$ then*

(i) $\langle f, W_{q,\alpha}f\rangle_{F_{q,\alpha}} = (1+q)(1+q^{2\alpha})\langle N_q f, B_q f\rangle_{F_{q,\alpha}} + [2\alpha+1]_q|D_q f(0)|^2 c_1(\alpha;q^2)$,

(ii) $\|\ell_{q,\alpha}f\|^2_{F_{q,\alpha}} = \|N_q f\|^2_{F_{q,\alpha}} + [2\alpha]_q\langle N_q f, B_q f\rangle_{F_{q,\alpha}} - [2\alpha+1]_q|D_q f(0)|^2 c_1(\alpha;q^2)$,

(iii) $\|Mf\|^2_{F_{q,\alpha}} = \|N_q f\|^2_{F_{q,\alpha}} + (1+q)[2\alpha+2]_q\|B_q f\|^2_{F_{q,\alpha}} + (1+q+[2\alpha+2]_q)\langle N_q f, B_q f\rangle_{F_{q,\alpha}}$,

(iv) $\|Mf\|^2_{F_{q,\alpha}} = \|\ell_{q,\alpha}f\|^2_{F_{q,\alpha}} + (1+q)[2\alpha+2]_q\|B_q f\|^2_{F_{q,\alpha}} + \langle f, W_{q,\alpha}f\rangle_{F_{q,\alpha}}$.

Proof. Let $f(z) = \sum_{n=0}^{\infty} a_n z^n \in F_{q,\alpha}$.

(i) Follows from (8), (9) and (10).

(ii) From (12), we get

$$\|\ell_{q,\alpha}f\|^2_{F_{q,\alpha}} = \sum_{n=0}^{\infty} |a_n|^2[n]_q[n+2\alpha]_q\, c_n(\alpha;q^2) - [2\alpha+1]_q|D_q f(0)|^2 c_1(\alpha;q^2).$$

Using the fact $[n+2\alpha]_q = [n]_q + q^n[2\alpha]_q$, we deduce

$$\|\ell_{q,\alpha}f\|^2_{F_{q,\alpha}} = \|N_q f\|^2_{F_{q,\alpha}} + [2\alpha]_q\langle N_q f, B_q f\rangle_{F_{q,\alpha}} - [2\alpha+1]_q|D_q f(0)|^2 c_1(\alpha;q^2).$$

(iii) By (14) and using the fact that

$$[n+2]_q[n+2\alpha+2]_q = ([n]_q)^2 + (1+q+[2\alpha+2]_q)q^n[n]_q + (1+q)[2\alpha+2]_q q^{2n},$$

we obtain

$$\|Mf\|^2_{F_{q,\alpha}} = \|N_q f\|^2_{F_{q,\alpha}} + (1+q)[2\alpha+2]_q\|B_q f\|^2_{F_{q,\alpha}} + (1+q+[2\alpha+2]_q)\langle N_q f, B_q f\rangle_{F_{q,\alpha}}.$$

(iv) Follows directly from (i), (ii) and (iii). □

Remark 5. (a) *Let $f(z) = \sum_{n=0}^{\infty} a_n z^n \in F_{q,\alpha}$. Since $\langle f, W_{q,\alpha}f\rangle_{F_{q,\alpha}} \ge 0$, then*

$$\|Mf\|^2_{F_{q,\alpha}} \ge (1+q)[2\alpha+2]_q\|B_q f\|^2_{F_{q,\alpha}}.$$

Therefore $Mf = 0$ implies that $f = 0$. Then $M : F_{q,\alpha} \to F_{q,\alpha}$ is injective continuous operator on $F_{q,\alpha}$.

(b) *In the classical Fock spaces F_α the norm inequalities of Theorem 3.2 are realized on $\mathcal{D}(M)$, and therefore $M : \mathcal{D}(M) \to F_{q,\alpha}$ is injective continuous operator on $\mathcal{D}(M)$.*

In the classical Fock spaces F_α, for $f \in \mathcal{D}(M)$ and $g \in \mathcal{D}(\ell_\alpha)$, we have

$$\langle Mf, g\rangle_{F_\alpha} = \langle f, \ell_\alpha g\rangle_{F_\alpha}.$$

But in $F_{q,\alpha}$, and since $\mathcal{D}(M) = \mathcal{D}(\ell_{q,\alpha}) = F_{q,\alpha}$, we obtain the following.

Proposition 3.1. *The operators M and $\ell_{q,\alpha}$ are adjoint-operators on $F_{q,\alpha}$; and for all $f, g \in F_{q,\alpha}$, we have*

$$\langle Mf, g\rangle_{F_{q,\alpha}} = \langle f, \ell_{q,\alpha}g\rangle_{F_{q,\alpha}}.$$

Proof. Consider $f(z)=\sum_{n=0}^{\infty}a_nz^n$ and $g(z)=\sum_{n=0}^{\infty}b_nz^n$ in $F_{q,\alpha}$. From (11) and (13),

$$\ell_{q,\alpha}g(z)=\sum_{n=0}^{\infty}b_{n+2}\frac{c_{n+2}(\alpha;q^2)}{c_n(\alpha;q^2)}z^n,$$

and

$$Mf(z)=\sum_{n=2}^{\infty}a_{n-2}\,z^n.$$

Thus from (4), we get

$$\langle Mf,g\rangle_{F_{q,\alpha}}=\sum_{n=2}^{\infty}a_{n-2}\overline{b_n}\,c_n(\alpha;q^2)=\sum_{n=0}^{\infty}a_n\overline{b_{n+2}}\,c_{n+2}(\alpha;q^2)=\langle f,\ell_{q,\alpha}g\rangle_{F_{q,\alpha}},$$

which gives the result. □

In the next part of this section we study a generalized translation and multiplication operators on $F_{q,\alpha}$. We begin by the following definition.

Definition 3.1. *For $f\in F_{q,\alpha}$ and $w,z\in D(0,\frac{1}{1-q})$, we define:*
- The q-translation operators on $F_{q,\alpha}$, by

$$T_zf(w):=\sum_{n=0}^{\infty}\frac{\ell^n_{q,\alpha}f(w)}{c_n(\alpha;q^2)}z^n. \tag{15}$$

- The generalized multiplication operators on $F_{q,\alpha}$, by

$$M_zf(w):=\sum_{n=0}^{\infty}\frac{M^nf(w)}{c_n(\alpha;q^2)}z^n. \tag{16}$$

For $w,\ z\in D(0,\frac{1}{1-q})$, the function $S_\alpha(.;q^2)$ satisfies the following product formulas:

$$T_zS_\alpha(.;q^2)(w)=S_\alpha(z;q^2)S_\alpha(w;q^2),$$
$$M_zS_\alpha(.;q^2)(w)=S_\alpha(zw^2;q^2)S_\alpha(w;q^2).$$

Proposition 3.2. *Let $f(z)=\sum_{n=0}^{\infty}a_nz^n\in F_{q,\alpha}$ and $z,w\in D(0,\frac{1}{1-q})$. Then*

(i) $T_zf(w)=\sum_{n=0}^{\infty}a_n\Big[\sum_{k=0}^{[n/2]}\gamma(n,k;q^2)\frac{\Gamma_{q^2}(\alpha+1)\Gamma_{q^2}(\frac{n}{2}+\alpha+1)}{\Gamma_{q^2}(\frac{k}{2}+\alpha+1)\Gamma_{q^2}(\frac{n}{2}-k+\alpha+1)}(\frac{z}{w^2})^k\Big]w^n,$ *where $[n/2]$ is the integer part of $n/2$ and*

$$\gamma(n,k;q^2)=\frac{(1+q)^k\Gamma_{q^2}(\frac{n}{2}+1)}{\Gamma_{q^2}(\frac{k}{2}+1)\Gamma_{q^2}(\frac{n}{2}-k+1)}.$$

(ii) $M_zf(w)=\sum_{n=0}^{\infty}\Big[\sum_{k=0}^{[n/2]}\frac{a_{n-2k}}{c_k(\alpha;q^2)}z^k\Big]w^n.$

Proof. Let $f(z)=\sum_{n=0}^{\infty}a_nz^n\in F_{q,\alpha}$.
(i) From (15), we have

$$T_zf(w)=\sum_{n=0}^{\infty}\frac{\ell^n_{q,\alpha}f(w)}{c_n(\alpha;q^2)}z^n;\quad w,\ z\in D(0,\frac{1}{1-q}).$$

Since from Lemma 2.3,

$$\ell^n_{q,\alpha}w^k=\frac{c_k(\alpha;q^2)}{c_{k-2n}(\alpha;q^2)}w^{k-2n},\quad k\geq 2n,$$

we can write

$$\ell^n_{q,\alpha} f(w) = \sum_{k=2n}^{\infty} a_k \frac{c_k(\alpha;q^2)}{c_{k-2n}(\alpha;q^2)} w^{k-2n}.$$

Thus we obtain

$$T_z f(w) = \sum_{n=0}^{\infty} a_n \sum_{k=0}^{[n/2]} \frac{c_n(\alpha;q^2)}{c_k(\alpha;q^2)c_{n-2k}(\alpha;q^2)} w^{n-2k} z^k.$$

On the other hand from (1), we get

$$\frac{c_n(\alpha;q^2)}{c_k(\alpha;q^2)c_{n-2k}(\alpha;q^2)} = \gamma(n,k;q^2)\frac{\Gamma_{q^2}(\alpha+1)\Gamma_{q^2}(\frac{n}{2}+\alpha+1)}{\Gamma_{q^2}(\frac{k}{2}+\alpha+1)\Gamma_{q^2}(\frac{n}{2}-k+\alpha+1)},$$

which gives the (i).

(ii) From (16), we have

$$M_z f(w) = \sum_{n=0}^{\infty} \frac{M^n f(w)}{c_n(\alpha;q^2)} z^n; \quad w,\, z \in D(0, \frac{1}{1-q}).$$

But from (13), we have

$$M^n f(w) = \sum_{k=2n}^{\infty} a_{k-2n} w^k.$$

Thus we obtain

$$M_z f(w) = \sum_{n=0}^{\infty} \Big[\sum_{k=0}^{[n/2]} \frac{a_{n-2k}}{c_k(\alpha;q^2)} z^k \Big] w^n,$$

which completes the proof of the proposition. □

In the classical Fock spaces F_α, if $f \in F_\alpha$, then $\tau_z f$ and $M_z f$, $z \in \mathbb{C}$ are not necessarily elements of F_α. But on $F_{q,\alpha}$, and according to Theorem 3.1 we can study the continuous property of the operators T_z and M_z on $F_{q,\alpha}$, for $z \in D(0, \frac{1}{1-q})$.

Theorem 3.3. *If $f \in F_{q,\alpha}$ and $z \in D(0, \frac{1}{1-q})$, then $T_z f$ and $M_z f$ belong to $F_{q,\alpha}$, and*

(i) $\|T_z f\|_{F_{q,\alpha}} \le S_\alpha(\frac{|z|}{1-q}; q^2)\|f\|_{F_{q,\alpha}}$,

(ii) $\|M_z f\|_{F_{q,\alpha}} \le S_\alpha(\frac{|z|}{1-q}; q^2)\|f\|_{F_{q,\alpha}}$.

Proof. From (15) and Theorem 3.1 (i), we deduce

$$\|T_z f\|_{F_{q,\alpha}} \le \sum_{n=0}^{\infty} \|\ell^n_{q,\alpha} f\|_{F_{q,\alpha}} \frac{|z|^n}{c_n(\alpha;q^2)} \le \sum_{n=0}^{\infty} \frac{|z|^n}{(1-q)^n c_n(\alpha;q^2)} \|f\|_{F_{q,\alpha}}.$$

Therefore,

$$\|T_z f\|_{F_{q,\alpha}} \le S_\alpha(\frac{|z|}{1-q}; q^2)\|f\|_{F_{q,\alpha}},$$

which gives the first inequality, and as in the same way we prove the second inequality of this theorem. □

From Proposition 3.1 we deduce the following results.

Proposition 3.3. *For all $f, g \in F_{q,\alpha}$ and $z \in D(0, \frac{1}{1-q})$, we have*

$$\langle M_z f, g\rangle_{F_{q,\alpha}} = \langle f, T_{\overline{z}} g\rangle_{F_{q,\alpha}},$$

$$\langle T_z f, g\rangle_{F_{q,\alpha}} = \langle f, M_{\overline{z}} g\rangle_{F_{q,\alpha}}.$$

We denote by R_z, $z \in D(0, \frac{1}{1-q})$ the following operator defined on $F_{q,\alpha}$ by

$$R_z := T_{\overline{z}} M_z - M_{\overline{z}} T_z = S_\alpha(\overline{z}\ell_{q,\alpha}; q^2) S_\alpha(zM; q^2) - S_\alpha(\overline{z}M; q^2) S_\alpha(z\ell_{q,\alpha}; q^2).$$

Then, we prove the following theorem.

Theorem 3.4. *For all $f \in F_{q,\alpha}$ and $z \in D(0, \frac{1}{1-q})$, we have*

(i) $\|M_z f\|^2_{F_{q,\alpha}} = \|T_z f\|^2_{F_{q,\alpha}} + \langle f, R_z f\rangle_{F_{q,\alpha}}$,

(ii) $\|R_z f\|_{F_{q,\alpha}} \leq 2[S_\alpha(\frac{|z|}{1-q}; q^2)]^2 \|f\|_{F_{q,\alpha}}$.

Proof. (i) From Proposition 3.3, we get

$$\|M_z f\|^2_{F_{q,\alpha}} = \langle f, T_{\overline{z}} M_z f\rangle_{F_{q,\alpha}} = \langle f, (M_{\overline{z}} T_z + R_z) f\rangle_{F_{q,\alpha}} = \|T_z f\|^2_{F_{q,\alpha}} + \langle f, R_z f\rangle_{F_{q,\alpha}}.$$

(ii) From Theorem 3.3, we have

$$\|R_z f\|_{F_{q,\alpha}} \leq \|T_{\overline{z}} M_z f\|_{F_{q,\alpha}} + \|M_{\overline{z}} T_z\|_{F_{q,\alpha}} \leq 2[S_\alpha(\frac{|z|}{1-q}; q^2)]^2 \|f\|_{F_{q,\alpha}},$$

which completes the proof. □

4. Application: Extremal function on $F_{q,\alpha}$

In this section we shall give an application of the theory of reproducing kernels to the Tikhonov regularization, which gives the approximate solutions for bounded linear operator equation on the Fock spaces $F_{q,\alpha}$.

Definition 4.1. *Let $\lambda > 0$ and let $L : F_{q,\alpha} \to F_{q,\alpha}$ be a bounded linear operator from $F_{q,\alpha}$ into itself. We denote by $\langle ., .\rangle_{\lambda, F_{q,\alpha}}$ the inner product defined on the space $F_{q,\alpha}$ by*

$$\langle f, g\rangle_{\lambda, F_{q,\alpha}} := \lambda \langle f, g\rangle_{F_{q,\alpha}} + \langle Lf, Lg\rangle_{F_{q,\alpha}}, \tag{17}$$

and the norm $\|f\|_{\lambda, F_{q,\alpha}} := \sqrt{\langle f, f\rangle_{\lambda, F_{q,\alpha}}}$.

As examples of the operator L we can choose the precedent operators B_q, N_q, $W_{q,\alpha}$, $\ell_{\alpha,q}$, M, T_z, M_z and R_z, when $z \in D(0, \frac{1}{1-q})$.

Remark 6. *Let $\lambda > 0$ and let $f \in F_{q,\alpha}$. The two norms $\|.\|_{F_{q,\alpha}}$ and $\|.\|_{\lambda, F_{q,\alpha}}$ are equivalent, and*

$$\sqrt{\lambda}\, \|f\|_{F_{q,\alpha}} \leq \|f\|_{\lambda, F_{q,\alpha}} \leq \sqrt{\lambda + \|L\|^2}\, \|f\|_{F_{q,\alpha}}.$$

Lemma 4.1. *Let $\lambda > 0$. The Fock space $(F_{q,\alpha}, \langle ., .\rangle_{\lambda, F_{q,\alpha}})$ possesses a reproducing kernel $K_L(w, z)$; $w, z \in D(0, \frac{1}{1-q})$ which satisfying the equation*

$$(\lambda I + L^* L) K_L(w, .) = K_{q,\alpha}(w, .), \tag{18}$$

where L^* is the adjoint of L in $(F_{q,\alpha}, \langle ., .\rangle_{F_{q,\alpha}})$.

Proof. Let $f \in F_{q,\alpha}$. From relation (6) and Remark 6, we have

$$|f(w)| \leq \|K_{q,\alpha}(w, .)\|_{F_{q,\alpha}} \|f\|_{F_{q,\alpha}} = \Big[\frac{S_\alpha(|w|^2; q^2)}{\lambda}\Big]^{1/2} \|f\|_{\lambda, F_{q,\alpha}}.$$

Then, the map $f \to f(w)$, $w \in D(0, \frac{1}{1-q})$ is a continuous linear functional on $(F_{q,\alpha}, \langle ., .\rangle_{\lambda, F_{q,\alpha}})$. Thus from [1], $(F_{q,\alpha}, \langle ., .\rangle_{\lambda, F_{q,\alpha}})$ has a reproducing kernel denoted

by $K_L(w,z)$.
On the other hand,

$$\begin{aligned} f(w) &= \lambda\langle f, K_L(w,.)\rangle_{F_{q,\alpha}} + \langle Lf, L[K_L(w,.)]\rangle_{F_{q,\alpha}} \\ &= \langle f, (\lambda I + L^*L)[K_L(w,.)]\rangle_{F_{q,\alpha}}. \end{aligned}$$

Thus,

$$(\lambda I + L^*L)K_L(w,.) = K_{q,\alpha}(w,.).$$

This clearly yields the result. □

Example 4.1. *Let* $w, z \in D(0, \frac{1}{1-q})$.

(*a*) *If* $L = M$, *then*

$$K_L(w,z) = \sum_{n=0}^{\infty} \frac{(\overline{w}z)^n}{(\lambda + [n+2]_q[n+2\alpha+2]_q)c_n(\alpha;q^2)}.$$

(*b*) *If* $L = \ell_{q,\alpha}$, *then*

$$K_L(w,z) = \frac{1}{\lambda c_0(\alpha;q^2)} + \frac{\overline{w}z}{\lambda c_1(\alpha;q^2)} + \sum_{n=2}^{\infty} \frac{(\overline{w}z)^n}{(\lambda + [n]_q[n+2\alpha]_q)c_n(\alpha;q^2)}.$$

(*c*) *If* $L = B_q$, *then*

$$K_L(w,z) = \sum_{n=0}^{\infty} \frac{(\overline{w}z)^n}{(\lambda + q^{2n})c_n(\alpha;q^2)}.$$

(*d*) *If* $L = N_q$, *then*

$$K_L(w,z) = \sum_{n=0}^{\infty} \frac{(\overline{w}z)^n}{(\lambda + ([n]_q)^2)c_n(\alpha;q^2)}.$$

(*e*) *If* $L = W_{q,\alpha}$, *then*

$$K_L(w,z) = \frac{1}{\lambda c_0(\alpha;q^2)} + \frac{\overline{w}z}{([2\alpha+1]_q)^2 c_1(\alpha;q^2)} + \sum_{n=1}^{\infty} \frac{(\overline{w}z)^n}{(\lambda + \eta([n]_q)^2 q^{2n})c_n(\alpha;q^2)},$$

where

$$\eta = (1+q)^2(1+q^{2\alpha})^2.$$

The main result of this section can then be stated as follows.

Theorem 4.1. *For any* $h \in F_{q,\alpha}$ *and for any* $\lambda > 0$, *there exists a unique function* $f^*_{\lambda,h}$, *where the infimum*

$$\inf_{f\in F_{q,\alpha}} \left\{ \lambda\|f\|^2_{F_{q,\alpha}} + \|h - Lf\|^2_{F_{q,\alpha}} \right\} \tag{19}$$

is attained. Moreover, the extremal function $f^*_{\lambda,h}$ *is given by*

$$f^*_{\lambda,h}(w) = \langle h, L[K_L(w,.)]\rangle_{F_{q,\alpha}},$$

where K_L *is the kernel given by* (18).

Proof. The existence and unicity of the extremal function $f^*_{\lambda,h}$ satisfying (19), is given by [9, 11, 13]. Moreover, by Lemma 4.1 we deduce that

$$f^*_{\lambda,h}(w) = \langle h, L[K_L(w,.)]\rangle_{F_{q,\alpha}}, \tag{20}$$

where K_L is the kernel given by (18).
This clearly yields the result. □

Remark 7. *The extremal function* $f^*_{\lambda,h}$ *satisfies the following inequalities*

$$\begin{aligned}|f^*_{\lambda,h}(w)| &\le \|L\|\|h\|_{F_{q,\alpha}}\|K_L(w,.)\|_{F_{q,\alpha}}\\ &\le \frac{\|L\|}{\sqrt{\lambda}}\|h\|_{F_{q,\alpha}}\|K_L(w,.)\|_{\lambda,F_{q,\alpha}}\\ &\le \frac{\|L\|}{\sqrt{\lambda}}[K_L(w,w)]^{1/2}\|h\|_{F_{q,\alpha}}.\end{aligned}$$

If we take in (20), $h = Lf$, where $f \in F_{q,\alpha}$, we obtain the following Calderón's reproducing formula.

Theorem 4.2. *(**Calderón's formula**). Let* $\lambda > 0$ *and* $f \in F_{q,\alpha}$*. The extremal function* f^*_λ *given by*

$$f^*_\lambda(w) = \langle Lf, L[K_L(w,.)]\rangle_{F_{q,\alpha}},$$

satisfies

$$f(w) = \lim_{\lambda\to 0^+} f^*_\lambda(w) = \lim_{\lambda\to 0^+}\langle Lf, L[K_L(w,.)]\rangle_{F_{q,\alpha}}.$$

Proof. Let $f \in F_{q,\alpha}$, $h = Lf$ and $f^*_\lambda = f^*_{\lambda,Lf}$. Then

$$f^*_\lambda(w) = \langle f, L^*L[K_L(w,.)]\rangle_{F_{q,\alpha}}. \tag{21}$$

But from (18), we have

$$\lim_{\lambda\to 0^+} L^*L[K_L(w,.)] = K_{q,\alpha}(w,.).$$

Thus,

$$\lim_{\lambda\to 0^+} f^*_\lambda(w) = \langle f, K_{q,\alpha}(w,.)\rangle_{F_{q,\alpha}} = f(w),$$

which ends the proof. □

Remark 8. *Let* $w \in D(0, \frac{1}{1-q})$*. From* (18) *and* (21)*, the extremal function* f^*_λ *satisfies*

$$f^*_\lambda(w) = f(w) - \lambda\langle f, K_L(w,.)\rangle_{F_{q,\alpha}}.$$

Thus we obtain

$$\lim_{\lambda\to 0^+}\lambda K_L(w,.) = 0,$$

and

$$\begin{aligned}|f^*_\lambda(w) - f(w)| &\le \lambda\|f\|_{F_{q,\alpha}}\|K_L(w,.)\|_{F_{q,\alpha}}\\ &\le \sqrt{\lambda}\|f\|_{F_{q,\alpha}}\|K_L(w,.)\|_{\lambda,F_{q,\alpha}}\\ &\le \sqrt{\lambda}[K_L(w,w)]^{1/2}\|f\|_{F_{q,\alpha}}.\end{aligned}$$

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